PROCEEDINGS OF SCIENCE

Variational and Dyson–Schwinger equations of QCD

Davide Campagnari*, **Hugo Reinhardt**
Institut für Theoretische Physik, Eberhard-Karls-Universität Tübingen
Auf der Morgenstelle 14, 72076 Tübingen, Germany

Dyson–Schwinger equations are an established, powerful non-perturbative tool for QCD. In the Hamiltonian formulation of QCD they allow for variational calculations with non-Gaussian wave functionals: by means of DSEs the various n-point functions, needed in expectation values of observables like the Hamiltonian, can be expressed in terms of the variational kernels of the trial Ansatz for the vacuum wave functional. Equations of motion for these variational kernels are derived by minimizing the energy density, renormalized, and solved numerically. We determine the chiral condensate from the renormalized quark propagator.

XIII Quark Confinement and the Hadron Spectrum - Confinement2018
31 July – 6 August 2018
Maynooth University, Ireland

*Speaker.

PoS(Confinement2018)041

1. Introduction

The two most outstanding features of Quantum Chromodynamics (QCD) at ordinary density and temperature are colour confinement and the spontaneous breaking of chiral symmetry. Both phenomena are strongly intertwined and, since they originate in the low-energy sector of the theory, they cannot be approached by means of perturbative methods. Although we still lack a rigorous understanding of these phenomena, we have been able to gain a good deal of insight by means of lattice [1, 2] and continuum [3–10] studies.

In this talk we present recent results for the quark propagator in the Hamiltonian approach to QCD in Coulomb gauge. It has been known for quite some time [11–16] that a linearly rising potential between colour charges can trigger chiral symmetry breaking. A vacuum wave functional which includes the coupling of the quarks to the transverse gluons improves the results towards the phenomenological findings [10, 17–19]. Here we present a refinement of these investigations based on the application of Dyson–Schwinger equations (DSEs) to treat non-Gaussian wave functionals in a systematic way [20–22].

2. Hamiltonian Approach to Quantum Chromodynamics

The canonical quantization of QCD is performed in the temporal gauge $A_0^a = 0$ and results in a functional Schrödinger equation. The Gauss law operator enforces gauge invariance as a constraint on the wave functionals. This constraint can be explicitly solved in Coulomb gauge and results in the gauge-fixed Hamiltonian [23]

$$\begin{aligned} H = H_{\mathrm{YM}} &+ \int \mathrm{d}^3x\, \hat{\psi}^\dagger(\vec{x}) \Big[-\mathrm{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m - g\vec{\alpha}\cdot\vec{A}^a(\vec{x}) t_a \Big] \hat{\psi}(\vec{x}) \\ &+ \frac{g^2}{2} \int \mathrm{d}^3x\, \mathrm{d}^3y\, J_A^{-1}\, \rho^a(\vec{x})\, J_A\, F_A^{ab}(\vec{x},\vec{y})\, \rho^b(\vec{y}), \end{aligned} \tag{2.1}$$

where H_{YM} is the Hamiltonian of pure Yang–Mills theory; β and α_i are the usual Dirac matrices; t_a are the generators of the gauge group in the fundamental representation; and $J_A = \mathrm{Det}(G_A^{-1})$ is the Faddeev–Popov determinant, with

$$\left[G_A^{ab}(\vec{x},\vec{y}) \right]^{-1} = \left(-\delta^{ab}\nabla^2 - g f^{acb} A_i^c(\vec{x}) \partial_i \right) \delta(\vec{x}-\vec{y})$$

being the Faddeev–Popov operator. Here, g is the coupling constant, and f^{abc} are the structure constants of the $\mathfrak{su}(N_c)$ algebra. The term in the second line of Eq. (2.1) represents a two-body interaction of the total colour charge

$$\rho^a = \hat{\psi}^\dagger t_a \hat{\psi} - \mathrm{i} f^{abc} A_i^b \frac{\delta}{\delta A_i^c}$$

mediated by the so-called Coulomb kernel

$$F_A^{ab}(\vec{x},\vec{y}) \equiv \int \mathrm{d}^3z\, G_A^{ac}(\vec{x},\vec{z}) \left(-\nabla_z^2 \right) G_A^{cb}(\vec{z},\vec{y}). \tag{2.2}$$

