We study tetraquark resonances for a pair of static antiquarks $\bar{b}\bar{b}$ in presence of two light quarks $ud$ based on lattice QCD potentials. The system is treated in the Born-Oppenheimer approximation and we use the emergent wave method. We focus on the isospin $I = 0$ channel but take different angular momenta $l$ of the heavy antiquarks $\bar{b}\bar{b}$ into account. Further calculations have already predicted a bound state for the $l = 0$ case with quantum numbers $I(J^P) = 0(1^+)$. Performing computations for several angular momenta, we extract the phase shifts and search for $T$ and $S$ matrix poles in the second Riemann sheet. For angular momentum $l = 1$, we predict a tetraquark resonance with quantum numbers $I(J^P) = 0(1^-)$, resonance mass $m = 10576^{+4}_{-4}$ MeV and decay width $\Gamma = 112^{+90}_{-103}$ MeV, which decays into two $B$ mesons.
1. Introduction

A very challenging problem in particle physics is understanding exotic hadrons. In this work, we investigate a tetraquark system with two heavy antiquarks $\bar{b}\bar{b}$ and two light quarks $qq$ with $q \in \{u,d,s,c\}$. While the existence of bound states has been studied in recent years applying lattice QCD potentials and the Born-Oppenheimer approximation, a stable tetraquark state has been predicted with quantum numbers $I(J^P) = 0(1^+)$ [1, 2, 3, 4, 5, 6, 7, 8, 9]. This state has been confirmed using heavy quarks of finite mass [10]. The current work extends this investigation by a new technique from scattering theory, the emergent wave method [11], and we search for possibly existing tetraquark resonances (cf. [12]).

2. Lattice QCD Potentials of two Static Heavy Antiquarks $\bar{Q}\bar{Q}$ in the Presence of two Light Quarks $qq$

In previous studies, we have computed the potentials $V(r)$ for two static antiquarks $\bar{Q}\bar{Q}$ in the presence of two light quarks $qq$ applying methods of lattice QCD. Calculations have been performed for different light quark flavour combinations i.e. $qq$ with $q \in \{u,d,s,c\}$. Moreover, several values for the parity $P$ and the total angular momentum of the light quarks and gluons $j$ (cf. e.g. [7, 8]) have been studied. For these wide range of quantum numbers, there are attractive as well as repulsive channels. There have been identified two attractive potentials with $q \in \{u,d\}$ which are quite wide and deep. These are most promising when investigating the existence of bound tetraquark states or resonances. The two attractive potentials are characterised by the quantum numbers $(I = 0, j = 0)$ and $(I = 1, j = 1)$. The potentials are illustrated in Fig. 1 for lattice spacing $a \approx 0.079 \text{ fm}$ and u/d quark masses corresponding to a pion mass $m_\pi \approx 340 \text{ MeV}$.

It is well-known, that the binding energy and hence the existence or non-existence of a stable tetraquark state depends on the light quark mass [7]. Thus, we have performed calculations for three different light quark masses u/d corresponding to $m_\pi \in \{340 \text{ MeV}, 480 \text{ MeV}, 650 \text{ MeV}\}$. These results are used to perform an extrapolation to $m_\pi = 140 \text{ MeV}$. Moreover, a study of discretisation errors and final volume effects shows that these uncertainties are negligible compared to statistical uncertainties (cf. [8]). Searching for bound states as well as resonances, we parametrize the

![Figure 1: (left) $(I = 0, j = 0)$ potential. (right) $(I = 1, j = 1)$ potential.](image-url)
potential by a screened Coulomb potential:

\[ V(r) = -\frac{\alpha}{r} e^{-r^2/d^2}. \]  

This ansatz is motivated by a one-gluon exchange for small \( \bar{Q}Q \) separations \( r \) and the formation of \( b\bar{b} \) tetraquark resonances.

![Diagram](image)

Figure 2: (a) At small separations the static antiquarks \( \bar{Q}Q \) interact by perturbative one-gluon exchange. (b) At large separations the light quarks \( qq \) screen the interaction and the four quarks form two rather weakly interacting \( B \) mesons.

