PROCEEDINGS OF SCIENCE

Fermions on simplicial lattices and their dual lattices

Alan Horowitz*

E-mail: alhorow@yahoo.com

I show how naive fermions can be formulated on simplicial lattices and also on their dual lattices in any dimension in a way that easily enables a reduction to staggered fermions. Despite the absence of an exact chiral symmetry, there is no additive mass renormalization. There is an interesting duality between vector and axial vector currents paralleling the duality between the lattices.

The 36th Annual International Symposium on Lattice Field Theory - LATTICE2018
22-28 July, 2018
Michigan State University, East Lansing, Michigan, USA

*Speaker.

1. Introduction

Isotropic lattices that occur in every dimension are the hypercubic, the A_d, its dual, the A_d^*, and the D_n and its dual, the D_n^* (which is equivalent to D_n in 4 dimensions). (See, e.g., ref. [1].) Studies of pure gauge models on A_4 [2] and D_4 lattices [3] were done a long time ago, claiming advantages over hypercubic in efficiency and in the size of the scaling region, stemming from better rotational symmetry. Nevertheless, alternative lattices never gained any traction in the lattice gauge community. The situation is even worse for fermions. While Wilson fermions are easily formulated on any lattice, it seems no calculations have ever been done except on hypercubic. There was some work on the 2-dimensional triangular lattice by Chodos and Healy [4] and by Goeckeler [5]. The construction of Chodos and Healy is related to the fact that fermions occur naturally on a hexagonal grid [6], which is a triangular lattice with a 2-point basis, corresponding to the 2 components of a Dirac fermion in 2 dimensions. Goeckeler pursued a rather complicated transcription of the Dirac-Kaehler formalism wherein degrees of freedom live on sites, links and cells. The purpose of this report is to show that staggered fermions can be formulated on A_d and A_d^* lattices as easily as on hypercubic. The construction is interesting and sheds light on lattice fermions and may also prove useful in calculations.

It was noticed a long time ago [7] that the obvious choice for the kinetic part of a naive action on a non-hypercubic lattice of the form

$$\sum_{\mathbf{n},i} \bar{\psi}_{\mathbf{n}}\, \mathbf{e}_i \cdot \gamma(\psi_{\mathbf{n}+\mathbf{e}_i} - \psi_{\mathbf{n}-\mathbf{e}_i})$$

where $\mathbf{e}_i$ is the vector from $\mathbf{n}$ to its i'th nearest neighbor, generates non-rotationally invariant doublers. Of course, a Wilson term can be added to rid of them, but an action of this form fails if one wants to get from naive to staggered fermions. Rotational invariance of a fermion action only guarantees that the mode near zero momentum obeys the Dirac equation. A symmetry connecting doublers is also needed to get full rotational invariance and a route to staggered fermions.

2. The A_4 Lattice

A site on an A_d lattice, also known as a "simplicial" lattice in the physics literature, is defined by $d+1$ integers n_i such that $\Sigma n_i = 0$. In other words, the A_d lattice lies in the hyperplane, $\Sigma n_i = 0$, of a $(d+1)$-dimensional hypercubic lattice. Each lattice site has $d(d+1)$ nearest neighbors. The dual lattice A_d^* has $2(d+1)$ nearest neighbors. In 3 dimensions, A_3 and A_3^* are the face-centered and body-centered cubic lattices, respectively. In 2 dimensions they are both equivalent to the triangular lattice. In this paper, for definiteness, I will set $d=4$. The Bravais group of the A_4 and A_4^* lattices has 240 elements, smaller than the 384 elements of the 4-dimensional hypercubic lattice, despite having more nearest neighbors.

