

Influence of quark masses and strangeness degrees of freedom on inhomogeneous chiral phases

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Several model calculations of the QCD phase structure at nonzero temperature and density suggest that in certain regions of the phase diagram inhomogeneous condensates are favored over homogeneous ones. In particular, in a two-flavor NJL model in the chiral limit the would-be first-order boundary associated with the chiral phase transition is entirely covered by an inhomogeneous phase, characterized by a spatially modulated chiral condensate, whose tip coincides exactly with the chiral critical point. In this contribution we discuss how this result is altered by the inclusion of nonzero quark masses and strange quarks in the model.

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1. Introduction

The possible existence of a chiral critical point in the QCD phase diagram at nonzero temperature T and quark chemical potential μ has stimulated a large amount of theoretical and experimental efforts during the last two decades (see e.g. [1, 2, 3, 4, 5]). At low T and nonvanishing μ , where lattice QCD techniques cannot be applied, QCD-inspired models, like the Nambu–Jona-Lasinio (NJL) model or the quark-meson (QM) model, typically find a first-order chiral phase transition. Beyond a given temperature, this transition line eventually terminates at a critical endpoint (CEP) [6, 7], above which only a crossover is found, in agreement with the lattice result for vanishing densities [8]. If these model calculations are performed considering only up and down quarks with vanishing bare masses (the so-called chiral limit) one finds instead a second-order transition in the low- μ -high- T regime which is connected to the first-order phase boundary at low T in a tricritical point (TCP).

The underlying assumption which has been used to obtain the aforementioned picture is that the chiral order parameter is spatially homogeneous. If one instead allows for spatial modulations, inhomogeneous phases are found to be energetically favored over the homogeneous ones in certain regions of the phase diagram (for a review see [9]). In particular, for a two-flavor NJL model in the chiral limit one finds that the first-order phase boundary separating the homogeneous chirally broken phase from the restored one is completely covered by an inhomogeneous phase. As a consequence, a Lifshitz point (LP), which is the point where three different phases (the homogeneous and inhomogeneous chirally broken phases, together with the restored one) meet, appears in the phase diagram. In the simplest realization of a two-flavor NJL model in the chiral limit, it was shown within a Ginzburg-Landau (GL) analysis that the LP is located at the same position in the T - μ plane as the TCP in the case when inhomogeneous phases are not considered [10].

In this contribution we discuss how the presence of bare quark masses and of strange quarks affects this result. Inhomogeneous phases in a two-flavor NJL model with nonzero current masses have already been studied in Ref. [11]. There it was found numerically that the inhomogeneous region shrinks with increasing quark masses but still covers the entire first-order phase boundary between the homogeneous chirally broken and restored phases. In Ref. [12] we have recently confirmed this more rigorously within a GL analysis, showing that the tip of the inhomogeneous phase, which we termed “pseudo-Lifshitz point” (PLP), exactly coincides with the CEP. This will be discussed in section 2.

In section 3 we extend our GL study to a three-flavor NJL model. A first investigation of inhomogeneous phases in three-flavor matter has been performed in Ref. [13], albeit only for vanishing temperatures and employing a simple explicit ansatz for the spatial dependence of the chiral condensate. Our analysis will instead allow us to investigate the relation between the CEP and the (P)LP, an aspect which is of particular interest: Indeed, our motivation for including strange quarks is not only that they may play a role under realistic conditions, e.g., in astrophysical scenarios, but also the fact that in the limit of three very light quark flavors the first-order phase boundary (and hence the CEP) is expected to eventually reach the T axis [3]. Therefore, if the (P)LP still coincides with the CEP for three flavors and assuming that the same holds in QCD, this particular limit would open the possibility to study inhomogeneous phases at $\mu = 0$ on the lattice.

2. Inhomogeneous phases away from the chiral limit

We consider the standard NJL-model Lagrangian

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + G \{ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{\tau}\psi)^2 \}, \quad (2.1)$$

describing quark fields ψ with bare mass m and $N_f = 2$ and $N_c = 3$ color degrees of freedom, interacting via scalar-isoscalar and pseudoscalar-isovector four-point vertices proportional to a coupling constant G .