3. Dyson–Schwinger Equations in the Hamiltonian Approach

Matrix elements of operators O in the Hamiltonian approach have the form

$$\langle \Phi | O | \Psi \rangle = \int \mathcal{D}A\, J_A\, \mathcal{D}\xi^\dagger\, \mathcal{D}\xi\, \mathrm{e}^{-\mu[\xi^\dagger,\xi]} \Phi^*[A,\xi^\dagger,\xi]\, O\, \Psi^*[A,\xi^\dagger,\xi]. \tag{3.1}$$

The gauge field integration runs over transverse configurations only and, in principle, should be restricted to the first Gribov region; the presence of the Faddeev–Popov determinant J_A is a consequence of the gauge fixing. The fermion degrees of freedom are expressed through the Grassmann fields $\xi^\dagger$, ξ, and μ is the corresponding integration measure (in Ref. [21] a coherent-state representation was chosen). For the vacuum wave functional we make an Ansatz of the form

$$\Psi[A,\xi^\dagger,\xi] =: \exp\left\{-\frac{1}{2}\, S_A[A] - S_f[A,\xi^\dagger\xi]\right\}, \tag{3.2}$$

where S_A is a functional of the gauge field only, while S_f contains both the fermion and the gluon fields. Once functional derivatives which might be present in the operator O in Eq. (3.1) have acted onto the wave functional, the expectation values in the Hamiltonian approach reduce to a path integral of the form

$$\langle f \rangle = \int \mathcal{D}A\, J_A\, \mathcal{D}\xi^\dagger\, \mathcal{D}\xi\, \mathrm{e}^{-S_A - S_f - S_f^* - \mu}\, f[A,\xi^\dagger,\xi]$$

which looks like a correlation function of a Euclidean field theory with "action"

$$S = S_A + S_f + S_f^* + \mu - \mathrm{Tr}\ln G_A^{-1}.$$

Correspondingly, we can derive Dyson–Schwinger-like equations by starting from the identity

$$0 = \int \mathcal{D}A\, \mathcal{D}\xi^\dagger\, \mathcal{D}\xi\, \frac{\delta}{\delta\phi}\left\{ f[A,\xi^\dagger,\xi]\, \mathrm{e}^{-S}\right\}. \tag{3.3}$$

The "Dyson–Schwinger" equations derived from Eq. (3.3) are not quite equations of motion in the usual sense but rather relate the Green functions of the theory to the (so far undetermined) vacuum wave functional. To avoid possible confusion with the common terminology we name them canonical recursive Dyson–Schwinger equations (CRDSEs).

4. The Vacuum Wave Functional

The explicit form of the vacuum wave functional is, of course, unknown. As already pointed out, a perturbative evaluation, though possible [24], cannot grasp chiral symmetry breaking. We will therefore solve the Schrödinger equation by means of the variational principle: the expectation value of the Hamilton operator Eq. (2.1) is taken with an appropriately chosen Ansatz for the wave functional, which depends on some variational kernels. Using the CRDSEs, the resulting vacuum energy density is minimized with respect to the variational kernels, yielding a set of gap equations.

In this work we concentrate on the quark sector of the theory. For the fermionic part of the vacuum wave functional Eq. (3.2) we make the Ansatz

$$S_f = \int \mathrm{d}^3x\, \mathrm{d}^3y\, \xi_+^\dagger(\vec{x})\left\{\beta\, s(\vec{x},\vec{y}) + g \int \mathrm{d}^3z \left[V(\vec{x},\vec{y};\vec{z})\,\alpha_i + W(\vec{x},\vec{y};\vec{z})\,\beta\,\alpha_i\right] A_i^a(\vec{z}) t_a\right\}\xi_-(\vec{y}), \tag{4.1}$$

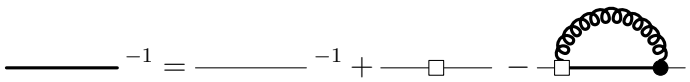

Figure 1: CRDSE for the quark propagator. Empty boxes stand for the variational kernels, fat dots for the full (one-particle irreducible) vertices, and fat lines for fully dressed propagators.

