

## Low temperature condensation and scattering data

---

**Oliver Orasch\***

*University of Graz, Institute of Physics, 8010 Graz, Austria*

*E-mail: [oliver.orasch@uni-graz.at](mailto:oliver.orasch@uni-graz.at)*

**Christof Gattringer**

*University of Graz, Institute of Physics, 8010 Graz, Austria*

*E-mail: [christof.gattringer@uni-graz.at](mailto:christof.gattringer@uni-graz.at)*

**Mario Giuliani**

*University of Graz, Institute of Physics, 8010 Graz, Austria*

*E-mail: [mario.giuliani@uni-graz.at](mailto:mario.giuliani@uni-graz.at)*

We study  $\phi^4$  lattice field theory at finite chemical potential  $\mu$  in two and four dimensions, using a worldline representation that overcomes the complex action problem. We compute the particle number at very low temperature as a function of  $\mu$  and determine the first three condensation thresholds, where the system condenses 1, 2 and 3 particles. The corresponding critical values of the chemical potential can be related to the 1-, 2- and 3-particle energies of the system, and we check this relation with a direct spectroscopy determination of the  $n$ -particle energies from  $2n$ -point functions. We analyze the thresholds as a function of the spatial size of the system and use the known finite volume results for the  $n$ -particle energies to relate the thresholds to scattering data. For four dimensions we determine the scattering length from the 2-particle threshold, while in two dimensions the full scattering phase shift can be determined. In both cases the scattering data computed from the 2-particle threshold already allow one to determine the 3-particle energy. In both, two and four dimensions we find very good agreement of this "prediction" with direct determinations of the 3-particle energy from either the thresholds or the 6-point functions. The results show that low temperature condensation is indeed governed by scattering data.

*The 36th Annual International Symposium on Lattice Field Theory - LATTICE2018*

*22-28 July, 2018*

*Michigan State University, East Lansing, Michigan, USA.*

---

\*Speaker.

## 1. Introduction

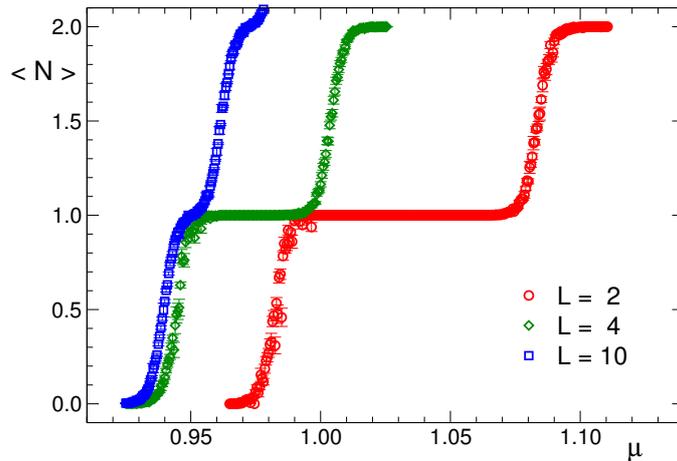
In recent years considerable progress was made with overcoming the complex action problem at finite density for several lattice field theories. It was possible to exactly map the partition sum of these systems to a representation in terms of worldlines (and/or worldsheets) where all contributions to the partition sum are real and positive, such that a Monte Carlo simulation can be done directly in terms of the worldlines (see, e.g., [1, 2, 3, 4] for work on the  $\phi^4$  theory studied here). With the worldline approach it is possible to address new physics questions related to finite density. An example is condensation of particles at low temperatures, which is the topic of this contribution.

To illustrate the condensation phenomenon we study here, in Fig. 1 we show the results for the expectation value of the particle number  $\langle N \rangle$  versus the chemical potential  $\mu$ . The results are for  $\phi^4$  theory in 2d on a  $L \times N_t$  lattice with  $N_t = 400$  and three different values of  $L$ . The temperature is very low ( $T = 1/N_t = 0.0025$  in lattice units) and indeed we observe condensation as a function of  $\mu$  (details see below). For each of the three values of  $L$  the particle number  $\langle N \rangle$  quickly rises from  $\langle N \rangle = 0$  to  $\langle N \rangle = 1$  at some critical chemical potential value  $\mu_1(L)$ , then further to  $\langle N \rangle = 2$  at a second critical value  $\mu_2(L)$  and similar for higher particle number sectors. Note that at zero temperature ( $N_t = \infty$ ) one expects discontinuous jumps of  $\langle N \rangle$  which are here rounded by temperature effects. Nevertheless we can identify the critical values  $\mu_n(L)$  and determine their values as a function of  $L$ . The values  $\mu_n(L)$  correspond to the values of the chemical potential where we observe condensation of another particle visible in the step from  $\langle N \rangle = n - 1$  to  $\langle N \rangle = n$ .