We perform a mean-field approximation, linearizing the interaction in the presence of a scalar condensate $\langle \bar{\psi}\psi \rangle$. In order to describe inhomogeneous phases we generally allow the condensate to depend on the spatial coordinate \mathbf{x} but we assume it to be time independent. For simplicity we restrict ourselves to scalar condensates. The inclusion of pseudoscalar condensates is straightforward but it is well known for homogeneous phases that they are disfavored against scalar condensates if $m \neq 0$. In Ref. [12] we have shown that this is also true for inhomogeneous condensates, at least in the regime where the GL analysis we discuss below is valid. The mean-field Lagrangian then takes the form

$$\mathcal{L}_{\text{MF}} = \bar{\psi} (i\gamma^\mu \partial_\mu - M(\mathbf{x})) \psi - \frac{(M(\mathbf{x}) - m)^2}{4G}, \quad (2.2)$$

where

$$M(\mathbf{x}) = m - 2G \langle \bar{\psi}\psi \rangle(\mathbf{x}) \quad (2.3)$$

can be interpreted as a space-dependent constituent quark mass.

Noting that \mathcal{L}_{MF} is bilinear in the quark fields, the mean-field thermodynamic potential per volume $\Omega(T, \mu) = -T/V \log \mathcal{Z}(T, \mu)$ can be obtained by standard path-integral techniques. One finds

$$\Omega(T, \mu) = -\frac{T}{V} \mathbf{Tr} \log S^{-1} + \frac{1}{V} \int d^3x \frac{(M(\mathbf{x}) - m)^2}{4G}, \quad (2.4)$$

where

$$S^{-1}(x) = i\gamma^\mu \partial_\mu + \mu\gamma^0 - M(\mathbf{x}) \quad (2.5)$$

is the inverse dressed quark propagator, V is a quantization volume, and the functional trace \mathbf{Tr} runs over the Euclidean 4-volume $V_4 = [0, \frac{1}{T}] \times V$ as well as over Dirac, color and flavor degrees of freedom.

2.1 Ginzburg-Landau analysis of critical and Lifshitz points

In order to find the thermodynamically favored state at given T and μ , Ω has to be minimized with respect to the mass function $M(\mathbf{x})$. This is obviously a nontrivial task. The position of the (P)LP, on the other hand, can be determined within a GL analysis, which is possible without knowing the explicit form of $M(\mathbf{x})$.

In the chiral limit, $m = 0$, the GL expansion corresponds to an expansion of the thermodynamic potential about the chirally restored phase, $M(\mathbf{x}) = 0$ in terms of powers and gradients of M :

$$\Omega[M] = \Omega[0] + \frac{1}{V} \int d^3x \left(\alpha_2 M^2(\mathbf{x}) + \alpha_{4,a} M^4(\mathbf{x}) + \alpha_{4,b} (\nabla M(\mathbf{x}))^2 + \dots \right). \quad (2.6)$$

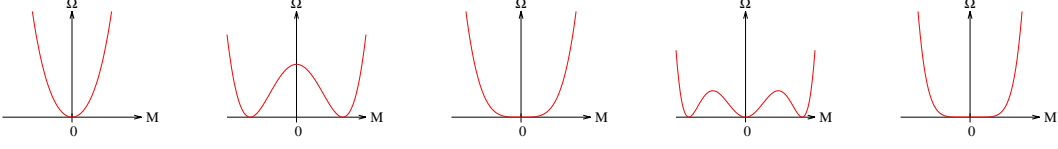


Figure 1: Qualitative behavior of the thermodynamic potential Ω in the chiral limit as a function of a spatially homogeneous order parameter M (from left to right): (i) restored phase, (ii) broken phase, (iii) second-order phase transition, (iv) first-order phase transition, (v) TCP.

Here odd powers are forbidden by chiral symmetry, which implies that Ω is invariant under $M \rightarrow -M$. The GL coefficients α_i are functions of T and μ and thus determine the phase structure.

In the following we assume that M and ∇M are small and that the coefficients of all higher-order terms (indicated by the ellipsis in Eq. (2.6)) are positive. If $\alpha_{4,b}$ is positive as well, gradients are suppressed and the situation reduces to the standard case of a GL analysis for homogeneous phases, which is illustrated in Fig. 1: If all coefficients are positive, the minimum of Ω is the restored phase, $M = 0$ (i), while for $\alpha_2 < 0$ a solution with $M \neq 0$ is favored (ii). If $\alpha_{4,a} > 0$ we thus have a second-order phase transition at $\alpha_2 = 0$ (iii). For $\alpha_{4,a} < 0$, on the other hand, there can be a minimum with $M \neq 0$, even for $\alpha_2 > 0$, so that we can have a first-order phase transition in this case (iv). The TCP, where the first-order phase boundary goes over into a second-order one is thus given by the condition $\alpha_2 = \alpha_{4,a} = 0$ (v).