where $\xi_\pm$ are the positive/negative energy components of the Grassmann spinor field. The functions s, V and W are the variational kernels. The quark-gluon coupling in Eq. (4.1) contains both the leading-order term $\propto \alpha_i$ known from perturbation theory [25] as well as a further Dirac structure $\propto \beta\alpha_i$, which will turn out to be realized only in the chirally broken phase. The first term in Eq. (4.1) is a BCS-type of wave functional, already used in the Coulomb-gauge pairing model and its extensions [11–16]. Early calculations [17, 18] in the quark sector used an Ansatz similar to Eq. (4.1) where, however, only the leading-order Dirac structure α_i was considered ($W = 0$). The resulting equations were plagued by linear divergences, while keeping both Dirac structures ($V \neq 0$, $W \neq 0$) removes all UV divergences from the quark gap equation. With the Ansatz given by Eqs. (3.2), (4.1) we find the quark propagator equation represented diagrammatically in Fig. 1.

5. Quark Mass Function and Chiral Condensate

In the evaluation of the energy density one major approximation is made: we replace the full quark-gluon vertex by the bare one, which is the kernel appearing in the vacuum wave functional Eq. (4.1).[1] While this might look like an excessively crude approximation, especially given our experience with Landau gauge calculations, one must keep in mind that in Coulomb gauge the dominating interaction is mediated by the (strongly IR divergent) Coulomb kernel Eq. (2.2). The CRDSE for the quark-gluon vertex is nevertheless currently under investigation. The Coulomb kernel is approximated by its Yang–Mills expectation value and parametrized by the form

$$g^2 F(\vec{p}) = \frac{8\pi\sigma_{\mathrm{C}}}{(\vec{p}^2)^2} + \frac{\alpha_{\mathrm{s}}}{\vec{p}^2}, \tag{5.1}$$

which is supported by the variational calculation in the Yang–Mills sector [9].

The variational equations for the vector kernels V and W can be explicitly solved in terms of the scalar kernel s and the gluon propagator; the remaining gap equation for the scalar kernel s must be solved numerically. It turns out that the second vector kernel W is crucial not only to eliminate the divergences from the gap equation but also to render the physical quark propagator multiplicatively renormalizable.

The renormalization point dependent quark condensate is usually quoted in the $\overline{\mathrm{MS}}$ scheme at the renormalization scale $\mu = 2\,\mathrm{GeV}$. Solving the gap equation for the value $\alpha_s(2\,\mathrm{GeV}) = 0.30(1)$ [27] yields the quark mass function shown in Fig. 2. The resulting chiral condensate and IR value of the mass are

$$M(0) = 0.19\sqrt{\sigma_{\mathrm{C}}}, \qquad \langle\bar{q}q\rangle^{\overline{\mathrm{MS}}}_{\mu=2\,\mathrm{GeV}} = \left(-0.31\sqrt{\sigma_{\mathrm{C}}}\right)^3.$$

[1] Note that "bare" must be understood in Dyson–Schwinger sense: the variational kernel is "bare" in the sense that it represents the leading-order term in the DSE for the quark-gluon vertex. The kernel itself is, however, determined by the minimization of the energy density and is thus principally non-perturbative.

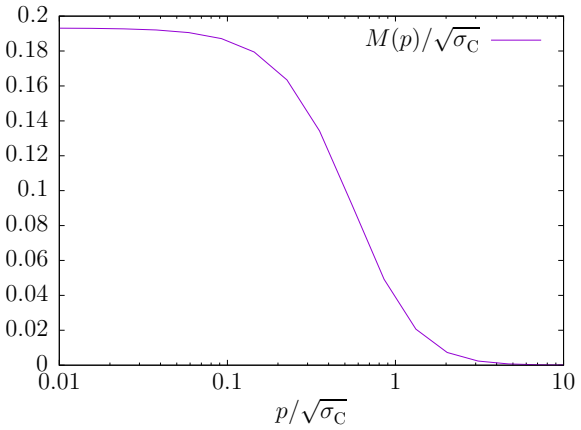

Figure 2: Mass function of the renormalized quark propagator.

