PROCEEDINGS OF SCIENCE

Neutrino Theory

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The possible manifestations of New Physics beyond that predicted by the Standard Theory, which can be related to the existence of non-zero neutrino masses and neutrino mixing, are briefly discussed. The phenomenology of 3-neutrino mixing, the present status of our knowledge about the 3-neutrino mixing parameters, including the absolute neutrino mass scale, and of the Dirac and Majorana CP violation in the lepton sector, are summarised. The current theoretical ideas about the origins i) of neutrino masses and of the enormous disparity between their values and the values of the charged lepton and quark masses, and ii) of the pattern of neutrino mixing revealed by the neutrino oscillation experiments, are reviewed, with the non-Abelian discrete symmetry approach considered in somewhat greater detail. The possibilities to test these ideas are also briefly discussed.

The 39th International Conference on High Energy Physics (ICHEP2018)
4-11 July, 2018
Seoul, Korea

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PoS(ICHEP2018)699

1. Introduction (Preliminary Remarks)

Understanding the origin of the patterns of neutrino mixing and of neutrino mass squared differences that emerged from the neutrino oscillation data is one of the most challenging problems in neutrino physics. It is part of the more general fundamental problem in particle physics of understanding the origins of flavour in the quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing.

Of critical importance for making progress towards the solution of lepton flavour problem and the understanding of the mechanism giving rise to neutrino masses and mixing are (see, e.g., [1]):
– determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos;
– determination of the status of CP symmetry in the lepton sector;
– determination of the type of spectrum neutrino masses possess, or the "neutrino mass ordering";
– determination of absolute neutrino mass scale or the value of the lightest neutrino mass;
– high precision measurement of neutrino mixing parameters (see further).

The discovery of neutrino oscillations [2, 3, 4] – transitions in flight between the different flavour neutrinos ν_e, ν_μ, ν_τ (antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$) caused by non-zero neutrino masses and neutrino mixing – opened up a new field of research in elementary particle physics. In the 20 years after the first compelling evidence for oscillations of atmospheric muon neutrinos and antineutrinos ν_μ and $\bar{\nu}_\mu$ was provided by the Super Kamiokande experiment in 1998 [5], oscillation of the solar ν_e neutrinos, accelerator ν_μ and $\bar{\nu}_\mu$ and reactor $\bar{\nu}_e$ neutrinos were also observed and the neutrino oscillation phenomenon was extensively studied (see, e.g., [1]). In this period a remarkable progress has been made in the measurement of the parameters which drive the oscillations.

The neutrino oscillation data imply, as is well known, the presence of neutrino mixing in the leptonic part of the weak charged current interaction Lagrangian:

$$\mathscr{L}_{\rm CC} = -\,\frac{g}{\sqrt{2}}\sum_{l=e,\mu,\tau}\overline{l_L}(x)\,\gamma_\alpha\,\nu_{lL}(x)\,W^{\alpha\dagger}(x) + h.c.\,,\;\; \nu_{l{\rm L}}(x) = \sum_{j=1}^{n} U_{lj}\,\nu_{j{\rm L}}(x)\,. \tag{1.1}$$

Here $\nu_{lL}(x)$ are the left-handed (LH) fields of flavour neutrinos, $\nu_{j{\rm L}}(x)$ is the LH field of the neutrino ν_j having a mass m_j, and U is a unitary matrix - the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) neutrino mixing matrix [2, 3, 4], $U \equiv U_{\rm PMNS}$. On the basis of the existing data it is impossible to determine the nature of massive neutrinos ν_j, which can be Dirac or Majorana particles. It follows from the current data that at least 3 of the neutrinos ν_j, say ν_1, ν_2, ν_3, must be light, $m_{1,2,3} \lesssim 0.5$ eV, and must have different masses, $m_1 \neq m_2 \neq m_3$. This implies that light neutrino masses are much smaller than the masses of charged leptons and quarks. If we take as an indicative upper limit $m_j \lesssim 0.5$ eV, we have $m_j/m_{l,q} \lesssim 10^{-6}$, where m_l and m_q are the charged lepton and quark masses, $l = e, \mu, \tau$, $q = d, s, b, u, c, t$.

The remarkable disparity between the light neutrino masses and the masses of the charged leptons and quarks suggests (but is not a proof) that the values of neutrino masses are related to the existence of a new fundamental mass scale in particle physics, and thus to New Physics beyond that predicted by the Standard Theory. The New Physics can manifest itself:
– In the existence of more than 3 massive neutrinos: $n > 3$ ($n = 4$, or $n = 5$, or $n = 6$,...).
– In the pattern of neutrino mixing and the values of CP violation phases in the PMNS matrix.

– In the Majorana nature of massive neutrinos, and total lepton charge non-conservation, $L \neq const$.
– In the existence of new particles, e.g., at the TeV scale: heavy Majorana neutrinos N_j, doubly and singly charged scalars, H^{--}, H^{-}, etc.
– In the existence of new (flavour changing and/or flavour conserving but flavour non-symmetric) nonstandard neutrino interactions (NSI) [6, 7] (for recent discussions see, e.g., [8, 9, 10]).
– In the existence of charged lepton flavour violating (ChLFV) processes, $\mu \to e + \gamma$ and $\mu \to 3e$ decays, μ - e conversion on nuclei, etc., having rates close to the existing stringent upper limits.
– In the existence of unforeseen new phenomena.