In [5] it was shown that at very low temperature the condensation thresholds  $\mu_n$  are related to the physical mass  $m(L)$  and the  $n$ -particle energies  $W_n(L)$  via the relations

$$m(L) = \mu_1(L), \quad W_n(L) = \sum_{k=1}^n \mu_k(L), \quad (1.1)$$

where we now made explicit, that not only the  $\mu_n(L)$ , but also the physical mass  $m(L)$  and the  $n$ -particle energies  $W_n(L)$  depend on the spatial extent  $L$ . It has been known since the pioneering paper [6] that the dependence of the 2- and 3-particle energies  $W_2(L)$  and  $W_3(L)$  on the spatial extent  $L$



**Figure 1:** The expectation value of the particle number  $\langle N \rangle$  as a function of the chemical potential  $\mu$  (in lattice units). We show the results for the 2d case at  $N_t = 400$  and different values of  $L$ .

can be parameterized in terms of scattering data of the underlying theory. Thus we conclude from (1.1) that the condensation thresholds  $\mu_n(L)$  are governed by the scattering data.

Demonstrating and analyzing the connection between low temperature condensation and scattering data in  $\phi^4$  theory in two and four dimensions is the topic of this contribution (see also [7]).

## 2. Worldline representation and Monte Carlo simulation

The system where we explore the relation between condensation and scattering data is the complex  $\phi^4$  field in  $d = 2$  and  $d = 4$  dimensions. The lattice action is given by

$$S[\phi] = \sum_{x \in \Lambda} \left( \eta |\phi_x|^2 + \lambda |\phi_x|^4 - \sum_{\hat{v}=1}^d \left[ e^{\mu \delta_{v,d}} \phi_x^* \phi_{x+\hat{v}} + e^{-\mu \delta_{v,d}} \phi_{x+\hat{v}}^* \phi_x \right] \right), \quad (2.1)$$

where  $\eta \equiv 2d + m_b^2$  with  $m_b$  the bare mass parameter.  $\lambda$  is the quartic coupling and  $\mu$  the chemical potential. The fields  $\phi_x$  are assigned to the sites  $x$  of a lattice of size  $L^{d-1} \times N_t$ .

At  $\mu \neq 0$  the action is complex and the Boltzmann factor  $e^{-S[\phi]}$  cannot be used as a probability in a Monte Carlo simulation. This complex action problem of the conventional representation (2.1) can be solved by exactly mapping the system to a worldline representation (see, e.g., [3] for a derivation of the form we use here). In the worldline representation the partition sum reads ( $\beta \equiv N_t$ )

$$Z = \sum_{\{k\}} \left[ \prod_x \delta(\vec{\nabla} \cdot \vec{k}_x) \right] e^{\mu \beta \omega[k]} B[k]. \quad (2.2)$$

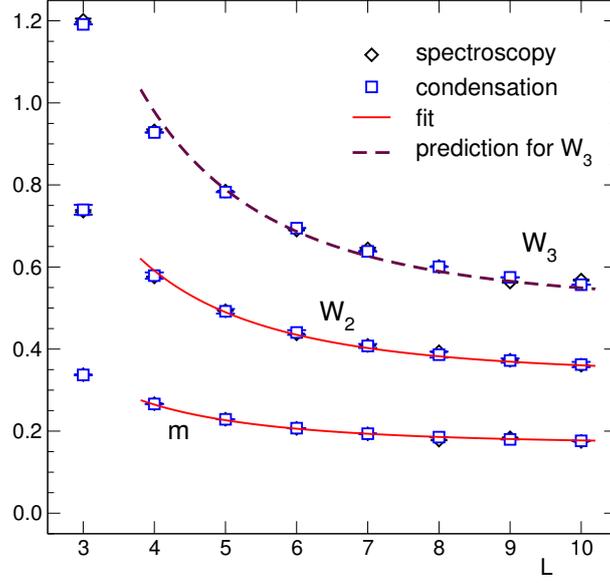
$Z$  is a sum over configurations of the worldline variables  $k_{x,v} \in \mathbb{Z}$  assigned to the links of the lattice. They have to obey constraints which have the form of a product over Kronecker deltas  $\delta(j) \equiv \delta_{j,0}$  at all sites  $x$ . At each  $x$  the Kronecker deltas enforce  $\vec{\nabla} \cdot \vec{k}_x \equiv \sum_v (k_{x,v} - k_{x-\hat{v},v}) = 0$ , i.e., zero divergence for  $k_{x,v}$ , and as a consequence the worldline variables  $k_{x,v}$  must form closed loops of conserved flux. By  $\omega[k]$  we denote the total winding number of the  $k$ -flux around the compact time direction and the chemical potential couples to  $\omega[k]$  in the form  $e^{\mu \beta \omega[k]}$ . The observable we need for our analysis is the expectation value of the particle number  $\langle N \rangle = \partial \ln Z / \partial \beta \mu = \langle \omega[k] \rangle_{wl}$ , where  $\langle \cdot \rangle_{wl}$  denotes the vacuum expectation in the worldline representation.