The scale in our calculations is fixed by the Coulomb string tension σ_C occurring in the colour Coulomb potential Eq. (5.1). Lattice and continuum calculations [28–30] quote values of the Coulomb string tension from 2.5 to as large as 4 times the Wilson string tension $\sigma = (440\,\mathrm{MeV})^2$, which gives us $\sqrt{\sigma_C}$ in the range from 696 MeV to 880 MeV. This yields

$$M(0) = 135 \text{ to } 170\,\mathrm{MeV}, \qquad \langle\bar{q}q\rangle^{\overline{\mathrm{MS}}}_{\mu=2\,\mathrm{GeV}} = (-216\,\mathrm{MeV})^3 \text{ to } (-270\,\mathrm{MeV})^3.$$

6. Mass Function in the Full and Static Propagator

While our result for the chiral condensate is in good agreement with findings from lattice simulations and chiral perturbation theory calculations [1, 2, 31–33], the infrared value of the mass function is much smaller than the expected value of roughly 300 MeV, i.e. around the value of the constituent quark mass. However, this naive expectation is misleading, since the Hamiltonian approach deals with *static* propagators.

Let us clarify this point with an example in Landau gauge, where the (Euclidean) quark propagator can be written as

$$S(p) = \frac{1}{-\mathrm{i}\not{p}A(p^2) + B(p^2)} = \frac{1}{A(p^2)}\frac{\mathrm{i}\not{p} + M(p^2)}{p^2 + M^2(p^2)}.$$

The quark mass function M is defined as $M(p^2) = B(p^2)/A(p^2)$, and at tree level we have $A = 1$ and $B = M = m$, with m being the bare current quark mass. The static, i.e. equal-time propagator $S_3(\vec{p})$ is obtained from the full one $S(p)$ by integrating out the energy component p_4 of the four-momentum

$$S_3(\vec{p}) = \int \frac{\mathrm{d}p_4}{2\pi} S(p) = \mathrm{i}\vec{\gamma}\cdot\vec{p}\int \frac{\mathrm{d}p_4}{2\pi}\frac{1}{A(p_4^2+\vec{p}^2)}\frac{1}{p_4^2+\vec{p}^2+M^2(p_4^2+\vec{p}^2)} + \int \frac{\mathrm{d}p_4}{2\pi}\frac{1}{A(p_4^2+\vec{p}^2)}\frac{M(p_4^2+\vec{p}^2)}{p_4^2+\vec{p}^2+M^2(p_4^2+\vec{p}^2)}.$$

In complete analogy to the definition of the quark mass function M we can introduce the equal-time mass function $M_3(\vec{p}^2)$ as ratio of the coefficients of the $\mathbb{1}$ and γ^i terms of the equal-time propagator,

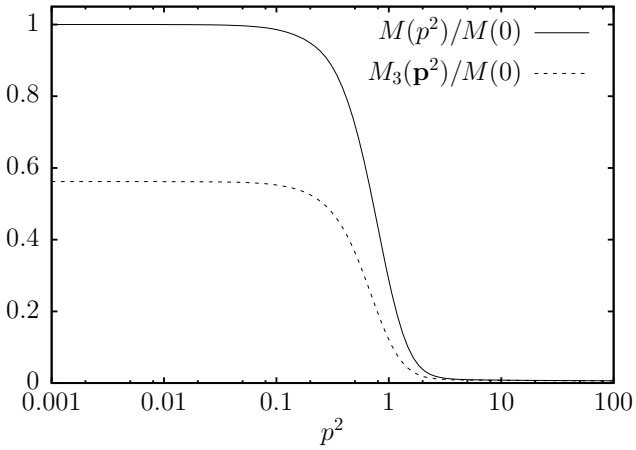

Figure 3: Comparison between the full mass function $M(p^2)$ in Landau gauge (continuous line) and the mass function $M_3(\vec{p}^2)$ of the equal-time propagator (dashed line). (Landau gauge data courtesy of M. Huber.)

yielding

$$M_3(\vec{p}^2) = \frac{\displaystyle\int_0^\infty \mathrm{d}p_4 \, \frac{1}{A(p_4^2+\vec{p}^2)} \, \frac{M(p_4^2+\vec{p}^2)}{p_4^2+\vec{p}^2+M^2(p_4^2+\vec{p}^2)}}{\displaystyle\int_0^\infty \mathrm{d}p_4 \, \frac{1}{A(p_4^2+\vec{p}^2)} \, \frac{1}{p_4^2+\vec{p}^2+M^2(p_4^2+\vec{p}^2)}}. \tag{6.1}$$

For typical results for the Landau gauge quark propagator we find that $M_3(0)$ lies between 50% and 60% of $M(0)$, see Fig. 3.

