Determining the Efficacy of Different Parameterizations of the z-expansion

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We develop a methodology to test the accuracy of lattice extrapolations of the form factors in B decays only using experimental data. We test this methodology by comparing the BGL parameterization proposed by [1] and the BCL parameterization proposed by [2] in the context of $B \to \pi \ell \nu$. Our results our inconclusive as to whether the BGL or BCL parameterizations provide a better description of the high energy pion regime.

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1. Introduction

It is important to have an accurate description of the form factors f_+ and f_0 in $B \to \pi \ell \nu$ because this decay has a complicated dependence on the lepton transfer momentum (q^2). Furthermore lattice computations are limited in the energy regions that can be probed and therefore are limited in the q^2 region where values of the form factors can be computed. The first step to handle the q^2 dependence is using the conformal mapping variable:

$$z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}} \tag{1.1}$$

first introduced by [3, 4]. This transformation maps the cut q^2 plane into the unit disk and is the foundation for the parameterizations of the form factors using the z-expansion.

There are two important parameterizations of the z-expansion. The parameterization proposed by [1],

$$f_+^{\text{BGL}}(q^2; t_0) = \frac{1}{B(q^2)\phi(q^2)} \sum_i^N a_n z(q^2; t_0)^n, \tag{1.2}$$

uses the function $\phi(q^2)$ to impose a simple constraint on the coefficients a_n, and uses the Blashke factor $B(q^2)$ to account for any resonance poles. A second parameterization proposed by [2],

$$f_+^{\text{BCL}}(q^2; t_0) = \frac{1}{1 - q^2/m_{B^*}^2} \sum_n^{N-1} b_n (z^n - (-1)^{N-n} \frac{n}{N} z^N), \tag{1.3}$$

addresses the issues of the $(1/q^2)^{1/2}$ falloff that is introduced by 1.2 and the threshold constraint imposed by conservation of angular momentum [2].

2. Methodology

Our method of testing different parameterizations is straightforward. The first step involves ensuring that the parameterizations satisfactorily describe the form factor over the full kinematic region. This is done by fitting the parameterization to all available experimental data related to the decay process. Once it is clear that the parameterizations can adequately describe the decay rate over the full kinematic region, several smaller regions of phase space are chosen, including a region where lattice calculations have determined values of the form factor. The decay rate is fitted to each expansion in these Contribution titleregions and then the fits are compared inside and outside of the fitted region. Comparisons inside the fitted region use the minimized reduced χ^2 value while the comparisons outside the fitted region use the following quantity which we call the predictive measure:

$$X_p^2 = \frac{1}{N_{\text{unfitted}}} \sum_i^{N_{\text{unfitted}}} \frac{(\Delta B_i^{\text{exp}} - \Delta B_i^{\text{fit}})^2}{\sigma_i^{\Delta B}}. \tag{2.1}$$

N_{unfitted} is the number of unfitted data points, ΔB_i^{exp} and ΔB_i^{fit} are, respectively, the experimental and fitted partial branching fractions corresponding to the q_i^2 to q_{i+1}^2 bin. In addition to comparing the accuracy of the different fits, it is also important to examine the stability of the coefficients in the parameterization. We examine the coefficients stability by using the best fit parameterization over

the full kinematic range to generate a large number of bootstrap datasets. The parameterizations are fitted to these "synthetic" data sets to establish how variable these coefficients will be to new measurements of the partial branching fractions.

3. Comparison of BGL and BCL parameterizations

We examined the BGL and BCL parameterizations of the vector form factor $f_+(q^2)$ at 3 and 4 parameters fitted (maximal order z^2 and z^3) for the BGL parameterization and 2 and 3 fitted parameters (maximal order z^2 and z^3) for the BCL parameterization. It is possible to use 4 fitted parameters for the BCL expansion in our analysis, however the high q^2 data is so noisy that there is substatial overfitting. We only implemented the weak unitarity constraint in both the BGL expansion ($\sum |a_i|^2 \leq 1$) and BCL expansion ($\sum b_i b_j B_{ij} \leq 1$), this assumes the $B \to \pi$ channel saturates the constraint.

The choices for order comparison were made because they correspond to the minimal number of parameters and the next highest order that can adequately describe the form factors in the lattice regime; [5] argue that a curvature term (z^2) is necessary to describe the lattice determinations of the form factors - this necessity is still be seen in [6]. Although these parameterizations could be compared at equal number of parameters this choice biases the comparison because the curvature is different for the two parameterizations if the same number of parameters are used because the BCL expansion has an extra order in z that is included by the conservation of angular momentum [2]. For example, the 2 parameter BCL expansion has the z^2 term included in the b_1 coefficient dependence:

$$f_+^{BCL}(q^2) = \frac{1}{1 - q^2/m_{B^*}^2}(b_0 + b_1(z + \frac{1}{2}z^2)) \tag{3.1}$$

while the 2 parameter BGL expansion,

$$f_+^{BGL}(q^2) = \frac{1}{B(q^2)\phi(q^2)}(a_0 + a_1 z) \tag{3.2}$$

does not possess a z^2 term.