For $\alpha_{4,b} < 0$ inhomogeneous phases can become favored over homogeneous ones. Its boundary to the restored phase is found to be second order and determined by a balance between the free-energy gain due to the negative $\alpha_{4,b}$ term and a free-energy loss caused by a positive α_2 . As a consequence, while the amplitude of the space-dependent modulation vanishes on the phase boundary, its wave number stays finite and goes to zero only at the LP where both α_2 and $\alpha_{4,b}$ vanish.

Away from the chiral limit, i.e., for $m \neq 0$ the situation is more complicated since there is no exactly restored phase. We therefore expand the thermodynamic potential about an a priori unknown mass M_0 , which may depend on T and μ but which we assume to be constant in space. Then, writing $M(\mathbf{x}) = M_0 + \delta M(\mathbf{x})$ and assuming that the fluctuations $\delta M(\mathbf{x})$ and their gradients are small, the expansion reads

$$\Omega[M] = \Omega[M_0] + \frac{1}{V} \int d^3x (\alpha_1 \delta M + \alpha_2 \delta M^2 + \alpha_3 \delta M^3 + \alpha_{4,a} \delta M^4 + \alpha_{4,b} (\nabla \delta M)^2 + \dots), \quad (2.7)$$

with coefficients α_i which now depend on T , μ , and M_0 . In contrast to the chiral limit, the integrand now contains both even and odd powers of δM . In the following we will assume that M_0 corresponds to a stationary point of the thermodynamic potential at given values of T and μ . This implies that the linear term vanishes, $\alpha_1(T, \mu; M_0) = 0$, which corresponds to a gap equation for $M_0(T, \mu)$. As we will see below, it is however crucial to keep the α_3 term.

Again, we first consider the homogeneous case, illustrated in Fig. 2. Although there is no strictly restored phase for $m \neq 0$ we can have a first-order phase transition where Ω as a function of M has two degenerate minima separated by a maximum (i). If we now move along the phase boundary towards the CEP, the maximum gets more and more shallow until the three extrema merge to a single minimum at the CEP (ii). Alternatively we can approach the CEP along the

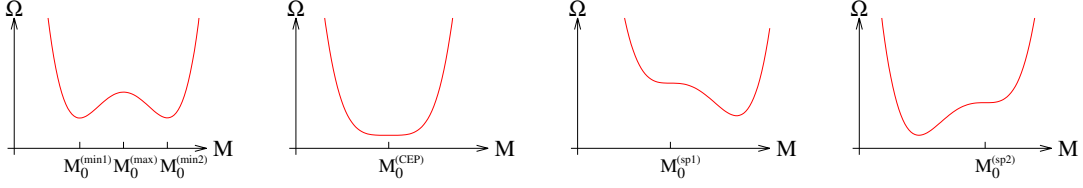


Figure 2: Qualitative behavior of the thermodynamic potential Ω for $m \neq 0$ as a function of a spatially homogeneous order parameter M (from left to right): (i) at a first-order phase boundary, (ii) at the CEP, (iii) at the left spinodal, (iv) at the right spinodal.

spinodals, which correspond to the lines in the T - μ plane where one of the two minima merges with the maximum to a saddle point, see Fig. 2 (iii) and (iv). Either way one can convince oneself that the CEP is characterized by $\alpha_2 = \alpha_3 = 0$. For instance on the saddlepoint of the left spinodal (iii) the second derivative of Ω with respect to δM vanishes while the third derivative is negative, corresponding to $\alpha_2 = 0$ and $\alpha_3 < 0$ if we expand about this point. On the other hand, if we expand about the saddlepoint of the right spinodal, we get $\alpha_2 = 0$ and $\alpha_3 > 0$, and therefore both α_2 and α_3 vanish at the CEP where the two spinodals meet.

Lacking a chirally restored phase away from the chiral limit, there also cannot exist a Lifshitz point, where the second-order boundary between the homogeneous chirally broken and restored phases meets the boundaries of an inhomogeneous phase. It is still possible however to realize a second-order transition between a homogeneous and an inhomogeneous phase where the amplitude of the *oscillating* part of $M(\mathbf{x})$ goes to zero, while its wave number can stay finite. In analogy to the LP we then define the PLP as the point on that phase boundary where this wave number also vanishes. Since our expansion point M_0 is constant, the oscillating part is entirely contained in δM . In the same way as for the LP in the chiral limit we therefore find that $\alpha_2 = \alpha_{4,b} = 0$ at the PLP. Moreover, as argued in Ref. [12], the phase boundary cannot continue in a smooth way beyond this point. It is therefore most plausible to identify it with the tip of the inhomogeneous phase in the T - μ plane. This was also confirmed numerically.