The situation might be similar in Coulomb gauge, where the propagator takes the form

$$S^{-1}(p) = -\mathrm{i}\gamma_4 p_4 A_t(p_4,\vec{p}) - \mathrm{i}\vec{\gamma}\cdot\vec{p}\,A_s(p_4,\vec{p}) - \mathrm{i}\gamma_4 p_4 \vec{\gamma}\cdot\vec{p}\,A_d(p_4,\vec{p}) + B(p_4,\vec{p}).$$

Since Coulomb gauge is non-covariant, the propagator depends separately on p_4 and $\vec{p}$, and has therefore *four* Dirac components instead of two. The mixed structure $\gamma_4\gamma_i$ does not arise at one-loop level in perturbation theory [26] and is not found in lattice calculations [34, 35] either; therefore we will set $A_d = 0$ in the following. The propagator in Coulomb gauge takes therefore the form

$$S(p) = \frac{\mathrm{i}\gamma_4 p_4 A_t(p_4,\vec{p}) + \mathrm{i}\vec{\gamma}\cdot\vec{p}\,A_s(p_4,\vec{p}) + B(p_4,\vec{p})}{p_4^2 A_t^2(p_4,\vec{p}) + \vec{p}^2 A_s^2(p_4,\vec{p}) + B^2(p_4,\vec{p})}.$$

The mass function of the four-dimensional propagator can be defined by $B(p_4,\vec{p})/A_s(p_4,\vec{p})$. Analogously to Eq. (6.1) the equal-time mass function in Coulomb gauge is given by

$$M_3(\vec{p}) = \frac{\displaystyle\int_0^\infty \mathrm{d}p_4 \, \frac{B(p_4,\vec{p})}{p_4^2 A_t^2(p_4,\vec{p}) + \vec{p}^2 A_s^2(p_4,\vec{p}) + B^2(p_4,\vec{p})}}{\displaystyle\int_0^\infty \mathrm{d}p_4 \, \frac{A_s(p_4,\vec{p})}{p_4^2 A_t^2(p_4,\vec{p}) + \vec{p}^2 A_s^2(p_4,\vec{p}) + B^2(p_4,\vec{p})}}.$$

We expect that also in Coulomb gauge $M_3(\vec{p})$ is smaller than the mass extracted from the four-dimensional propagator.

7. Conclusions

The approach to Hamiltonian QCD by means of Dyson–Schwinger equations, presented first in Ref. [20] in the framework of pure Yang–Mills theory and generalized to full QCD in Ref. [21], has been applied to an Ansatz for the vacuum wave functional which in addition to the usual quark-gluon coupling includes a non-trivial Dirac structure. This additional term, which in non-zero only in the chirally broken phase, removes the divergences from the gap equation and ensures the multiplicative renormalizability of the quark propagator. We have also demonstrated that the mass function extracted from the static propagator is considerably smaller than the one extracted from the four-dimensional propagator.

Acknowledgments

This work was supported in part by the Deutsche Forschungsgemeinschaft (DFG) under contract No. Re856/10-1.

References

[1] S. Aoki *et al.*, Eur. Phys. J. **C77**, 112 (2017), arXiv:1607.00299.

[2] M. Faber and R. Höllwieser, Prog. Part. Nucl. Phys. **97**, 312 (2017).

[3] R. Alkofer and L. von Smekal, Phys. Rept. **353**, 281 (2001), arXiv:hep-ph/0007355.

[4] J. M. Pawlowski, Annals Phys. **322**, 2831 (2007), arXiv:hep-th/0512261.

[5] C. S. Fischer, J. Phys. **G32**, R253 (2006), arXiv:hep-ph/0605173.

[6] D. Binosi and J. Papavassiliou, Phys. Rept. **479**, 1 (2009), arXiv:0909.2536.