The choice not to implement the strong unitarity constraint for either case is due to the increased complexity of the constraint (see [7, 8] for implementation in $B \to D$)

3.1 Comparison of Lattice Region

The order z^2 fits corresponding to the lattice region in the BGL case overestimates the partial branching fractions more than the BCL parameterization; but the order z^3 fit in both cases these both over estimate the partial branching fractions outside of the lattice region (see 1). The predictive measures X_p^2 are given in 1. The significantly higher value for X_p^2 for the 4 parameter BGL fit and 3 parameter BCL fit compared to the 2 parameter BCL and 3 parameter BGL fits is likely due to overfitting. This can be seen in the bootstrap analysis; the constant, linear and quadratic terms for the 4 parameter BGL fit are gaussian but the cubic term is non normally distributed (see 3). It is surprising that the mean of the 2 parameter's values with the BCL fit accurately predicts the low q^2 region.
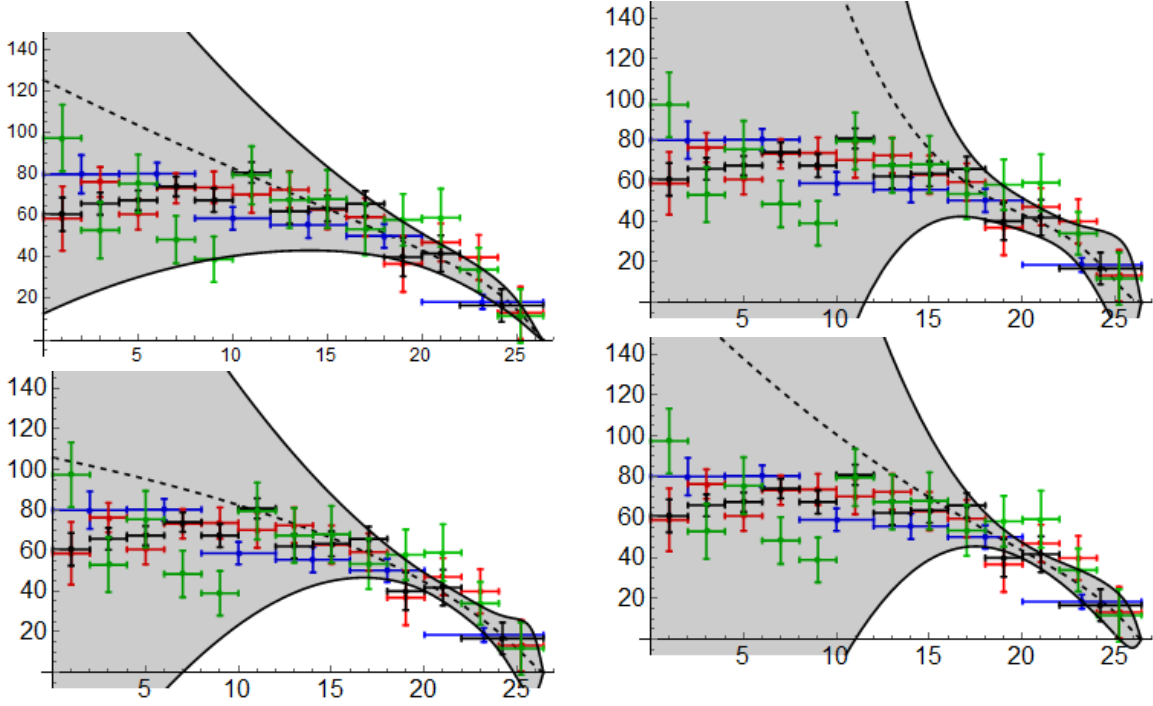
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Figure 1: fits to experimental data fitted $q^2 > 17$ GeV^2. Index: (Blue: BaBar 2011 [9], Red: Belle 2011 [10], Black: BaBar 2012 [11], Green: Belle 2013 results [12])

