Localization and topology in high temperature QCD

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At high temperature part of the spectrum of the quark Dirac operator is known to consist of localized states. This comes about because around the crossover temperature to the quark-gluon plasma localized states start to appear at the low end of the spectrum and as the system is further heated, states higher up in the spectrum also get localized. Since localization and the crossover to the chirally restored phase happen around the same temperature, the question of how the two phenomena are connected naturally arises. Here we speculate on the nature of possible gauge configurations that could support localized quark eigenmodes. In particular, by analyzing eigenmodes of the staggerd and overlap Dirac operator we show that the dilute gas of calorons in the high temperature phase is very unlikely to play a major role in localization.

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1. Introduction

At low temperature the eigenmodes of the Dirac operator are well known to be delocalized and in the epsilon regime the corresponding statistics of the low eigenvalues is described by Random Matrix Theory [1]. It is by now also well established that this picture drastically changes at the crossover to the quark-gluon plasma state, where the lowest eigenmodes become localized. If the temperature is further raised, the mobility edge, separating the localized modes from the delocalized ones moves up in the spectrum, further away from zero. How can we characterize the type of gauge field configurations that support such localized modes?

By turning the question around we could also ask what sort of gauge field configurations are responsible for the delocalized low Dirac modes below the crossover temperature. A possible answer is given by the instanton liquid model (for a review see e.g. [2]). According to that the QCD vacuum is populated by a dense liquid of instantons and antiinstantons, each carrying topological charge of unit magnitude. According to the index theorem an instanton solution of the Yang-Mills equations is always accompanied by an exact zero eigenvalue of the corresponding covariant Dirac operator [3]. Since the instanton-antiinstanton liquid is not an exact solution, the zero modes are only approximate. The mixing of the approximate zero modes produces the so called zero mode zone in the Dirac spectrum, a finite density of low modes around zero virtuality, responsible for the breaking of chiral symmetry through the Banks-Casher relation [4].

Let us consider how this picture changes when the temperature is raised. In the Euclidean setup, as the temperature increases, the temporal extension of the system decreases and when its linear dimension reaches the diameter of typical instantons, they are gradually squeezed out of the box. As a result, the instanton density and the topological susceptibility falls sharply across the crossover, leaving only a dilute instanton gas on the quark-gluon plasma side of the transition. In the end, the approximate zero modes of the remaining instantons (calorons) are spatially too far away and cannot mix sufficiently to support a finite spectral density at zero virtuality, and chiral symmetry is restored.

Since in this way instantons play a crucial role in chiral symmetry breaking and restoration, it is natural to expect that they are also related to the localization transition [5]. However, since zero modes are especially sensitive to chiral properties of the Dirac operator, for a direct study of the relationship between localization and the zero mode zone in the Dirac spectrum on the lattice, a chirally symmetric Dirac operator is needed. In the present paper we use staggered 2-stout [6] and overlap quarks [7] on quenched SU(3) gauge backgrounds to study the connection between localization and the zero mode zone. It was already observed in the early days of the overlap that on quenched gauge configurations above T_c the zero mode zone can be identified as a "bump" near zero virtuality in the overlap spectral density [8]. Here we show that stout smearing and finer lattices allow a similar identification also in the staggered Dirac spectrum. Our main result is that localized modes extend well beyond the zero mode zone and for increasing temperatures a smaller and smaller fraction of localized modes is contained in the zero mode zone. Therefore, a simple understanding of localized modes as non-mixing approximate topological zero modes is not possible.

2. The zero mode zone

In this section we show how the zero mode zone can be identified and distinguished from the rest of the Dirac spectrum. This is a nontrivial task since at low temperature the spectral density increases monotonically across the spectrum and it has no special feature that would allow the identification of the zero mode zone. This is not unexpected since at low temperature the instanton liquid is dense, the distance between neighboring topological objects is comparable to their size. It is known that in the field of an instanton-antiinstanton pair the two would be zero modes split into a pair of complex conjugate Dirac eigenvalues and their splitting is controlled by the distance (compared to their size) of the two topological objects[1][2]; the closer the instanton and the antiinstanton are, the larger the splitting is. Therefore, in a densely packed medium, the zero mode zone extends higher up in the spectrum and there is no well defined boundary between that and the bulk of the spectrum.

This situation, however, changes drastically through the crossover where the instanton density falls sharply. At high temperature the instanton gas becomes more dilute and also the typical instanton size decreases, as larger instantons are squeezed out of the system by the decreasing temporal size. As a result, the typical distance between neighboring topological objects becomes much larger than their size, which in turn will produce smaller splittings for the mixing approximate zero modes.