In summary, we find

$$\alpha_2 = \alpha_{4,a} = 0 \quad \text{at the TCP}, \quad \alpha_2 = \alpha_{4,b} = 0 \quad \text{at the LP} \quad (2.8)$$

in the chiral limit for an expansion around $M = 0$, and

$$\alpha_2 = \alpha_3 = 0 \quad \text{at the CEP}, \quad \alpha_2 = \alpha_{4,b} = 0 \quad \text{at the PLP} \quad (2.9)$$

away from the chiral limit for an expansion around a stationary point $M_0(T, \mu)$, which is obtained by simultaneously solving the gap equation $\alpha_1 = 0$.

2.2 Determination of the GL coefficients

For the explicit determination of the GL coefficients in the NJL model we basically follow Ref. [10]. Inserting our decomposition $M(\mathbf{x}) = M_0 + \delta M(\mathbf{x})$ of the constituent mass function into the mean-field thermodynamic potential, Eq. (2.4), one gets

$$\Omega = -\frac{T}{V} \text{Tr} \log(S_0^{-1} - \delta M) + \frac{1}{V} \int d^3x \frac{(M_0 - m + \delta M(\mathbf{x}))^2}{4G}, \quad (2.10)$$

where $S_0^{-1}(x) = i\gamma^\mu \partial_\mu + \mu\gamma^0 - M_0$ depends only on the constant mass M_0 . Expanding the logarithm into a Taylor series about S_0^{-1} this can be written as $\Omega = \sum_{n=0}^{\infty} \Omega^{(n)}$ where $\Omega^{(n)}$ is of the n th order in the fluctuating fields δM . Specifically one obtains

$$\Omega^{(1)} = \frac{T}{V} \mathbf{Tr} (S_0 \delta M) + \frac{M_0 - m}{2G} \frac{1}{V} \int d^3x \delta M(\mathbf{x}), \quad (2.11)$$

$$\Omega^{(2)} = \frac{1}{2} \frac{T}{V} \mathbf{Tr} (S_0 \delta M)^2 + \frac{1}{4G} \frac{1}{V} \int d^3x \delta M^2(\mathbf{x}), \quad (2.12)$$

$$\Omega^{(n)} = \frac{1}{n} \frac{T}{V} \mathbf{Tr} (S_0 \delta M)^n \quad \text{for } n \geq 3. \quad (2.13)$$

The functional traces are given by

$$\mathbf{Tr} (S_0 \delta M)^n = 2N_c \int \prod_{i=1}^n d^4x_i \text{tr}_D [S_0(x_n, x_1) \delta M(\mathbf{x}_1) S_0(x_1, x_2) \delta M(\mathbf{x}_2) \dots S_0(x_{n-1}, x_n) \delta M(\mathbf{x}_n)], \quad (2.14)$$

where the integrals are again over V_4 , tr_D indicates a trace in Dirac space, and we have already turned out the trivial traces in color and flavor space. Noting that S_0 is the standard propagator of a free fermion with mass M_0 at chemical potential μ , the evaluation of the Dirac trace is straightforward using the momentum-space representation of S_0 . After performing a gradient expansion of $\delta M(\mathbf{x}_i)$ about \mathbf{x}_1 , i.e., $\delta M(\mathbf{x}_i) = \delta M(\mathbf{x}_1) + \nabla \delta M(\mathbf{x}_1) \cdot (\mathbf{x}_i - \mathbf{x}_1) + \dots$, one can also perform the integrations over all space-time variables $x_i \neq x_1$ and then compare the results with Eq. (2.7) to read off the GL coefficients. One finds:

$$\alpha_1 = \frac{M_0 - m}{2G} + M_0 F_1, \quad (2.15)$$

$$\alpha_2 = \frac{1}{4G} + \frac{1}{2} F_1 + M_0^2 F_2, \quad (2.16)$$

$$\alpha_3 = M_0 \left(F_2 + \frac{4}{3} M_0^2 F_3 \right), \quad (2.17)$$

$$\alpha_{4,a} = \frac{1}{4} F_2 + 2M_0^2 F_3 + 2M_0^4 F_4, \quad (2.18)$$

$$\alpha_{4,b} = \frac{1}{4} F_2 + \frac{1}{3} M_0^2 F_3, \quad (2.19)$$

where we have introduced the functions

$$F_n = 8N_c \int \frac{d^3p}{(2\pi)^3} T \sum_j \frac{1}{[(i\omega_j + \mu)^2 - \mathbf{p}^2 - M_0^2]^n} \quad (2.20)$$

with fermionic Matsubara frequencies $\omega_j = (2j+1)\pi T$.