[7] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, Prog. Part. Nucl. Phys. **91**, 1 (2016), arXiv:1606.09602.

[8] C. Feuchter and H. Reinhardt, Phys. Rev. **D70**, 105021 (2004), arXiv:hep-th/0408236.
H. Reinhardt and C. Feuchter, Phys. Rev. **D71**, 105002 (2005), arXiv:hep-th/0408237.

[9] D. Epple, H. Reinhardt, and W. Schleifenbaum, Phys. Rev. **D75**, 045011 (2007), arXiv:hep-th/0612241.

[10] P. Vastag, H. Reinhardt, and D. Campagnari, Phys. Rev. **D93**, 065003 (2016), arXiv:1512.06733.

[11] J. R. Finger and J. E. Mandula, Nucl. Phys. **B199**, 168 (1982).

[12] A. Amer, A. Le Yaouanc, L. Oliver, O. Pene, and J. c. Raynal, Phys. Rev. Lett. **50**, 87 (1983).

[13] A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, Phys. Rev. **D29**, 1233 (1984).

[14] S. L. Adler and A. Davis, Nucl. Phys. **B244**, 469 (1984).

[15] R. Alkofer and P. Amundsen, Nucl. Phys. **B306**, 305 (1988).

[16] P. J. d. A. Bicudo and J. E. F. T. Ribeiro, Phys. Rev. **D42**, 1611 (1990).

[17] M. Pak and H. Reinhardt, Phys. Lett. **B707**, 566 (2012), arXiv:1107.5263.

[18] M. Pak and H. Reinhardt, Phys. Rev. **D88**, 125021 (2013), arXiv:1310.1797.

[19] D. R. Campagnari, E. Ebadati, H. Reinhardt, and P. Vastag, Phys. Rev. **D94**, 074027 (2016), arXiv:1608.06820.

[20] D. R. Campagnari and H. Reinhardt, Phys. Rev. **D82**, 105021 (2010), arXiv:1009.4599.

[21] D. R. Campagnari and H. Reinhardt, Phys. Rev. **D92**, 065021 (2015), arXiv:1507.01414.

[22] D. Campagnari and H. Reinhardt, Phys. Rev. **D97**, 054027 (2018), arXiv:1801.02045.

[23] N. H. Christ and T. D. Lee, Phys. Rev. **D22**, 939 (1980).

[24] D. Campagnari, A. Weber, H. Reinhardt, F. Astorga, and W. Schleifenbaum, Nucl. Phys. **B842**, 501 (2011), arXiv:0910.4548.

[25] D. R. Campagnari and H. Reinhardt, Int. J. Mod. Phys. **A30**, 1550100 (2015), arXiv:1404.2797.

[26] C. Popovici, P. Watson, and H. Reinhardt, Phys. Rev. **D79**, 045006 (2009), arXiv:0810.4887.

[27] A. Buckley *et al.*, Eur. Phys. J. **C75**, 132 (2015), arXiv:1412.7420.

[28] Y. Nakagawa, A. Nakamura, T. Saito, H. Toki, and D. Zwanziger, Phys. Rev. **D73**, 094504 (2006), arXiv:hep-lat/0603010.

[29] M. Golterman, J. Greensite, S. Peris, and A. P. Szczepaniak, Phys. Rev. **D85**, 085016 (2012), arXiv:1201.4590.

[30] J. Greensite and A. P. Szczepaniak, Phys. Rev. **D91**, 034503 (2015), arXiv:1410.3525.

[31] S. Borsanyi *et al.*, Phys. Rev. **D88**, 014513 (2013), arXiv:1205.0788.

[32] G. Cossu, H. Fukaya, S. Hashimoto, T. Kaneko, and J.-I. Noaki, PTEP **2016**, 093B06 (2016), arXiv:1607.01099.

[33] JLQCD, M. Tomii *et al.*, Phys. Rev. **D96**, 054511 (2017), arXiv:1703.06249.

[34] G. Burgio, M. Schrock, H. Reinhardt, and M. Quandt, Phys. Rev. **D86**, 014506 (2012), arXiv:1204.0716.

[35] M. Pak and M. Schröck, Phys. Rev. **D91**, 074515 (2015), arXiv:1502.07706.