On the A_4 lattice, it is easier and more transparent to work with 5-dimensional vectors, whose components sum to zero. The unit vectors from a site to its nearest neighbors are

$$\boldsymbol{\varepsilon}_{12} = \frac{1}{\sqrt{2}}(1,-1,0,0,0),\ \boldsymbol{\varepsilon}_{13} = \frac{1}{\sqrt{2}}(1,0,-1,0,0),\ldots,\boldsymbol{\varepsilon}_{45} = \frac{1}{\sqrt{2}}(0,0,0,1,-1) \tag{2.1}$$

along with their negatives. There are 20 nearest neighbors and thus 10 links per site. The 4 primitive lattice vectors, $\boldsymbol{\tau}_\mu$, can be chosen to be $\boldsymbol{\varepsilon}_{\mu 5}$. Sites of the lattice are located at $\sum n_\mu \boldsymbol{\tau}_\mu$ where n_μ are integers. The reciprocal lattice vectors, $\mathbf{b}_\mu$, defined by $\mathbf{b}_\mu \cdot \boldsymbol{\tau}_\nu = 2\pi\delta_{\mu\nu}$, are

$$\mathbf{b}_1 = \kappa(4,-1,-1,-1,-1),\ldots,\mathbf{b}_4 = \kappa(-1,-1,-1,4,-1) \tag{2.2}$$

with $\kappa = \sqrt{8}\pi/5$. They generate the lattice A_4^*.

We also need a set of orthonormal vectors on the 4 dimensional hypersurface. For example: $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1,-1,0,0,0)$, $\mathbf{e}_2 = \frac{1}{\sqrt{6}}(1,1,-2,0,0)$, $\mathbf{e}_3 = \frac{1}{\sqrt{12}}(1,1,1,-3,0)$, $\mathbf{e}_4 = \frac{1}{\sqrt{20}}(1,1,1,1,-4)$.

3. Fermions on the A_4 Lattice

The proposed action on an A_4 lattice of a Dirac fermion coupled to a gauge field is (with lattice spacing a set to 1)

$$S = \frac{\sqrt{5}}{4}\sum_{\mathbf{n}}\Big[\frac{i}{\sqrt{10}}\sum_{j>i}\bar{\psi}_{\mathbf{n}}\,\gamma_i\gamma_j\,(U_{\mathbf{n},ij}\psi_{\mathbf{n}+\boldsymbol{\varepsilon}_{ij}} - U^\dagger_{\mathbf{n}-\boldsymbol{\varepsilon}_{ij},ij}\psi_{\mathbf{n}-\boldsymbol{\varepsilon}_{ij}}) + m\bar{\psi}_{\mathbf{n}}\psi_{\mathbf{n}}\Big]\,. \tag{3.1}$$

The range of i and j is 1 to 5, γ_i are the Euclidean Dirac matrices (all squaring to one), and $U_{ij} = \exp[iA_{ij}(n)]$ is an element of the gauge group. The factors are chosen so that the naive continuum limit of S is $\int d^4x(\bar{\psi}\Gamma_\mu\partial_\mu\psi + m\bar{\psi}\psi)$. They take into account the volume of $\sqrt{5}/4$ per site and a normalization factor so that the effective Dirac matrices, Γ_μ, square to one (to be seen below). This construction works in any dimension d because the number of links/site is the same as the number of matrices $\gamma_i\gamma_j$ for indices from 1 to $d+1$, namely $d(d+1)/2$.

3.1 Propagator, Symmetries, and Modes

The free propagator for $m = 0$ is

$$S(k) = 2\sqrt{2}\sum_{j>i}\gamma_i\gamma_j\sin(\mathbf{k}\cdot\boldsymbol{\varepsilon}_{ij})/\sum_{j>i}\sin^2(\mathbf{k}\cdot\boldsymbol{\varepsilon}_{ij})$$

which has poles at $\mathbf{k} = \mathbf{b}_\mu/2$ and sums of 2, 3 and all 4 of these, 16 in total as on a hypercubic lattice.

