PROCEEDINGS OF SCIENCE

$\theta = \pi$ in $\mathrm{SU}(N)/\mathbb{Z}_N$ Theory

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We discuss the spontaneous CP violation in Yang-Mills theory in four dimensions with the gauge group $\mathrm{SU}(N)$. For this purpose, we investigate Yang-Mills theory on T^4 with the gauge group $\mathrm{SU}(N)/\mathbb{Z}_N$ instead. The finite volume correction in this theory strongly suggests that the free energy density has a cusp in the range $(0,\pi)$ of the θ-angle. This then implies the existence of the same cusp at $\theta=\pi$ in $\mathrm{SU}(N)$ theory, indicating the spontaneous CP violation.

The 39th International Conference on High Energy Physics (ICHEP2018)
4-11 July, 2018
Seoul, Korea

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PoS(ICHEP2018)619

1. Introduction

A quantum field theory may have several different phases. The phase structure usually reflects non-perturbative aspects of the theory. The knowledge of non-perturbative effects in the theory may help us understand the phase structure, and vice versa.

Recently, there appeared many researches on the phase structure of quantum field theories. One example is the study on bosonic Yang-Mills theory in four dimensions with the gauge group $\mathrm{SU}(N)$ [1]. In [1], an important role is played by a recently found tool, the 't Hooft anomaly matching for generalized global symmetries [2]. It was argued in [1] that there exists a mixed anomaly for CP symmetry and the center symmetry, the latter of which is a 1-form symmetry [2]. Based on this anomaly, it is argued that CP symmetry is spontaneously broken at $\theta = \pi$.

In [3], we gave an alternative argument to the spontaneous CP violation at $\theta = \pi$ in bosonic Yang-Mills theory claimed in [1]. Our argument is based on the knowledge on confinement in gauge theories. To show the presence of the spontaneous CP violation in $\mathrm{SU}(N)$ theory, we start our argument with $\mathrm{SU}(N)/\mathbb{Z}_N$ Yang-Mills theory. Since the free energy density of $\mathrm{SU}(N)/\mathbb{Z}_N$ theory approaches that of $\mathrm{SU}(N)$ theory in the infinite volume limit, we can say something on $\mathrm{SU}(N)$ theory. We define the $\mathrm{SU}(N)/\mathbb{Z}_N$ theory on T^4 with the volume V, and investigate the *finite volume correction* to the free energy density. As a result, we obtain a strong evidence for the presence of a cusp of the free energy density at $\theta = \pi$ in the large V limit, provided that the mass gap exists. This implies that CP symmetry is spontaneously broken in $\mathrm{SU}(N)$ theory.

This paper is organized as follows. In the next section, we review known results on the spontaneous CP violation in Yang-Mills theories with the gauge group $\mathrm{SU}(N)$. Then, we explain how finite volume corrections can be used to show the presence of the spontaneous CP violation in section 3. In section 4, we investigate the partition function of $\mathrm{SU}(N)/\mathbb{Z}_N$ Yang-Mills theory on T^4, and extract information on the finite volume corrections from the knowledge on confinement. Section 5 is devoted to discussion. More details can be found in [3].

2. Spontaneous CP violation

The spontaneous CP violation at $\theta = \pi$ has been already discussed in the literature. In [4, 5], the bosonic Yang-Mill theory in four dimensions with the gauge group $\mathrm{SU}(N)$ is discussed in the large N limit. The action is given as

$$S = \int d^4x \,\mathrm{Tr}\left[-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \right]. \tag{2.1}$$

In the large N limit, it is know that the free energy $F(\theta)$ scales as N^2. The θ-angle should enter in the combination θ/N, just like the gauge coupling constant enters as g^2N. Then, the only contribution of θ to F which survives in the large N limit takes the form θ^2. However, it is also known that the theory is periodic in θ with 2π periodicity. To implement the periodicity, F should be of the form

$$F \propto \min_{k\in\mathbb{Z}} (\theta + 2\pi k)^2 . \tag{2.2}$$

This function has cusps at $\theta = \pi$ mod 2π. At each cusp, the derivative of F with respect to θ changes discontinuously. Since $\partial_\theta F$ is given as

$$\partial_\theta F \ \sim\ \left\langle \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \right\rangle, \tag{2.3}$$

the discontinuity of $\partial_\theta F$ implies that, at $\theta = \pi$, $\left\langle \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \right\rangle$ has two values corresponding to two vacuum states exchanged by CP symmetry. This implies that CP is spontaneously broken at $\theta = \pi$.