Even without further evaluation, we can spot several interesting consequences of the above results:

- In the chiral limit, $m = 0$, the restored phase $M_0 = 0$ is a solution of the gap equation $\alpha_1 = 0$. Expanding about this solution, α_3 vanishes as well, in agreement with Eq. (2.6). Moreover, we reproduce the result $\alpha_{4,a} = \alpha_{4,b}$ of Ref. [10], meaning that the LP coincides with the TCP in the chiral limit.

- Considering $M_0 \neq 0$, but taking the limit $M_0 \rightarrow 0$ the $\alpha_3 = 0$ line in the T - μ plane approaches the $\alpha_{4,a} = 0$ line, and hence the CEP converges against the TCP (coinciding with the LP).
- For arbitrary M_0 we have $\alpha_3 = 4M_0\alpha_{4,b}$. For $M_0 \neq 0$ this implies that the PLP coincides with the CEP.

Hence, since the PLP corresponds to the tip of the inhomogeneous phase, the latter “ends” exactly at the same point as the first-order phase boundary in a purely homogeneous treatment of the same model. This is analogous to the coincidence of the LP with the TCP in the chiral limit. Of course, the GL study does not prove the existence of the CEP in the first place but only predicts that *if* there is a first-order phase transition in a homogeneous treatment of the model, then there should be an inhomogeneous phase, at least in the vicinity of the CEP. In fact, numerically one finds that the inhomogeneous phase covers the entire first-order boundary. We also found that the inhomogeneous phase shrinks as one moves away from the chiral limit, but survives even at significantly large values of the current quark mass [12].

On the other hand, even in the chiral limit the LP and TCP do not necessarily coincide anymore, if the model is modified, e.g., by adding a vector term [14]. In the next section we will study whether this is the case when strange quarks are added to the model.

3. Including strange quarks

We now consider the Lagrangian

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - \hat{m}) \psi + \mathcal{L}_4 + \mathcal{L}_6 \quad (3.1)$$

where $\psi = (u, d, s)^T$ denotes a quark field with three flavor degrees of freedom and \hat{m} is the corresponding bare mass matrix. The last two terms describe a $U(3)_L \times U(3)_R$ invariant four-point interaction

$$\mathcal{L}_4 = G \sum_{a=0}^8 [(\bar{\psi} \tau_a \psi)^2 + (\bar{\psi} i\gamma_5 \tau_a \psi)^2], \quad (3.2)$$

and a six-point (“Kobayashi - Maskawa - ’t Hooft”, KMT) interaction

$$\mathcal{L}_6 = -K [\det_f \bar{\psi} (1 + \gamma_5) \psi + \det_f \bar{\psi} (1 - \gamma_5) \psi], \quad (3.3)$$

which is $SU(3)_L \times SU(3)_R$ symmetric but breaks the $U(1)_A$ symmetry, mimicking the axial anomaly. In the former $\tau_a, a = 1, \dots, 8$, denote Gell-Mann matrices in flavor space while $\tau_0 = \sqrt{2/3} \mathbb{1}$ is proportional to the unit matrix.

Starting from these Lagrangians, we perform again a mean-field approximation, considering the non-strange and strange condensates

$$\sigma_\ell(\mathbf{x}) \equiv \langle \bar{u}u \rangle(\mathbf{x}) = \langle \bar{d}d \rangle(\mathbf{x}) \quad \text{and} \quad \sigma_s(\mathbf{x}) \equiv \langle \bar{s}s \rangle(\mathbf{x}), \quad (3.4)$$

which may be space dependent. Proceeding analogously as in the two-flavor case, the mean-field thermodynamic potential becomes

$$\Omega(T, \mu) = -\frac{T}{V} \text{Tr} \log S^{-1} + \frac{1}{V} \int d^3x \{ 2G(2\sigma_\ell^2 + \sigma_s^2) - 4K\sigma_\ell^2\sigma_s \}, \quad (3.5)$$

where the inverse dressed quark propagator is now given by $S^{-1} = \text{diag}_f(S_u^{-1}, S_d^{-1}, S_s^{-1})$ with the flavor components

$$S_f^{-1}(x) = i\gamma^\mu \partial_\mu + \mu\gamma^0 - M_f(\mathbf{x}). \quad (3.6)$$

and the constituent mass functions

$$M_\ell(\mathbf{x}) = m_\ell - 4G\sigma_\ell(\mathbf{x}) + 2K\sigma_s(\mathbf{x})\sigma_\ell(\mathbf{x}) \quad \text{and} \quad M_s(\mathbf{x}) = m_s - 4G\sigma_s(\mathbf{x}) + 2K\sigma_\ell^2(\mathbf{x}). \quad (3.7)$$

Here we assumed isospin symmetry in the light sector, $m_u = m_d \equiv m_\ell$, and thus $M_u = M_d \equiv M_\ell$.