These modes are connected by symmetries of the action

$$\psi_{\mathbf{n}} \to T(n)\psi_{\mathbf{n}},\;\; \bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}}T(n)$$

where $T(n) = (-1)^{n_\mu}\gamma_\mu$ and all possible products of these. Thus the propagator and its inverse are the same around each mode. So to see the Dirac structure of each mode it is sufficient to focus only on the one near $k = 0$ for which the inverse propagator is

$$D(k) \propto i\sum_{\mu=1}^{4}\sum_{j>i}\gamma_i\gamma_j\varepsilon^\mu_{ij}k_\mu \equiv c\sum_{\mu=1}^{4}\Gamma_\mu k_\mu$$

where $\varepsilon^\mu_{ij} = \boldsymbol{\varepsilon}_{ij}\cdot\mathbf{e}_\mu = (e^i_\mu - e^j_\mu)/\sqrt{2}$. Setting $c = \sqrt{5/2}$

$$\Gamma_\mu = i\sqrt{\tfrac{2}{5}}\sum_{j>i}\gamma_i\gamma_j\varepsilon^\mu_{ij} = i\sum_{i=1}^{5}e^i_\mu\gamma_i\,A \tag{3.2}$$

PoS(LATTICE2018)235

where

$$A = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i .$$

The Γ_μ comprise a set of Euclidean Dirac matrices: $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$. Thus the action describes 16 Dirac fermions. We also have $\Gamma_5 = A$.

Equation (3.2) can be inverted to express γ_i and $\gamma_i\gamma_j$ in terms of Γ:

$$\gamma_i = -i \sum_\mu e^i_\mu \Gamma_\mu \Gamma_5 + \frac{1}{\sqrt{5}} \Gamma_5 \tag{3.3}$$

$$\gamma_i \gamma_j = -i\sqrt{\tfrac{2}{5}}\; \varepsilon^\mu_{ij} \Gamma_\mu + \sum_{\nu>\mu} (e^i_\mu e^j_\nu - e^j_\mu e^i_\nu) \Gamma_\mu \Gamma_\nu \; . \tag{3.4}$$

The symmetry group of the A_4 lattice consists of all permutations of the 5 coordinates, known as the "symmetric" group, S_5, as well as a negation of all the coordinates. So the group is $S_5 \times (\pm 1)$ which has $5! \times 2 = 240$ elements. The subgroup of even permutations, known as the "alternating" group, A_5, are proper rotations. Odd permutations involve reflections. The negation of all the coordinates are rotations by 180^o in even dimensions while it is an inversion in odd ones. Elements of S_5 can be generated by single exchanges. An example is the exchange of the first and second elements of a coordinate vector, denoted as (21345). The action is invariant provided

$$\psi_{\mathbf{n}} \to \tau \psi_{\mathbf{n}'}, \;\; \bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}'} \tau, \;\; U_{12} \to U^\dagger_{12}, \;\; U_{1j} \leftrightarrow U_{2j} \tag{3.5}$$

for $j > 2$ and $\tau = \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2)$.

The $\boldsymbol{\varepsilon}_{ij}$, the $\gamma_i\gamma_j$, and the gauge fields, A_{ij}, transform as 10-dimensional representations of S_5, comprised of 4 and 6-dimensional irreducible representations. This reduction is apparent in (3.4) expressing $\gamma_i\gamma_j$ in terms of the 4-dimensional Γ_μ and the 6-dimensional $\Gamma_\mu\Gamma_\nu$. Similarly, we can express A_{ij} in terms of a 4-vector B_μ and an antisymmetric tensor $Y_{\mu\nu}$ as

$$A_{ij} = \varepsilon^\mu_{ij} B_\mu + \sum_{\nu>\mu} (e^i_\mu e^j_\nu - e^j_\mu e^i_\nu) Y_{\mu\nu} \tag{3.6}$$

Combining (3.1), (3.4), and (3.6) the naive continuum limit of the action is seen to be

$$\int d^4x \; \bar{\psi} \{ \Gamma_\mu (\partial_\mu - igB_\mu) + g\sigma_{\mu\nu} Y_{\mu\nu} \} \psi + m\bar{\psi}\psi$$

Since $Y_{\mu\nu}$ is short range [2], it can to a good approximation be integrated out, leading essentially to a four-fermion interaction with coupling of order $a^2 g^2$.