The configurations of the worldline variables  $k_{x,v}$  come with a real and positive weight factor

$$B[k] = \sum_{\{a\}} \prod_{x,v} \frac{1}{(a_{x,v} + |k_{x,v}|)! a_{x,v}!} \prod_x I(s_x) \quad \text{with} \quad I(s_x) = \int_0^\infty dr r^{s_x+1} e^{-\eta r^2 - \lambda r^4}. \quad (2.3)$$

$B[k]$  is a sum over configurations  $\sum_{\{a\}}$  of auxiliary link variables  $a_{x,v} \in \mathbb{N}_0$ , and by  $s_x$  we denote the non-negative integer combination  $s_x = \sum_v [ |k_{x,v}| + |k_{x-\hat{v},v}| + 2(a_{x,v} + a_{x-\hat{v},v}) ]$ , which appears as an argument in the integrals  $I(s_x)$  that come from integrating out the radial degrees of freedom of the original field variables at site  $x$ . They are pre-calculated and stored for the simulations.

All weight factors in the worldline representation are real and positive such that the complex action problem is solved. Concerning the details of the updates for the worldline variables  $k_{x,v}$  and the auxiliary variables  $a_{x,v}$  we refer to [8, 9]. In 4d we use lattices with  $N_t = 320$  and  $640$ , and  $L$  between 3 and 10 at coupling values of  $\eta = 7.44$  and  $\lambda = 1.0$ , with a statistics of  $2 \times 10^5$  configurations. In 2d the corresponding parameters are  $N_t = 400$ ,  $L$  between 2 and 16 with  $\eta = 2.6$ ,  $\lambda = 1.0$  and a statistics of  $4 \times 10^5$ .



**Figure 2:** The physical mass  $m(L)$  and the 2- and 3-particle energies  $W_2(L)$  and  $W_3(L)$  for the 4d case as a function of the lattice extent  $L$  (figure from [7]). We show the results determined from the condensation steps (blue squares) and compare them to the results from spectroscopy (black diamonds). The full red curves are the fits of  $m$  and  $W_2$  with (3.1) and (3.2). The dashed maroon curve is the function  $W_3$  from Eq. (3.3) when using the scattering length  $a$  from the fit of  $W_2$  as input.

### 3. Analysis of the 4d case

After computing  $\langle N \rangle$  as a function of  $\mu$  we identify the steps where  $\langle N \rangle$  transits from  $\langle N \rangle = n - 1$  to  $\langle N \rangle = n$  (compare Fig. 1). To determine the corresponding critical values  $\mu_n$  we fit the data for  $\langle N \rangle$  in the vicinity of the steps with the logistic function  $\langle N \rangle = [1 + \exp(-a_n[\mu - \mu_n])]^{-1} + n - 1$ . Using (1.1) we then compute  $m(L)$ ,  $W_2(L)$  and  $W_3(L)$  from the critical values  $\mu_n(L)$ .

In Fig. 2 we show the results for  $m(L)$ ,  $W_2(L)$  and  $W_3(L)$  determined from the critical chemical potential values  $\mu_n(L)$  as squares. To test the relations (1.1) and the reliability of our determination of the critical values  $\mu_n(L)$ , we computed  $m(L)$ ,  $W_2(L)$  and  $W_3(L)$  also in a spectroscopy analysis based on  $2n$ -point functions calculated at  $\mu = 0$  in the conventional representation (2.1). The corresponding results are shown as diamonds in Fig. 2 and coincide almost perfectly with the data from the condensation steps. This cross check confirms the interpretation of the critical chemical potential values as combinations of multi-particle energies.