From the above expressions we can see that the different flavors only couple through the six-point interaction and decouple for $K = 0$. This is well known from earlier studies of this model but will also play an important role in our analysis below.

3.1 Ginzburg-Landau analysis

The GL expansion of the thermodynamic potential introduced above is again complicated by the fact that in the presence of nonvanishing bare quark masses there is no chirally restored solution. On the other hand, at least for the strange quark, neglecting the mass would be a rather unrealistic approximation. We therefore consider a partially simplified problem with $m_s \neq 0$ but $m_\ell = 0$. In this case a two-flavor restored solution $\sigma_\ell = 0$ exists, which we take as the expansion point of our GL analysis. In the strange-quark sector we proceed similarly as in the previous section and expand about a T and μ dependent but spatially constant condensate $\sigma_s^{(0)}$, corresponding to a stationary point of Ω at $\sigma_\ell = 0$. We thus write

$$\sigma_s(\mathbf{x}) = \sigma_s^{(0)} + \delta\sigma_s(\mathbf{x}) \quad (3.8)$$

and expand

$$\Omega[\sigma_\ell, \sigma_s] = \Omega[0, \sigma_s^{(0)}] + \frac{1}{V} \int d^3x \omega_{GL}(\Delta_\ell, \Delta_s), \quad (3.9)$$

where

$$\Delta_\ell(\mathbf{x}) = -4G\sigma_\ell(\mathbf{x}) \quad \text{and} \quad \Delta_s(\mathbf{x}) = -4G\delta\sigma_s(\mathbf{x}) \quad (3.10)$$

are proportional to the fluctuations. The GL function takes the form

$$\begin{aligned} \omega_{GL}(\Delta_\ell, \Delta_s) = & \alpha_2\Delta_\ell^2 + \alpha_{4,a}\Delta_\ell^4 + \alpha_{4,b}(\nabla\Delta_\ell)^2 + \dots \\ & + \beta_1\Delta_s + \beta_2\Delta_s^2 + \beta_3\Delta_s^3 + \beta_{4,a}\Delta_s^4 + \beta_{4,b}(\nabla\Delta_s)^2 + \dots \\ & + \gamma_3\Delta_\ell^2\Delta_s + \gamma_4\Delta_\ell^2\Delta_s^2 + \dots, \end{aligned} \quad (3.11)$$

where the α - and β -terms correspond to the contributions from the light and strange condensates, respectively, and the γ terms describe the mixing. Since we expand about the two-flavor restored phase, only even powers of Δ_ℓ are allowed, and the structure of the α -terms is the same as in the GL expansion of the two-flavor model in the chiral limit, Eq. (2.6). For Δ_s , on the other hand, we can also have odd terms, so that the β -terms have a structure as in Eq. (2.7). Again, for a stationary expansion point, the linear coefficient β_1 has to vanish, giving rise to a gap equation for $\sigma_s^{(0)}$ at given T and μ .

For vanishing γ_i the GL analysis of the non-strange sector would be analogous to the two-flavor case in the chiral limit, i.e., the tricritical and Lifshitz points are related to the α coefficients

as in Eq. (2.8). In order to study how these relations get modified by $\gamma_i \neq 0$, we eliminate Δ_s by extremizing the thermodynamic potential with respect to this function. To this end we employ the Euler-Lagrange equations,

$$\frac{\partial \omega_{GL}}{\partial \Delta_s} - \partial_i \frac{\partial \omega_{GL}}{\partial \partial_i \Delta_s} = 0, \quad (3.12)$$

which yields

$$\Delta_s = -\frac{\gamma_3}{2\beta_2} \Delta_\ell^2 + \dots \equiv \Delta_s^{extr}. \quad (3.13)$$

Here we have already used the gap equation $\beta_1 = 0$. The ellipsis indicates higher orders in Δ_ℓ and gradients, which we treat equally, $\mathcal{O}(\nabla^n) = \mathcal{O}(\Delta_\ell^n)$. From the above equation we can then see that Δ_s is of the order $\mathcal{O}(\Delta_\ell^2)$. Inserting Eq. (3.13) into Eq. (3.11) and keeping only terms up to the order $\mathcal{O}(\Delta_\ell^4)$, one then obtains

$$\omega_{GL}(\Delta_\ell, \Delta_s^{extr}) = \alpha_2 \Delta_\ell^2 + \left(\alpha_{4,a} - \frac{\gamma_3^2}{4\beta_2} \right) \Delta_\ell^4 + \alpha_{4,b} (\nabla \Delta_\ell)^2 + \dots \quad (3.14)$$