3.2 Absence of Additive Mass Renormalization

Although the A_4 action has no exact chiral symmetry it has no additive mass renormalization. Under an inversion, that is $(12345) \to -(12345)$, $\boldsymbol{\varepsilon}_{ij}$ and A_{ij} switch sign. There is no matrix which anicommutes with all ten $\gamma_i\gamma_j$ and thus the action, (3.1), is not invariant. The transformation

$$\psi_{\mathbf{n}} \to \psi_{\mathbf{n}'}; \quad \bar{\psi}_{\mathbf{n}} \to -\bar{\psi}_{\mathbf{n}'}$$

leaves the kinetic term invariant but switches the sign of the mass term. It follows that when $m=0$ the full propagator in momentum space satisfies $S(-p) \to -S(p)$ which of course is also satisfied by the inverse propagator. Thus, when $m=0$, no mass term can be generated.

In order to have exact invariance of the action under the full symmetry group of the lattice (i.e., also under inversion) it seems we must double the fermions: $\psi \to (\psi_1,\ \psi_2)$. Then a Pauli matrix σ_3 can be inserted into the kinetic (or the mass term) and the action is invariant under $\mathbf{n} \to -\mathbf{n}$ provided

$$\psi_{\mathbf{n}} \to \sigma_i \psi_{\mathbf{n}'}, \quad \bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}'} \sigma_i$$

with $i=1$ or 2.

3.3 Reduction to Staggered Fermions

The reduction to staggered fermions proceeds as on cubic lattices. The action is diagonalized by

$$\psi_{\mathbf{n}} \to \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\gamma_5^{(n_1+n_2+n_3+n_4)}\psi_{\mathbf{n}}$$

leading to the staggered fermion action

$$S_{st} = \frac{\sqrt{5}}{4}\sum_{\mathbf{n}}\Big[\frac{1}{\sqrt{10}}\sum_{j>i}\bar{\chi}_{\mathbf{n}}\ \eta_i(n)\eta_j(n)(\chi_{\mathbf{n}+\boldsymbol{\varepsilon}_{ij}} - \chi_{\mathbf{n}-\boldsymbol{\varepsilon}_{ij}}) + m\bar{\chi}_{\mathbf{n}}\chi_{\mathbf{n}}\Big] \tag{3.7}$$

where χ_n is a single anticommuting variable and the phases are

$$\eta_1 = 1,\ \ \eta_2 = (-1)^{n_1},\ \ \eta_3 = (-1)^{n_1+n_2},\ \ \eta_4 = (-1)^{n_1+n_2+n_3},\ \ \eta_5 = (-1)^{n_1+n_2+n_3+n_4}.$$

Note that the first 4 of these have the same form as on a hypercubic lattice. As on a hypercubic lattice we can form a lattice with 16 basis points containing the degrees of freedom of 4 Dirac fermions. These cells of 16 points have the same connectivity as the A_4 lattice. That is, the $\bar{\chi}_n$ in a cell couple to the χ_n in 20 neighboring cells.

Like the naive action, the staggered action has no chiral symmetry yet it has no additive mass renormalization.