The next step is to invoke the finite volume relations for  $m(L)$  [10], the result [6, 11] for the 2-particle energy  $W_2(L)$  (using the notation of [13]) and the results [12, 13, 14, 15, 16] for the 3-particle energy  $W_3(L)$  (the numerical constants  $\mathcal{I}$  and  $\mathcal{J}$  are given by  $\mathcal{I} = -8.914$ ,  $\mathcal{J} = 16.532$ ):

$$m(L) = m_\infty + \frac{A}{L^{\frac{3}{2}}} e^{-L m_\infty}, \quad (3.1)$$

$$W_2(L) = 2m + \frac{4\pi a}{mL^3} \left[ 1 - \frac{a}{L} \frac{\mathcal{I}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{I}^2 - \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3 \right], \quad (3.2)$$

$$W_3(L) = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \frac{a}{L} \frac{\mathcal{J}}{\pi} + \left(\frac{a}{L}\right)^2 \frac{\mathcal{J}^2 + \mathcal{J}}{\pi^2} + \mathcal{O}\left(\frac{a}{L}\right)^3 \right]. \quad (3.3)$$

Up to order  $1/L^5$  only three parameters are needed to describe the data: the infinite volume mass  $m_\infty$ , the amplitude  $A$  and the scattering length  $a$ . Fitting the data for  $m(L)$  with the relation (3.1) we find a value of  $m_\infty = 0.168(1)$  in lattice units. For fitting  $W_2(L)$  we use (3.2), with the mass parameter  $m$  on the rhs. replaced by the corresponding values  $m(L)$ . This gives rise to a value of  $a = -0.078(7)$  for the scattering length in lattice units and a value of  $am_\infty = -0.013(1)$  for the dimensionless product of  $a$  and  $m_\infty$ . The functions (3.1) and (3.2) with the fit values for  $m_\infty$ ,  $A$  and  $a$  are shown as full red curves in Fig. 2 and describe the data for  $m(L)$  and  $W_2(L)$  very well (with the exception of the smallest  $L$  where higher corrections in  $1/L$  would be necessary).

Having determined the mass and the scattering length, no further parameters are necessary to describe  $W_3(L)$  with (3.3). Inserting the fit value for  $a$  and again using  $m(L)$  in the rhs. of (3.3), we thus get a "prediction" for the data  $W_3(L)$ . This prediction is shown as a dashed curve in Fig. 2 and obviously describes the data for  $W_3(L)$  very well (again with the exception of the smallest  $L$ ).

This concludes the discussion of the 4d case and our results confirm the relations (1.1) of the condensation thresholds to multi-particle energies, which in turn are described by scattering data. Thus we have quantitatively established the connection of condensation and scattering data.

#### 4. Analysis of the 2d case

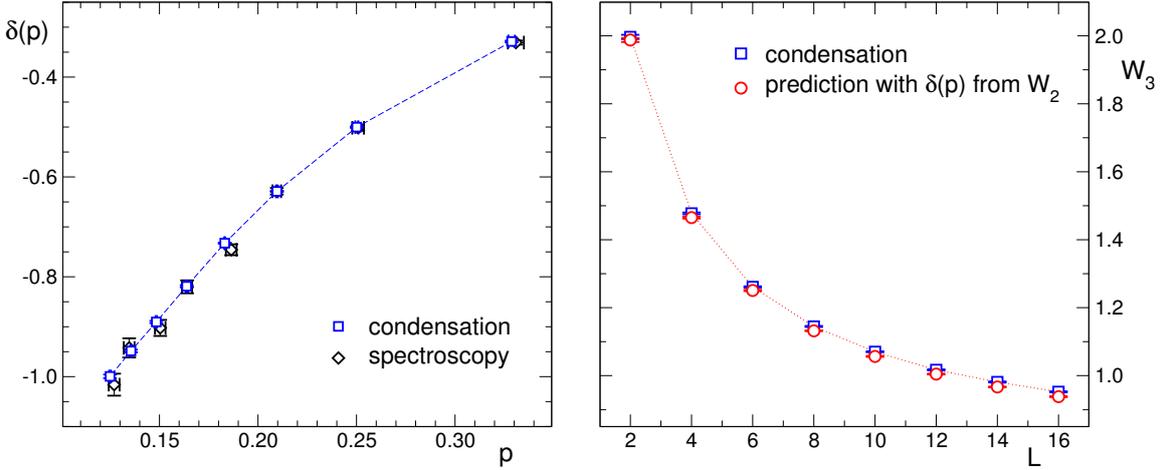
Also in the 2d case we determined the critical values  $\mu_n(L)$  from fitting the steps of  $\langle N \rangle$  and then computed  $m(L)$ ,  $W_2(L)$  and  $W_3(L)$  using the relations (1.1). We cross-checked these results with a spectroscopy calculation in the conventional representation and again found very good agreement between the condensation and the spectroscopy results.