However, to resolve these fine details of the spectrum, one needs a lattice Dirac operator that at least approximately respects chiral symmetry. This is because a lattice Dirac operator that explicitly breaks chiral symmetry does not have zero modes, the magnitude of its would be zero eigenvalues is controlled by the explicit chiral symmetry breaking of the operator. Thus, if the Dirac operator has bad chiral properties, the splittings in the zero mode zone are not governed by the physics of the instanton gas but by the explicit chiral symmetry breaking of the Dirac operator, a lattice artifact that will eventually disappear in the continuum limit.

For this reason we start our study by using the overlap Dirac operator that has an exact chiral symmetry and consequently exact topological zero modes. In Fig. 1 we plot the spectral density of the overlap Dirac operator on a quenched ensemble of lattices with $N_t = 6$, just above the critical temperature. In contrast to the low temperature case where the spectral density increases monotonically, here we see an accumulation of eigenvalues near zero, followed by a dip and then a rising spectral density in the bulk of the spectrum. We emphasize that the peak near zero is not due to exact zero modes. Those would show up as a trivial delta peak at zero, and we removed that from the plot. A similar accumulation of near zero modes was previously observed in Ref. [8] and more recently discussed in details [9].

Since this low part of the spectrum clearly separates from the bulk, it is a natural candidate for the zero mode zone. The question is whether it really contains only topology-related approximate zero modes and whether it contains all of them. Fortunately, the overlap spectrum provides a simple way of independently estimating the expected number of topological objects and their approximate zero modes. By the index theorem, on each configuration the number of zero modes gives the

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[1]The splitting also depends on the relative gauge orientation of the instanton and antiinstanton, but this is not relevant to the present argument.
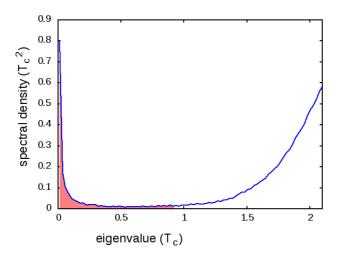
Figure 1: The spectral density of the overlap Dirac operator on quenched lattices with $N_t = 6$ at a temperature of $T = 1.06T_c$. The units on the axes are appropriate powers of T_c. The red shaded region shows how far up in the spectrum the zero mode zone extends. This is estimated based on the topological susceptibility, as given by the number of zero modes of the overlap Dirac operator.

net topological charge[2] which in turn determines the topological susceptibility. If we *assume* that above T_c the instanton gas is dilute enough so that interactions among topological objects can be neglected, then they occur independently. In this case a simple calculation shows that the density of topological objects (i.e. the instanton plus antiinstanton density) is equal to the susceptibility. Thus the susceptibility provides an independent estimate of the expected number of eigenmodes in the zero mode zone.

A key assumption of the above line of argument is that the interaction among topological objects can be neglected, they occur independently. This can be tested by comparing the distribution of the topological charge sectors found in lattice simulations to that of the free instanton gas with the same topological susceptibility. In Fig. 2 we compare the two distributions and find good agreement.

Having justified that our assumption about the independence of topological objects is valid, based on the topological susceptibility we can estimate how far up in the spectrum the zero mode zone extends. This is shown in Fig. 1, where we plot the overlap spectral density with the shaded region indicating the extension of the zero mode zone in the spectrum. We also performed simulations at a number of other temperatures ranging from $1.02T_c$ to $1.12T_c$ and the picture is qualitatively similar everywhere. This strongly suggests that the peak in the spectral density indeed represents the zero mode zone.

3. The zero mode zone and the mobility edge

A simple explanation for the appearance of localized modes above T_c would be that the local-

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[2]In principle the difference of the number of negative and positive chirality zero modes is equal to the topological charge. However, in practice one never encounters zero modes of opposite chirality on the same configuration.
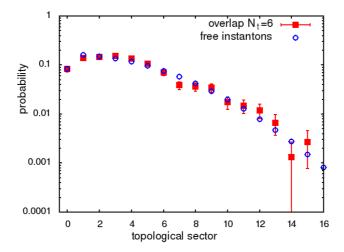
Figure 2: The probability of different topological charge sectors in the lattice simulation at $T = 1.06T_c$ and the corresponding distribution in a free instanton antiinstanton gas with the same topological susceptibility. Since the distribution is expected to be symmetric, the probabilities of positive and negative charges of the same magnitude have been added.
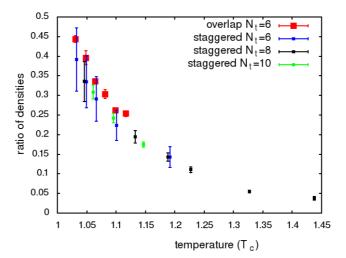