4. The A_4^* Lattice

As for the A_4 lattice it is convenient to work with 5-dimensional vectors whose components sum to zero. Nearest neighbor vectors, normalized to unity, are

$$\boldsymbol{f}_1 = (4,-1,-1,-1,-1)\sqrt{20},\ldots,\boldsymbol{f}_5 = (-1,-1,-1,-1,-4)\sqrt{20}$$

along with their negatives, 10 in all. The first 4 of these, denoted by $\boldsymbol{f}_\mu$ can be chosen as primitive lattice vectors, (differing only by a factor from the reciprocal lattice vectors above for the A_4 lattice). Sites of the lattice are located at $\sum n_\mu \boldsymbol{f}_\mu$ where n_μ are integers. The reciprocal lattice vectors are $\mathbf{b}_1 = \kappa(1,0,0,0,-1),\ldots,\mathbf{b}_4 = \kappa(0,0,0,1,-1)$, with $\kappa = 4\pi/\sqrt{5}$. They, of course, generate the A_4 lattice.

5. Fermions on the A_4^* Lattice

The action on an A_4^* lattice of a Dirac fermion coupled to a gauge field is (with lattice spacing $a \equiv 1$)

$$S = \frac{5\sqrt{5}}{16}\sum_{\mathbf{n}}\Big[\frac{1}{\sqrt{5}}\sum_{j=1}^{5}\bar{\psi}_{\mathbf{n}}\;\gamma_j\;(U_{\mathbf{n},j}\psi_{\mathbf{n}+\boldsymbol{f}_j} - U^{\dagger}_{\mathbf{n}-\boldsymbol{f}_j,j}\psi_{\mathbf{n}-\boldsymbol{f}_j}) + m\bar{\psi}_{\mathbf{n}}\psi_{\mathbf{n}}\Big]. \tag{5.1}$$

The factors are chosen so that the naive continuum limit of S is $\int d^4x(\bar{\psi}\Gamma_\mu\partial_\mu\psi + m\bar{\psi}\psi)$.

5.1 Propagator, Symmetries, and Modes

The free propagator in momentum space with $m = 0$ is

$$S(k) \propto \sum_j \gamma_j \sin(\mathbf{k}\cdot\boldsymbol{f}_j) / \sum_j \sin^2(\mathbf{k}\cdot\boldsymbol{f}_j)$$

which has the usual 16 poles within the Brillouin zone at $k = \mathbf{b}_\mu/2$ and sums of 2, 3 and all 4 of these. The symmetries connecting these modes are

$$\psi_{\mathbf{n}} \to T(n)\psi_{\mathbf{n}}, \quad \bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}}T(n)$$

where $T(n) = (-1)^{n_\mu} i\gamma_\mu\gamma_5$ and products of these. Near each pole, the fermions obey the (Euclidean) Dirac equation with Dirac matrices given by

$$\Gamma_\mu = \sum_{i=1}^{5} e^i_\mu \gamma_i \tag{5.2}$$

where the $\mathbf{e}_\mu$ are an orthonormal basis. As for the A_4 lattice, $\Gamma_5 = \frac{1}{\sqrt{5}}\sum_{i=1}^{5}\gamma_i$.

The nearest neighbor vectors $\boldsymbol{f}_j$ and the gauge field A_j transform as 5-dimensional representations of S_5, reducing to $4 \oplus 1$. So do the γ_j as seen from the inversion of (5.2):

$$\gamma_j = \sum_\mu e^j_\mu \Gamma_\mu + \frac{1}{\sqrt{5}}\Gamma_5 \; .$$

The symmetry group of the A_4^* lattice is the same as for A_4, namely the 240 element group $S_5 \times (\pm 1)$. As on A_4, the fermion action on A_4^* is invariant under only a 120 element subgroup, however it is not the same subgroup. The A_4^* action is invariant under negated odd elements, e.g., -(21345). For both lattices, the action is invariant under the subgroup, A_5, comprising purely rotational elements. Also on the A_4^* lattice, as on the A_4 lattice, the kinetic and mass terms are not both invariant under an inversion, implying the absence of additive mass renormalization even without a chiral symmetry.