The next step is the finite volume analysis of  $m(L)$ ,  $W_2(L)$  and  $W_3(L)$ . As before the mass  $m(L)$  can be described with a 2-parameter ansatz, which in 2d reads  $m(L) = m_\infty + A e^{-m_\infty L} / \sqrt{L}$ . For analyzing the 2-particle energy  $W_2(L)$  we follow the approach [17] that is applicable to short range potentials. Outside the interaction range the wave function is a 2-particle plane wave  $\psi = e^{-ix_1 p_1} e^{-ix_2 p_2}$  with momenta  $p_1$  and  $p_2$ . The corresponding energy is  $W_2(L) = \sum_{j=1}^2 \sqrt{m(L)^2 + p_j^2}$ . We rewrite the wave function  $\psi$  by using the center of mass coordinate  $(x_1 + x_2)/2$  and the relative coordinate  $r = x_1 - x_2$ . The energy values  $W_2(L)$  determined from the condensation steps correspond to vanishing total momentum  $p_1 + p_2 = 0$ , and we set  $p_1 = p = -p_2$ . For vanishing total momentum the wave function then has the form  $\psi = e^{-ipr}$  and the 2-particle energy is given by

$$W_2(L) = 2\sqrt{m(L)^2 + p^2}. \quad (4.1)$$

For finalizing the connection between the 2-particle energy and the scattering data we need to invoke the quantization for the momenta  $p$  in a finite box of size  $L$ . This condition is obtained from the boundary condition for  $\psi$  which connects the wave function at  $r = 0$  to its value at  $r = L$  and reads  $e^{-ipL} = e^{i2\delta(p)}$ . It expresses the fact that the plane wave solution is correct only outside the interaction range, and that when connecting  $r = 0$  with  $r = L$  one has to take into account the phase shift  $\delta(p)$  that is picked up when the two particles interact. Thus we obtain

$$\delta(p) = -\frac{pL}{2}. \quad (4.2)$$



**Figure 3:** Lhs.: The scattering phase shift  $\delta(p)$  versus  $p$ . We compare the results from the condensation threshold (blue squares) to reference data from spectroscopy (black diamonds). Rhs.: The 3-particle energy  $W_3$  as a function of  $L$ . We show the results of the direct determination from the condensation thresholds (blue squares) and compare it to a prediction based on  $\delta(p)$  determined from  $W_2$ .

The two equations (4.1) and (4.2) constitute the connection between scattering data and  $W_2(L)$ . We can use the numerically determined values for  $W_2(L)$  to compute from (4.1) the corresponding relative momenta  $p$ , and then use these to compute the scattering phase shift  $\delta(p)$ . We show the corresponding results in the lhs. plot of Fig. 3 and compare the data from a determination based on the condensation thresholds to those from a determination based on standard spectroscopy.

Before we discuss  $W_3(L)$ , we stress that at fixed couplings the phase shift is only a function of the lattice size  $L$ : the momentum  $p$  determined from (4.1) depends only on the lattice size  $L$ , i.e., we have  $p = p(L)$ . Thus the phase shift from (4.2) is given by  $\delta(p(L)) = -\frac{p(L)L}{2} \equiv \delta(L)$ .

Similar to the 4d case we now use the scattering data determined from  $W_2(L)$  to "predict"  $W_3(L)$  and thus the third critical chemical potential value  $\mu_3(L)$ . The approach is a generalization of the strategy [17] we have followed for analyzing  $W_2(L)$ . Again we make a plane wave ansatz  $\psi = e^{-ix_1 p_1} e^{-ix_2 p_2} e^{-ix_3 p_3}$  for three particles which describes the system when all three particles are sufficiently remote from each other. The corresponding energy is  $W_3(L) = \sum_{j=1}^3 \sqrt{m(L)^2 + p_j^2}$ .