We thus find that the quartic term in Δ_ℓ gets an additional contribution through the coupling to the strange quarks, while the gradient term does not. So, instead of Eq. (2.8), we now have

$$\alpha_2 = \alpha_{4,a} - \frac{\gamma_3^2}{4\beta_2} = 0 \quad \text{at the TCP}, \quad \alpha_2 = \alpha_{4,b} = 0 \quad \text{at the LP}, \quad (3.15)$$

and therefore, even if $\alpha_{4,a}$ and $\alpha_{4,b}$ were still equal (as they are in the two-flavor model), the TCP and the LP would no longer coincide for $\gamma_3 \neq 0$.

Of course, in order to be able to make definite statements we have to evaluate the relevant GL coefficients. This can be done in the same way as discussed in Sec. 2.2 for the two-flavor model. We find

$$\alpha_2 = (1 + 2\delta) \frac{1}{4G} + (1 + \delta)^2 \frac{1}{2} F_1^{(\ell)} + \frac{K}{16G^2} M_{s,0} F_1^{(s)}, \quad (3.16)$$

$$\alpha_{4,a} = (1 + \delta)^4 \frac{1}{4} F_2^{(\ell)} + \frac{K^2}{256G^4} \left(F_1^{(s)} + 2M_{s,0}^2 F_2^{(s)} \right), \quad (3.17)$$

$$\alpha_{4,b} = (1 + \delta)^2 \frac{1}{4} F_2^{(\ell)}, \quad (3.18)$$

$$\beta_2 = \frac{1}{8G} + \frac{1}{4} F_1^{(s)} + \frac{1}{2} M_{s,0}^2 F_2^{(s)}, \quad (3.19)$$

$$\gamma_3 = \frac{K}{2G^2} \left\{ \frac{1}{8G} + (1 + \delta) \frac{1}{4} F_1^{(\ell)} + \frac{1}{8} F_1^{(s)} + \frac{1}{4} M_{s,0}^2 F_2^{(s)} \right\}, \quad (3.20)$$

where $M_{s,0} = m_s - 4G\sigma_s^{(0)}$, and $F_n^{(\ell)}$ and $F_n^{(s)}$ are the functions defined in Eq. (2.20) with $M_0 = 0$ and $M_0 = M_{s,0}$, respectively. Furthermore we defined $\delta = -\frac{K}{2G}\sigma_s^{(0)}$.

As we have seen earlier, the different flavors are only coupled through the six-point interaction. Indeed, for $K = 0$ and thus $\delta = 0$, the flavor mixing coefficient γ_3 vanishes, and the α coefficients reduce to the corresponding two-flavor expressions in the chiral limit. In particular, we reproduce again the result of Ref. [10] that TCP and LP coincide in this case.

For $K \neq 0$, on the other hand, $\gamma_3 \neq 0$ and therefore TCP and LP would not even coincide if $\alpha_{4,a}$ and $\alpha_{4,b}$ were equal, as seen in Eq. (3.15). Moreover, $\alpha_{4,a}$ and $\alpha_{4,b}$ are *not* equal, and the two

effects are not found to cancel each other. We thus find that TCP and LP split for $K \neq 0$, i.e., as a consequence of the axial anomaly. A quantitative investigation of this effect will be presented in a forthcoming paper [15].

4. Conclusions

In this contribution we discussed how the introduction of nonzero bare quark masses and of strange quarks affects inhomogeneous phases in the NJL model. To this end we performed a Ginzburg-Landau analysis, which allows to investigate these effects without specifying the shape of the spatial modulations and is valid close to the LP. Strictly speaking, away from the chiral limit one cannot define a LP as the location where the second-order boundary between the homogeneous chirally broken and restored phases meets the boundaries of an inhomogeneous phase. We therefore introduce the pseudo-LP as the point on that phase boundary where both the amplitude and the wave number of the spatially modulated part of the chiral order parameter vanish. Our results for a two-flavor model show that the PLP exactly coincides with the CEP [12], supporting the numerical evidence found in [11].