Two \( B \) mesons at larger \( r \) as a consequence of a screened Coulomb potential (cf. Fig. 2). Even if this approach is phenomenologically motivated, it is consistent with our lattice QCD results. The numerical results for the parameters \( \alpha \) and \( d \) are listed in Table 1. Clearly, the \((I = 0, j = 0)\) potential is more attractive than the \((I = 1, j = 1)\) potential. Consequently, Eq. (2.1) describes the potential of two heavy antiquarks \( \bar{b}b \) in the presence of two light quarks \( ud \), so in other words, we apply a Born-Oppenheimer approximation [13]. We can use this potential in the Schrödinger equation to search for either bound states (cf. [5, 7, 9, 12]) or resonances (cf. Sec. 3 and 4).

Solving the Schrödinger equation for \((I = 0, j = 0)\), a bound state with binding energy \( 90^{+43}_{-36} \) MeV and quantum numbers \( I(J^P) = 0(1^+) \) has been found [8].

### 3. The Emergent Wave Method

In this section, we present the emergent wave method, which is a well suited approach to study phase shifts and resonances. More details can be found, for example, in [11]. First, we consider the well-known Schrödinger equation

\[ \left( H_0 + V(r) \right) \Psi = E \Psi. \]
We start by splitting the wave function into two parts,
\[ \Psi = \Psi_0 + X, \]  
where \( \Psi_0 \) is the incident wave, which solves the free Schrödinger equation \( H_0 \Psi_0 = E \Psi_0 \) and \( X \) indicates the emergent wave. As the next step, we insert Eq. (3.2) into Eq. (3.1). Using the free Schrödinger equation to simplify, we obtain:
\[ \left( H_0 + V(r) - E \right) X = -V(r) \Psi_0. \]  
Solving this equation for any arbitrary energy \( E \), we can compute the emergent wave \( X \) by providing the corresponding \( \Psi_0 \) and fixing the appropriate boundary conditions. The asymptotic behaviour of \( X \) determines the phase shifts, the S matrix and the T matrix. Continuing this problem to complex energies is possible without difficulties. We find the poles of the S matrix and the T matrix in the complex plane and identify them with a resonance, when located in the second Riemann sheet at \( m - \Gamma/2 \), where \( m \) is the mass and \( \Gamma \) is the decay width of the resonance.

3.1 Partial Wave Decomposition

The Hamiltonian describing the two heavy antiquarks \( \bar{b} \bar{b} \) at vanishing total momentum is
\[ H = H_0 + V(r) = -\frac{\hbar^2}{2\mu} \Delta + V(r) \]  
where \( \mu = M/2 \) is the reduced mass and \( M = 5280 \text{ MeV} \) is the mass of the \( B \) meson from the PDG [14]. We consider all results as energy differences with respect to \( 2M \), so we omit the additive constant \( 2M \) in Eq. (3.4). In the next step, we express the incident plane wave \( \Psi_0 = e^{ikr} \) as a sum of spherical waves,
\[ \Psi_0 = e^{ikr} = \sum_l (2l+1) j_l(kr) P_l(\hat{k} \cdot \hat{r}), \]  
where \( j_l \) are spherical Bessel functions, \( P_l \) are Legendre polynomials and the relation between energy and momentum is \( \hbar k = \sqrt{2\mu E} \). Since the potential \( V(r) \) in Eq. (2.1) is spherically symmetric, we can also expand the emergent wave \( X \) in terms of Legendre polynomials \( P_l \),
\[ X = \sum_l (2l+1) j_l(kr) P_l(\hat{k} \cdot \hat{r}). \]  
Inserting Eq. (3.5) and Eq. (3.6) into Eq. (3.3) leads to a set of ordinary one-dimensional differential equations for \( \chi_l \),
\[ \left( -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)}{2\mu r^2} + V(r) - E \right) \chi_l(r) = -V(r)kr j_l(kr). \]  

3.2 Solving the Differential Equations for the Emergent Wave

The potentials \( V(r) \), Eq. (2.1), are exponentially screened, i.e. \( V(r) \approx 0 \) for \( r \geq R \), where \( R \gg d \). Consequently, the emergent wave is a superposition of outgoing spherical waves for large separations \( r \geq R \) and can be expressed by the spherical Hankel functions of first kind \( h_l^{(1)} \):
\[ \frac{\chi_l(r)}{kr} = it_l h_l^{(1)}(kr). \]
If we compute the complex prefactors $t_l$, this will lead to the phase shifts. To this end we solve the ordinary differential equation (3.7) with the corresponding boundary conditions as follows:

- At $r = 0$: $\chi_l(r) \propto r^{l+1}$.
- For $r \geq R$: Eq. (3.8).