As before we introduce the center of mass coordinate  $(x_1 + x_2 + x_3)/3$ , as well as the relative coordinates  $r_2 = x_2 - x_1$  and  $r_3 = x_3 - x_1$ . Using these to parameterize  $\psi$  and demanding total vanishing momentum  $p_1 + p_2 + p_3 = 0$ , we find  $\psi = e^{-ir_2 p_2} e^{-ir_3 p_3}$  and  $p_1 = -p_2 - p_3$ . This 3-particle wave function has to obey two quantization conditions of the form (4.2) that contain  $p_2$  and  $p_3$ . Using the fact that the phase shift is only a function of  $L$  we can determine  $p_2$  and  $p_3$  as  $p_2 = p_3 = -2\delta(L)/L$ . Inserting these values and  $p_1 = -p_2 - p_3$  into  $W_3(L) = \sum_{j=1}^3 \sqrt{m(L)^2 + p_j^2}$  we obtain our prediction for  $W_3(L)$ . The corresponding values are shown as red circles in the rhs. plot of Fig. 3. We compare them to the results of a direct determination from all three condensation thresholds. The results agree very well and we conclude that the structure of the condensation thresholds  $\mu_n(L)$  can indeed be correctly described with the scattering data of the theory.

We remark that the 2- and 3-particle energies  $W_2(L)$  and  $W_3(L)$  can be analyzed with a different approach [18, 19], where one uses the exact solution for the scattering phase shift which depends on

a single parameter, the amplitude  $V_0$  of the point-like interaction. This parameter can be determined from  $W_2(L)$  and subsequently used for  $\delta$  in the 3-particle quantization conditions to determine the two independent momenta  $p_2$  and  $p_3$  needed to compute  $W_3(L)$ .

## 5. Concluding remarks

In this contribution we have shown for a simple scalar field theory in two and four dimensions that low temperature condensation is governed by the scattering data of the theory. This relation is expected to be a general non-perturbative feature, but in order to study it on the lattice usually a complex action problem has to be solved, which so far has been achieved for only a few systems. However, there exist interesting theories which are already free of the complex action problem. Examples are lattice field theories based on the gauge group  $SU(2)$ , and more interestingly, QCD with isospin chemical potential where the condensation of pions is expected to be related to pion scattering data. For these systems an analysis along the lines sketched here should be possible.

**Acknowledgements:** We thank F. Bruckmann, P. Guo, T. Kloiber, A. Maas, C.B. Lang, C. Marchis, S. Sharpe and T. Sulejmanpasic for discussions. This work is supported by the Austrian Science Fund FWF, grant I 2886-N27 and the FWF DK W 1203, "Hadrons in Vacuum, Nuclei and Stars".

## References

- [1] M.G. Endres, Phys. Rev. D **75**, 065012 (2007).
- [2] P. Weisz and U. Wolff, Nucl. Phys. B **846**, 316 (2011).
- [3] C. Gattringer, T. Kloiber, Nucl. Phys. B **869**, 56 (2013).
- [4] C. Gattringer, T. Kloiber, Phys. Lett. B **720**, 210 (2013).
- [5] F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, Phys. Rev. Lett. **115**, 231601 (2015).
- [6] K. Huang, C.N. Yang, Phys. Rev. **105**, 767 (1957).
- [7] C. Gattringer, M. Giuliani and O. Orasch, Phys. Rev. Lett. **120**, 241601 (2018).
- [8] M. Giuliani, C. Gattringer, arXiv:1702.04771.
- [9] M. Giuliani, O. Orasch, C. Gattringer, EPJ Web of Conferences 175, 07007 (2018).
- [10] K. Rummukainen, S.A. Gottlieb, Nucl. Phys. B **450**, 397 (1995).
- [11] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).
- [12] S.R. Beane, W. Detmold, M.J. Savage, Phys. Rev. D **76**, 074507 (2007).
- [13] S.R. Sharpe, Phys. Rev. D **96**, 054515 (2017).
- [14] M.T. Hansen, S.R. Sharpe, Phys. Rev. D **90**, 116003 (2014).
- [15] M.T. Hansen, S.R. Sharpe, Phys. Rev. D **92**, 114509 (2015).
- [16] M.T. Hansen, S.R. Sharpe, Phys. Rev. D **93**, 014506 (2016).
- [17] M. Lüscher, U. Wolff, Nucl. Phys. B **339**, 222 (1990).
- [18] P. Guo, T. Morris, arXiv:1808.07397.
- [19] P. Guo, Phys. Rev. D **95**, 054508 (2017).