When extending our analysis to include strange quarks we find on the other hand that the LP and the TCP split when the different flavors are coupled through a non-vanishing KMT vertex. Here we have considered massive strange quarks but massless up and down quarks. In a forthcoming publication we will perform a full quantitative investigation of this effect and corroborate it with numerical results away from the chiral limit. There we will also investigate the behavior of the LP for very small values of m_s . This is of special interest, as we expect that in this limit the TCP eventually reaches the T -axis. If the LP followed the same behavior, it would mean that it would be possible to realize an inhomogeneous phase all the way to the regime of vanishing densities, which, if also realized in QCD, could be investigated on the lattice. In the light of the above results we conclude that this scenario might however not be realized, as in the three-flavor case the LP does not coincide with the TCP anymore. A quantitative investigation of this effect is therefore required.

Finally, we recall that, while our analysis focused on the lowest-order GL coefficients which are relevant for pinpointing the location of the CP and the (P)LP, it would also be interesting to work out higher-order coefficients, as they can give informations on the favored shape of the spatially modulated chiral condensate [16, 17], as well as determine the phase boundary to the chirally restored phase [18].

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References

- [1] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, *Event-by-event fluctuations in heavy ion collisions and the QCD critical point*, Phys. Rev. **D60** (1999) 114028 [[hep-ph/9903292](https://arxiv.org/abs/hep-ph/9903292)].

- [2] Z. Fodor and S. D. Katz, *Critical point of QCD at finite T and μ , lattice results for physical quark masses*, JHEP **04** (2004) 050 [[hep-lat/0402006](#)].
- [3] M. A. Stephanov, *QCD phase diagram: An Overview*, PoS **LAT2006** (2006) 024 [[hep-lat/0701002](#)].
- [4] B. Friman, C. Höhne, J. Knoll, S. Leupold, J. Randrup et al., *The CBM physics book: Compressed baryonic matter in laboratory experiments*, Lect.Notes Phys. **814** (2011) 1.
- [5] L. Kumar, *Review of Recent Results from the RHIC Beam Energy Scan*, Mod. Phys. Lett. A **28** (2013) 1330033 [[1311.3426](#)].
- [6] M. Asakawa and K. Yazaki, *Chiral restoration at finite density and temperature*, Nucl. Phys. A **504** (1989) 668.
- [7] O. Scavenius, A. Mocsy, I. N. Mishustin and D. H. Rischke, *Chiral phase transition within effective models with constituent quarks*, Phys. Rev. C **64** (2001) 045202 [[nucl-th/0007030](#)].
- [8] Y. Aoki, G. Endrődi, Z. Fodor, S. D. Katz and K. K. Szabo, *The Order of the quantum chromodynamics transition predicted by the standard model of particle physics*, Nature **443** (2006) 675 [[hep-lat/0611014](#)].
- [9] M. Buballa and S. Carignano, *Inhomogeneous chiral condensates*, Prog. Part. Nucl. Phys. **81** (2015) 39 [[1406.1367](#)].
- [10] D. Nickel, *How many phases meet at the chiral critical point?*, Phys. Rev. Lett. **103** (2009) 072301 [[0902.1778](#)].
- [11] D. Nickel, *Inhomogeneous phases in the Nambu-Jona-Lasinio and quark-meson model*, Phys.Rev. **D80** (2009) 074025 [[0906.5295](#)].
- [12] M. Buballa and S. Carignano, *Inhomogeneous chiral phases away from the chiral limit*, Phys.Lett. **B791** (2019) 361 [[1809.10066](#)].
- [13] J. Moreira, B. Hiller, W. Broniowski, A. Osipov and A. Blin, *Nonuniform phases in a three-flavor Nambu–Jona-Lasinio model*, Phys. Rev. D **89** (2014) 036009 [[1312.4942](#)].
- [14] S. Carignano, D. Nickel and M. Buballa, *Influence of vector interaction and Polyakov loop dynamics on inhomogeneous chiral symmetry breaking phases*, Phys. Rev. **D82** (2010) 054009 [[1007.1397](#)].
- [15] M. Buballa and S. Carignano, in preparation.
- [16] H. Abuki, D. Ishibashi and K. Suzuki, *Crystalline chiral condensates off the tricritical point in a generalized Ginzburg-Landau approach*, Phys.Rev. **D85** (2012) 074002 [[1109.1615](#)].
- [17] S. Carignano, M. Schramm and M. Buballa, *Influence of vector interactions on the favored shape of inhomogeneous chiral condensates*, Phys. Rev. **D98** (2018) 014033 [[1805.06203](#)].
- [18] S. Carignano, M. Mannarelli, F. Anzuini and O. Benhar, *Crystalline phases by an improved gradient expansion technique*, Phys. Rev. **D97** (2018) 036009 [[1711.08607](#)].