num. coefs.	parameterization	X_p^2	χ^2/d.o.f	slope	curvature
2	BCL $(17 - 26.4\text{GeV}^2)$	7.97	1	-0.7(1.3)	[-0.37(67)]
3	BGL $(17 - 26.4\text{GeV}^2)$	17.95	0.96	-0.4(1.3)	-1.0(17)
3	BCL $(17 - 26.4\text{GeV}^2)$	48.5	0.99	-0.7(1.3)	2(18)
4	BGL $(17 - 26.4\text{GeV}^2)$	897	0.98	-0.8(1.3)	3(17)
2	BCL $(15 - 26.4\text{GeV}^2)$	2.62	0.94	-1.00(74)	[-0.50(37)]
3	BGL $(15 - 26.4\text{GeV}^2)$	3.23	0.91	-0.53(81)	-5.1(7.6)
3	BCL $(15 - 26.4\text{GeV}^2)$	4.33	0.96	-0.81(81)	-3.0(7.1)
4	BGL $(15 - 26.4\text{GeV}^2)$	1.90	0.96	-0.577(77)	-5.2(7.3)

Table 1: X_p^2, χ^2/d.o.f, slope (a_1/a_0 or b_1/b_0), and curvature (a_2/a_0 or b_2/b_0) for a 2 and 3 fitted parameter BCL expansions and 3 and 4 fitted parameter BGL expansions in the lattice and near lattice regions.

3.2 Near Lattice Region

A substantial difference in predictions occurs when the form factor parameterizations are fit to a region slightly larger (15 - 26.4 GeV^2) than the lattice region. The BCL and BGL parameterizations all reasonably predict the low q^2 region. The predictive measure (X_p^2) for all four choices lie approximately between 2.0 and 4.5. This likely indicates that either overfitting is no longer as significant an issue because the size of the data set has increased, or that this is an energy region where there is an interesting feature of the differential decay spectrum of $B \to \pi \ell \nu$ which occurs. It
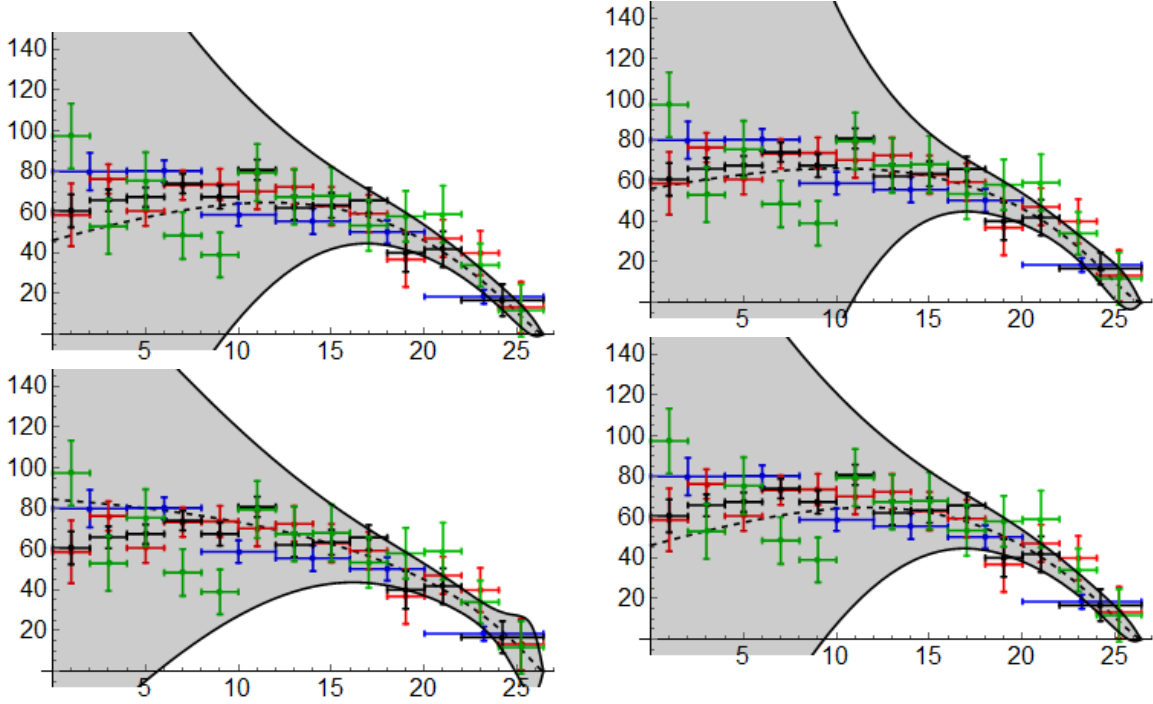
Figure 2: fits to experimental data fitted $q^2 > 15$ GeV^2. Index: (Blue: BaBar 2011 [9], Red: Belle 2011 [10], Black: BaBar 2012 [11], Green: Belle 2013 results [12])

is substantially suprising that the 4 parameter BGL fit's mean value most closely represents the low q^2 regime.