One might suspect that this is an artifact of the large N limit. However, it is not the case since there is another example without the large N limit [6, 7, 8, 9, 1]. Consider $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions. This theory has N vacua. Let us introduce a small gaugino mass m which breaks the supersymmetry. This induces the potential of the form:

$$V \ =\ -2m\mu^3 e^{-8\pi^2/(g(\mu)^2N)+i\theta/N} + \mathrm{c.c.} \tag{2.4}$$

This potential lifts all but one vacuum except when $\theta = \pi$, for which two vacua degenerate. These two vacua have different values of (2.3), implying again the spontaneous CP violation.

In the following, we will investigate whether a similar phenomenon happens also in the bosonic Yang-Mills theory in four dimensions with the gauge group $\mathrm{SU}(N)$ for finite N. It should be noted that the spontaneous CP violation at $\theta = \pi$ can be deduced if one shows that there is a single cusp of $F(\theta)$ in the range $\theta \in (0, 2\pi)$. Indeed, the CP symmetry at $\theta = 0$ and the 2π periodicity:

$$F(-\theta) \ =\ F(\theta), \qquad F(\theta + 2\pi) \ =\ F(\theta), \tag{2.5}$$

imply that the cusp must be at $\theta = \pi$.

3. Finite volume corrections

Our argument relies on a relation between first-order phase transitions and finite volume corrections. Suppose that the free energy density $F(\theta)$ has a parameter θ, and that it has a cusp at $\theta = \theta_*$. In general, it is quite difficult to confirm the presence of the cusp. To find the cusp, it is helpful to put the theory in a finite volume. Since any quantity of a theory in a finite volume is analytic, the cusp in the infinite volume theory is smoothed out. The deviation of $F(\theta)$ from its infinite volume limit is larger if θ is closer to θ_* since the approximation of a non-analytic function by an analytic function becomes worse in the vicinity of a cusp. Therefore, the growth of the finite volume corrections is a sign of the presence of a cusp.

Let us make the above argument more quantitative. Consider again the free energy density $F(\theta, V)$ as a function of the θ-angle and the space-time volume V. Since $\partial_\theta F(\theta, V)$ is given by a one-point function (2.3), the finite volume correction is estimated as

$$\partial_\theta F(\theta, V) - \partial_\theta F(\theta, \infty) \ \sim\ e^{-\Delta V^{1/4}}, \tag{3.1}$$

assuming that there is a mass gap $\Delta > 0$ [10, 11]. Integrating this relation, one obtains

$$g(2\pi, V) - g(0, V) \ \sim\ e^{-\Delta V^{1/4}} \tag{3.2}$$

for the finite volume correction $g(\theta,V) := F(\theta,V) - F(\theta,\infty)$ to the free energy density. Note that we assume the 2π periodicity of $F(\theta,\infty)$.

Suppose that the estimate (3.2) is not valid, for example, that the left-hand side decreases much slower than the exponential as V becomes large. This implies either

1. a cusp exists in the range $(0,2\pi)$, implying the spontaneous CP violation, or

2. the mass gap vanishes somewhere in the range $(0,2\pi)$.

In either case, this is a quite interesting and surprising result!

4. SU(N)/$\mathbb{Z}_N$ theory on T^4

In this section, we study SU(N)/$\mathbb{Z}_N$ Yang-Mills theory on T^4 with the volume V, instead of SU(N) theory. This is because (i) SU(N)/$\mathbb{Z}_N$ theory is more tractable than SU(N) theory, and (ii) the presence of a phase transition in the former theory implies that of the latter theory.

To explain the reason (ii) more clearly, let us review SU(N)/$\mathbb{Z}_N$ theory in detail. The action of this theory is the same as (2.1). The gauge group is the quotient group SU(N)/$\mathbb{Z}_N$ where the subgroup $\mathbb{Z}_N$ of SU(N) consists of elements of SU(N) proportional to the identity element. In SU(N)/$\mathbb{Z}_N$, two SU(N) elements which differ only by an element of $\mathbb{Z}_N$ are identified.

The SU(N)/$\mathbb{Z}_N$ theory on T^4 can be described in terms of the ordinary SU(N) theory. Due to the identification in SU(N)/$\mathbb{Z}_N$, twisted boundary conditions for the gauge field are allowed [12, 13, 14]. The partition function of SU(N)/$\mathbb{Z}_N$ theory is a sum of the partition functions of SU(N) theory with all possible twisted boundary conditions:

$$Z(\theta,V) := \sum_{\nu\in\mathbb{Z}}\sum_{\mathbf{k},\mathbf{m}} Z(\mathbf{k},\mathbf{m},\nu,V)e^{i\theta(\nu-\mathbf{k}\cdot\mathbf{m}/N)}, \tag{4.1}$$

where ν is the instanton number, and $\mathbf{k},\mathbf{m}$ label the twisted boundary conditions which will be explained shortly. Since the only differences from SU(N) theory are the boundary conditions, certain quantities, for example the free energy density,

$$F(\theta,V) := -\frac{1}{V}\log Z(\theta,V), \tag{4.2}$$

of SU(N)/$\mathbb{Z}_N$ theory should approach that of SU(N) theory in the large V limit. Therefore, if $F(\theta,\infty)$ has a cusp at $\theta\in(0,\pi)$, then so is for SU(N) theory.