Figure 3: The ratio of the number of modes in the zero mode zone and the number of localized modes as a function of the temperature (in units of T_c). Here we show data based on the 2-stout staggered Dirac operator along with the overlap data in the previously presented overlap data.

ized modes are nothing else but the modes in the zero mode zone that get separated from the bulk of the spectrum, as we saw above. Localized modes at the low end of the spectrum can be easily distinguished from the higher, delocalized part of the spectrum by tracing how the spectral statistics changes along the spectrum. For a detailed account of the determination of the mobility edge, separating localized and delocalized modes, see the contribution by R. A. Vig at this conference and also [10]. We can then count the average number of eigenvalues below the mobility edge and compare that to the number of eigenvalues in the zero mode zone to see whether the zero mode zone can account for localization. In Fig. 3 we plot the ratio of the number of modes in the zero mode zone to the number of localized modes as a function of the temperature. This shows that even right above T_c not more than 50% of the localized modes are contained in the zero mode zone and this fraction falls to 25% already at $T = 1.12T_c$.
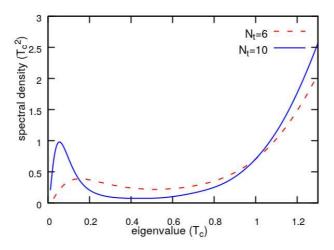

Figure 4: The spectral density of the 2-stout staggered Dirac operator on $N_t = 6$ and $N_t = 10$ lattices at the same physical temperature $T = 1.06T_c$.

Our limited data set for the overlap spectra does not allow us to follow how this ratio continues to change as the temperature increases further. However, this can be qualitatively seen using our more extensive set of staggered Dirac spectra. Indeed, on finer lattices the spectral density of our 2-stout smeared staggered Dirac operator shows features similar to those of the overlap and this allows the approximate identification of the zero mode zone here as well. We demonstrate this in Fig. 4 by showing the spectral density of the staggered operator on $N_t = 6$ and $N_t = 10$ lattices at the same physical temperature, $T = 1.06T_c$. It is easily seen that as the lattice becomes finer, the bump near zero becomes more pronounced and more separated from the bulk of the spectrum. We note that as the staggered operator does not provide unambiguous information about the topological susceptibility, the staggered zero mode zone can only be approximately identified using the location of the bump. However, this identification becomes increasingly precise in the continuum limit.

Keeping this in mind, in Fig. 3 we present staggered data for the ratio of the size of the zero mode zone and the localized region in the spectrum. The staggered data shown corresponds to three different lattice spacings ($N_t = 6, 8, 10$) and for comparison we included also the overlap data. The error bars for the staggered data also contain the uncertainty in estimating the size of the zero mode

zone. Note that due to the less expressed nature of the bump in the spectral density, this uncertainty is larger for the coarser lattices. This is also the reason why the error bars are decreasing towards higher temperatures since for a given N_t the temperature is set by changing the lattice spacing (the gauge coupling) and higher temperatures correspond to finer lattices.

The staggered data obtained at different lattice spacings are consistent and they also agree with the overlap data. This shows that our results can be safely considered to be a good qualitative estimate of the continuum limit. As the temperature increases, the ratio keeps falling and around $1.5T_c$ only about 5% of the localized modes is contained in the zero mode zone. This clearly rules out a simple identification of the zero mode zone with the region of localized modes, and makes a direct connection between localization and topology rather unlikely.

4. Conclusions

In the present paper we studied the question of whether the topology-related zero mode zone of the high temperature QCD Dirac operator can be identified with the region of localized modes at the low end of the spectrum. We showed that both with the overlap operator and the 2-stout staggered operator on fine enough lattices the zero mode zone can be identified as a bump in the spectral density near zero. However, the zero mode zone turns out to contain only a fraction of the localized modes and this fraction falls sharply at higher temperatures. Therefore, a simple identification of localized modes with approximate zero modes of topological origin is ruled out. Finally, we note that although the present study is based on the quenched approximation, we do not expect that the inclusion of dynamical quarks would change our main conclusion. Indeed, dynamical quarks suppress fluctuations of the topological charge and in the presence of dynamical quarks, topology is even less likely to provide an explanation for localization.

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