4. Conclusions and Outlook

Due to the large error bars of the extrapolation outside of the fitted region it is hard to draw statistically signifcant conclusions about which parameterizations better predict the low q^2 regime. From the conclusions that we can draw, the two parameter and three parameter fits are less susceptable to overfitting than four parameter fits, furthermore the χ^2/d.o.f. for each set are on par with each other (1). Therefore, the limiting factor in the efficacy of using the high q^2 partial branching fractions to predict the low q^2 partial branching fractions, for $B \to \pi\ell\nu$ is the substantial statistical noise associated with the high q^2 partial branching fractions measured by experiment. This experimental noise (greater than 20-30 per-cent) is much larger than the uncertainties associated with the lattice determinations of the form factors (between 1 - 9 per-cent [6]).

There are three possible methods to address this issue. The first method Contribution title-would be to wait until LHCb or Belle II publishes their analysis of $B \to \pi\ell\nu$ that has better statistics. The second method would be to use the lattice results from [6] to help constrain the coefficients and provide a priori estimates of the the ratios a_i/a_0 and b_i/b_0 in the z-expansion by using baysian analysis. The motivation for this comes from how the lattice and experimental results are determined. The partial branching fractions are binned; this results in a possible loss of some of the high q^2 behavior within each binned region. However the lattice results are not binned so more of
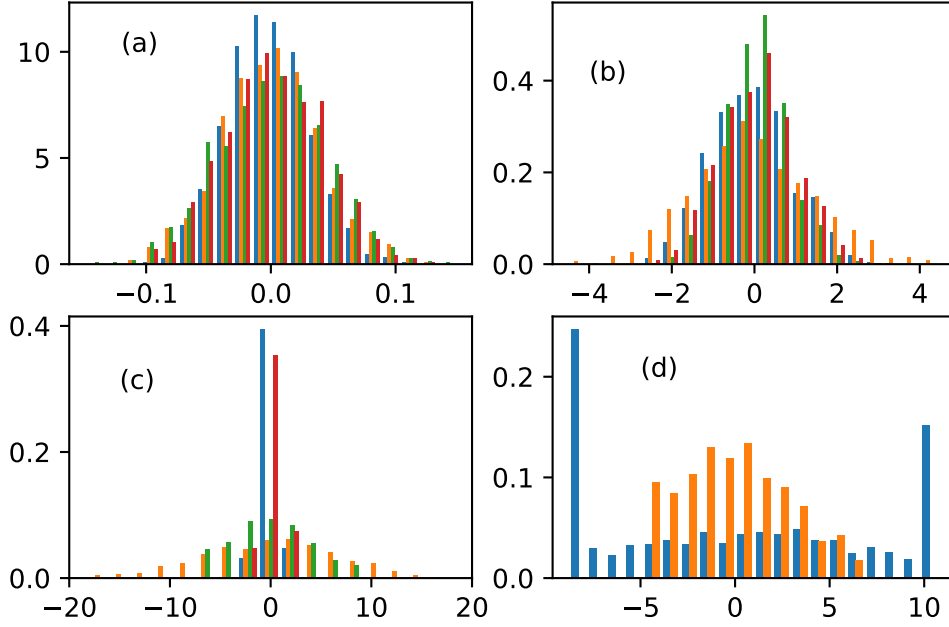
Figure 3: PDF corresponding to the fractional difference from the mean value of fitting to lattice region with bootstrapped data sets. We used 1000 bootstrapped samples. Data outside of 20x mean value for the theoretical value of b_2 for the are not shown so that the resolution of the fitted coefficients can be seen. (a) linear term; (b) z term; (c) z^2 term; (d) z^3 term. Blue bars: 4 parameter BGL coefficients; orange bars: 3 parameter BCL coefficients; red bars: 3 parameter BGL coefficients; green bars: 2 parameter BCL coefficients.

the variability is visible. The third option is to impose the strong unitarity constraint for the BGL expansion.

The substantial improvement of the predictions by slightly increasing the region to which we fit the data would suggest that a slight increase in the q^2 range for which we have lattice determinations of the form factors might provide a better idea on the shape of the form factor through the entire kinematic region. Therefore we suggest to find a method to extract a better signal for the high momentum modes. An attempt to do this in fact is already being done for $B \to \pi$ by [13].

It would be interesting to examine this process in other decay processes such as as $B \to D\ell\nu$ and $\Lambda_b \to \Lambda_c \ell\nu$. The each of these decays have analyses ([14] and [15]) with available covariance matrices and partial branching fractions, although the volume of publicly data available is less than that of $B \to \pi\ell\nu$, this methodology can be used to help rule out other parameterizations of the form factors for these decays.

5. Acknowledgements

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