We emphasize that the boundary condition for $r \geq R$ depends on $t_l$. Solving the differential equation for a given value of the energy $E$, this boundary condition is only fulfilled for a specific corresponding value of $t_l$. In other words the boundary condition for $r \geq R$ fixes $t_l$ as a function of $E$.

For the numerical solution of Eq. (3.7), we implement two different and independent approaches:

1. A fine uniform discretization of the interval $[0, R]$, which reduces the differential equation to a large set of linear equations, which can be solved rather efficiently, since the corresponding matrix is tridiagonal;

2. A standard 4-th order Runge-Kutta shooting method.

### 3.3 Phase Shifts, S and T Matrix Poles

We identify $t_l$ as an eigenvalue of the T matrix (cf. standard textbooks on quantum mechanics and scattering, e.g. [15]). Knowing $t_l$ we can determine the phase shift $\delta_l$ and also the corresponding S matrix eigenvalue $s_l$:

$$s_l = 1 + 2it_l = e^{2i\delta_l}.$$  

Note that both the S matrix and the T matrix are analytical in the complex plane and are also well-defined for complex energies $E$. Thus, we extend our numerical method to solve the differential Eq. (3.7) to complex $E$ and detect the S and T matrix poles by scanning the complex plane ($\text{Re}(E), \text{Im}(E)$) and applying Newton’s method to find the roots of $1/t_l(E)$. These poles correspond to complex energies of resonances and must be located in the second Riemann sheet with a negative imaginary part both for the energy $E$ and the momentum $k$.

### 4. Results for the Phase Shifts, S- and T-Matrix Poles and Prediction of Resonances

#### 4.1 Phase Shifts $\delta_l$ for Real Energies $E$

First, we consider the more attractive channel ($I = 0, j = 0$) of the $\bar{b}b\bar{d}u$ potential (cf. Sec 2). We compute $t_l$ for real energies $E$ and apply Eq. (3.9) to determine the phase shift $\delta_l$ for the different angular momenta $l = 0, 1, 2, 3, \ldots$. For a resonance, we expect a fast increasing of $\delta_l$ from 0 to $\approx \pi$ which is, however, not clearly found (cf. Fig. 3 (left)). Thus, we have to search more thoroughly if there exists a resonance or not. We consider the $l = 1$ channel and search for a clear resonance making the potential more attractive by increasing the parameter $\alpha$ while $d$ is kept fixed. We illustrate the phase shifts $\delta_l$ in Fig. 3 (right). For $\alpha \gtrsim 0.65$ we find a clear resonance while for $\alpha = 0.72$ a bound state is formed, i.e. the phase shift starts at $\pi$ and decreases monotonically. However, this observation does not allow to make a clear statement for which values of $\alpha$ a resonance exists or not.

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At large distances $r \geq R$, the radial wavefunction is $kr[j_l(kr) + it_l h_l^{(1)}(kr)] = (kr/2)[h_l^{(2)}(kr) + e^{2i\delta_l} h_l^{(1)}(kr)]$. 

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Figure 3: (left): Phase shift $\delta_l$ as a function of the energy $E$ for different angular momenta $l = 0, 1, 2, 3, 4$ for the $(I = 0, j = 0)$ potential ($\alpha = 0.34, d = 0.45\, \text{fm}$). (right): Phase shift $\delta_l$ as a function of the energy $E$ for different parameters for the potential. For illustration, we vary parameter $\alpha$ only while fixing $d = 0.45\, \text{fm}$ at the value of the $(I = 0, j = 0)$ potential. Fixing $d$ and varying $\alpha$ produces comparable results.