5.2 Reduction to Staggered Fermions

The naive action (5.1) is diagonalized by

$$\psi_{\mathbf{n}} \to \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\psi_{\mathbf{n}}$$

leading to the staggered fermion action

$$S_{st} = \frac{5\sqrt{5}}{16} \sum_{\mathbf{n}} \Big[\frac{1}{\sqrt{5}} \sum_{i=1}^{5} \bar{\chi}_{\mathbf{n}} \, \eta_i(n) (\chi_{\mathbf{n}+\boldsymbol{f}_i} - \chi_{\mathbf{n}-\boldsymbol{f}_i}) + m \bar{\chi}_{\mathbf{n}} \chi_{\mathbf{n}} \Big] \tag{5.3}$$

where χ_n is a single anticommuting variable and the phases are

$$\eta_1 = 1, \;\; \eta_2 = (-1)^{n_1}, \;\; \eta_3 = (-1)^{n_1+n_2}, \;\; \eta_4 = (-1)^{n_1+n_2+n_3}, \;\; \eta_5 = (-1)^{n_1+n_3}.$$

The first four of these are the same as the those for the A_4 and hypercubic lattices, while η_5 is the product of these, just as γ_5 is the product of the 4 Dirac matrices.

6. Other Notable Features and Final Remarks

There is an interesting duality between vector and axial-vector interactions, paralleling the duality between the A_4 and A_4^* lattices. On an A_4 lattice, an axial-vector interaction, identical for all the doublers, is

$$\sum_{\mathbf{n}} \sum_{i=1}^{5} (\bar{\psi}_{\mathbf{n}} \, \gamma_i \, \psi_{\mathbf{n}+\mathbf{b}_i} + \bar{\psi}_{\mathbf{n}+\mathbf{b}_i} \, \gamma_i \, \psi_{\mathbf{n}}) \, Z_i(\mathbf{n})$$

where $\mathbf{b}_1 = \frac{1}{\sqrt{2}}(4,-1,-1,-1,-1), \ldots, \mathbf{b}_5 = \frac{1}{\sqrt{2}}(-1,-1,-1,-1,4)$. The first 4 of these generate an A_4^* sublattice. In the naive continuum limit, this interaction, using (3.3), is actually an axial-vector plus a pseudoscalar interaction. Similarly, on an A_4^* lattice, an axial-vector interaction lives on an A_4 sublattice, along with an antisymmetric tensor interaction.

While naive fermions on A_4, A_4^* and hypercubic lattices all give rise to 16 fermions, a Wilson term reduces the degeneracy of the masses differently on each lattice. Recall that on a hypercubic, the degeneracies for each mass are 1, 4, 6, 4, and 1 from lightest to heaviest. On an A_4 lattice the corresponding pattern is 1, 5, 10 while on A_4^* it is 1, 10, 5.

One promising use for fermions on the A_4 lattice might be in calculations involving complex actions where simulations are difficult. With 10 links/site, compared to just 4 for hypercubic, a mean-field, $1/d$ expansion on A_4 should, already at low orders, give much better results.

The higher number of links/site and better rotational invariance might also be advantageous in simulations as claimed for pure gauge models [2, 3].

The D_4 lattice (also known as F_4) has more rotational symmetry than any other 4-dimensional lattice, with rotational invariance broken only at $O(a^4)$. Is there a simple way to formulate staggered fermions on D_4? I have not yet found one.

References

[1] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, 1999.

[2] J.M. Drouffe and K.J.M. Moriarty, Nucl. Phys. B **220** (1983) 253.

[3] W. Celmaster, Phys. Rev. Lett. **52** (1984) 403.

[4] A. Chodos, J.B. Healy, Nucl. Phys. B **127** (1977) 426.

[5] M. Gockeler, Z. Phys. C **18** (1983) 323.

[6] A. H. Castro Neto *et al*, Rev. Mod. Phys. **81** (2009) 109.

[7] W. Celmaster, F. Krausz, Nucl. Phys. B **220** (1983) 434.