Now let us investigate the partition function of SU(N)/$\mathbb{Z}_N$ theory on T^4 in more detail. The change of the gauge group from SU(N) to SU(N)/$\mathbb{Z}_N$ appears in

$$\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = \nu + \left(\frac{N-1}{N}\right)\mathbf{k}\cdot\mathbf{m}, \tag{4.3}$$

where $\mathbf{k} = (k_1,k_2,k_3)$ and $\mathbf{m} = (m_1,m_2,m_3)$ are three-vectors whose entries are integers modulo N. These vectors label the twisted boundary conditions mentioned above. Note that the right-hand side takes fractional values. Due to this, the periodicity of $Z(\theta,V)$ in θ is $2N\pi$, not 2π.

Physically, the presence of a non-zero $\mathbf{m}$ means that there are "magnetic fluxes" [15, 16]. On the other hand, a non-zero $\mathbf{k}$ does not correspond to "electric fluxes." The partition function, or the free energy density $F(\mathbf{e},\mathbf{m},\theta,V)$ in the presence of electric and magnetic fluxes are instead defined by a discrete Fourier transformation [15]:

$$e^{-VF(\mathbf{e},\mathbf{m},\theta,V)} = \frac{1}{N^3}\sum_{\nu,\mathbf{k}} e^{-2\pi i\mathbf{k}\cdot\mathbf{e}/N+i\theta(\nu-\mathbf{k}\cdot\mathbf{m}/N)} Z(\mathbf{k},\mathbf{m},\nu,V). \tag{4.4}$$

In terms of these free energy densities, the partition function (4.1) can be written as

$$Z(\theta,V) = N^3\sum_{\mathbf{m}} e^{-VF(\mathbf{0},\mathbf{m},\theta,V)}. \tag{4.5}$$

Now we can investigate the θ-dependence of the partition function $Z(\theta,V)$. The expression (4.5) shows that there are no electric flux in the sum. From now on, we use the common knowledge on confinement which can be applied to the case $\theta = 0$. The magnetic fluxes do not cost the energy for large V. In the large V limit, we find

$$Z(0,V) \sim N^6. \tag{4.6}$$

It should be noted that the large N limit is not taken here. The corrections to the above estimate is exponentially small in V.

Let us shift θ by 2π. It is known that the electric fluxes are induced by the Witten effect [17] if a non-zero magnetic flux exists. Because of the confinement, electric fluxes becomes infinitely heavy in the large V limit. Therefore, we obtain

$$Z(2\pi,V) = N^3\sum_{\mathbf{m}} e^{-VF(\mathbf{m},\mathbf{m},0,V)} \sim N^3. \tag{4.7}$$

Here, only the sector with $\mathbf{m} = \mathbf{0}$ contributes to the sum. Then, the above results imply

$$g(2\pi,V) - g(0,V) \sim \frac{1}{V}\log N^3. \tag{4.8}$$

Recall that the left-hand side should be exponentially small in V if a mass gap is present and the free energy density is analytic in θ. Since we found that the finite volume correction is of order $\mathcal{O}(1/V)$, we conclude that either there is a cusp for $F(\theta,\infty)$ in the range $\theta \in (0,2\pi)$, or the mass gap disappears at some value of θ. Since $F(\theta,\infty)$ coincides with the free energy density of $\mathrm{SU}(N)$ theory on $\mathbb{R}^4$, we conclude that there is a phase transition in $\mathrm{SU}(N)$ theory at $\theta = \pi$ (recall the argument at the end of section 2), as long as the mass gap exists.

5. Discussion

We have argued that CP symmetry in $\mathrm{SU}(N)$ Yang-Mills theory in four dimensions is spontaneously broken at $\theta = \pi$. This is deduced from the dynamics of the gauge theory: the behavior of electric and magnetic fluxes in the confinement phase. It would be interesting to extend this argument to more general theories, for example, Yang-Mills theory coupled to matter fields.

In [3], we also discussed the lattice simulation of $\mathrm{SU}(N)/\mathbb{Z}_N$ theory on T^4, and proposed how our argument can be verified. We hope to discuss this issue in a future publication.

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