4.2 Resonances as Poles of the $S$ and $T$ Matrices for Complex Energies $E$

Thus, we search directly for resonances as poles of the $T$ matrix eigenvalue $t_l$ in the complex energy plane. For angular momentum $l = 1$ and the physical $(I = 0, j = 0)$ potential, we clearly identify a pole which is shown in Fig. 4 (left) plotting $t_l$ as a function of the complex energy $E$. For a better understanding of the resonance and its dependence on the potential, we determine the pole of the $T$ matrix eigenvalue $t_l$ for various parameters $\alpha$. In Fig. 4 (right), we illustrate the location of the pole for different values of $\alpha$ in the $(\text{Re}(E), \text{Im}(E))$ plane. Indeed, starting with $\alpha = 0.21$ we find poles. Consequently, our prediction for a resonance at $\alpha = 0.34$ is confirmed. For angular momenta $l \neq 1$ as well as for the less attractive channel $(I = 1, j = 1)$ for all $l$, no pole has been found.

4.3 Statistical and Systematic Error Analysis

Finally, we perform a detailed statistical and systematic error analysis with regard to the pole of $t_l$ in the complex plane $(\text{Re}(E), \text{Im}(E))$ for angular momentum $l = 1$. We apply the same analysis methods as for our study of bound states presented in [7]. We parametrize the lattice QCD data for the potential $V^\text{lat}(r)$ with an uncorrelated $\chi^2$ minimizing fit with the ansatz of Eq. (2.1), i.e. we minimize the expression

$$\chi^2 = \sum_{r = r_{\text{min}} \ldots r_{\text{max}}} \left( \frac{V(r) - V^\text{lat}(r)}{\Delta V^\text{lat}(r)} \right)^2 \quad (4.1)$$

with respect to $\alpha$ and $d$ defined in Eq. (2.1). $\Delta V^\text{lat}(r)$ denotes the corresponding statistical errors.

To investigate systematic errors, we perform fits for various fit ranges. Besides, we vary the range of the temporal separation $t_{\text{min}} \leq t \leq t_{\text{max}}$ of the correlation function where $V^\text{lat}(r)$ is read off and the spatial separation $r_{\text{min}} \leq r \leq r_{\text{max}}$ denoting the $\bar{b}\bar{b}$ distance considered in the $\chi^2$ minimizing fit to determine the parameters $\alpha$ and $d$. 
Figure 4: (left): T matrix eigenvalue $t_1$ as a function of the complex energy $E$ for the $(l = 0, j = 0)$ potential ($\alpha = 0.34, d = 0.45\text{fm}$). Along the vertical axis we show the norm $|t_1|$, while the phase $\arg(t_1)$ corresponds to different colours. (right): Trajectory of the pole of the eigenvalue $t_1$ of the T matrix in the complex plane $(\text{Re}(E), \text{Im}(E))$, corresponding to a variation of parameter. We also illustrate with a cloud of diamond points the systematic error [7].

The statistical errors are included determining the jackknife errors of the medians of $\text{Re}(E)$ and $\text{Im}(E)$. Finally, systematic and statistical uncertainties are added quadratically.

Applying this combined systematic and statistical error analysis, we find a resonance energy $\text{Re}(E) = 17^{+4}_{-4}\text{MeV}$ and a decay width $\Gamma = -2\text{Im}(E) = 112^{+90}_{-103}\text{MeV}$. Studying the symmetries of the quarks with respect to colour, flavour, spin and their spatial wave function and considering the Pauli principle we determine the quantum numbers to be $I(J^P) = 0(1^-)$. The decay product of this resonance will be two $B$ mesons, so its mass is given by $m = 2M + \text{Re}(E) = 10576^{+4}_{-4}\text{MeV}$.

5. Conclusion and Outlook

We searched for resonances in the $\bar{b}b\bar{u}d$ system applying lattice QCD potentials for two static antiquarks in the presence of two light quarks, the Born-Oppenheimer approximation and the emergent wave method. First, we have considered the scattering phase shift for a $BB$ meson pair. Afterwards, we continued analytically the S matrix and T matrix to the second Riemann sheet and searched for poles in the complex plane.

After a solid statistical and systematic analysis, we predict a new resonance with quantum numbers $I(J^P) = 0(1^-)$, a resonance mass $\text{Re}(E) = 17^{+4}_{-4}\text{MeV}$ and a decay width $\Gamma = 112^{+90}_{-103}\text{MeV}$.

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