Apart obviously from the last item, the indicated possible manifestations of New Physics were extensively discussed at this Conference.

There can be more than 3 massive neutrinos, $n > 3$, for example, if there exist sterile right-handed (RH) neutrinos $\nu_{\tilde{l}R}$ and left-handed (LH) antineutrinos $\tilde{\nu}_{\tilde{l}L}$ (described by $SU(2) \times U(1)_{\rm Y_W}$ singlet RH neutrino fields $\nu_{\tilde{l}R}(x)$), which posses a Majorana mass term and couple via a Dirac mass term to the active flavour LH neutrinos ν_{lL} and RH antineutrinos $\tilde{\nu}_{lR}$ (LH flavour neutrino fields $\nu_{lL}(x)$). In what concerns the masses of the additional massive neutrino states ν_4, ν_5,..., m_4, m_5,..., there are a few possibilities.
i) They can be at the eV scale, $m_4, m_5, ... \sim 1$ eV. In this case active-sterile neutrino oscillations, $\nu_{e(\mu)} \to \nu_s (\equiv \tilde{\nu}_{\tilde{l}L})$ are possible. At present we have hints that such oscillations might take place from LSND and MiniBooNE experiments, re-analyses of short baseline (SBL) reactor neutrino oscillation data ("reactor neutrino anomaly") and data of radioactive source calibration of the solar neutrino SAGE and GALLEX experiments "Gallium anomaly") (see, e.g., [11]). However, as recent analyses have shown [11, 12], the global fits of the relevant data - positive evidence and negative results - have an extremely low quality, indicating the existence of inconsistencies between the different data sets. The possibility of active-sterile neutrino oscillations is being and will be tested in a large number of experiments with reactor and accelerator neutrinos (DANSS, NEOS, PROSPECT, STEREO, Neutrino-4, SOLID, Short Baseline Neutrino (SBN) program at Fermilab, JSNS^2, etc., see, e.g., [11]), some of which have already provided data and are taking further data (DANSS, NEOS, PROSPECT, STEREO, Neutrino-4), or are under preparation (e.g., SOLID, the SBN program at Fermilab, JSNS^2). Most of these experiments have been discussed at this Conference. It is foreseen that within the next 3-4 years we will have a definite answer about whether active-sterile neutrino oscillations take place in Nature or not.
ii) The additional states can have masses $M_{4,5,...} \sim (10^2 - 10^3)$ GeV, which corresponds to TeV scale type I seesaw models of neutrino mass generation; or $M_{4,5,...} \sim (10^9 - 10^{13})$ GeV which is related to the "classical" GUT scale type I seesaw models (see further).
Other scenarios for the masses of ν_4, ν_5,..., are also possible (e.g., the three masses of ν_4, ν_5, ν_6 being at very different scales).

All well understood and compelling neutrino oscillation data can be described within the reference scheme of 3-neutrino mixing in vacuum, which we will consider next.

2. The Three Neutrino Mixing

The PMNS matrix, as is well known, can be parametrised in the case of 3-neutrino mixing by 3 angles and, depending on whether the massive neutrinos ν_j are Dirac or Majorana particles, by

Parameter	Best fit value	2σ range	3σ range
$\Delta m^2_{31}/10^{-3}$ eV^2 (NO)	2.49	2.43 – 2.56	2.39 – 2.59
$(-\Delta m^2_{32})/10^{-3}$ eV^2 (IO)	2.48	2.41 – 2.54	2.38 – 2.58
$\Delta m^2_{21}/10^{-5}$ eV^2 (NO,IO)	7.34	7.05 – 7.69	6.92 – 7.91
$\sin^2\theta_{12}/10^{-1}$ (NO)	3.04	2.78 – 3.32	2.65 – 3.46
$\sin^2\theta_{12}/10^{-1}$ (IO)	3.03	2.77 – 3.31	2.64 – 3.45
$\sin^2\theta_{13}/10^{-2}$ (NO)	2.14	1.98 – 2.31	1.90 – 2.39
$\sin^2\theta_{13}/10^{-2}$ (IO)	2.18	2.02 – 2.35	1.95 – 2.43
$\sin^2\theta_{23}/10^{-1}$ (NO)	5.51	4.48 – 5.88	4.30 – 6.02
$\sin^2\theta_{23}/10^{-1}$ (IO)	5.57	4.86 – 5.89	4.44 – 6.03
δ/π (NO)	1.32	0.98 – 1.79	0.83 – 1.99
δ/π (IO)	1.52	1.22 – 1.79	1.07 – 1.92

Table 1: The best fit values, 2σ and 3σ ranges of the neutrino oscillation parameters obtained in the global analysis of the neutrino oscillation data performed in [18]. See text for further details. (The Table is taken from ref. [21]).

one Dirac, or one Dirac and two Majorana, CP violation (CPV) phases [13]:

$$U = VP\,, \quad P = \mathrm{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})\,, \tag{2.1}$$

where $\alpha_{21,31}$ are the two Majorana CPV phases. In the "standard" parametrisation [1] the matrix V has the form:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{2.2}$$

Here $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, the angles $\theta_{ij} = [0, \pi/2]$, and $\delta = [0, 2\pi)$ is the Dirac CPV phase.

In the case of 3-neutrino mixing, oscillations involving all flavour neutrinos ν_l (antineutrinos $\bar{\nu}_l$), $\nu_l \leftrightarrow \nu_{l'}$ ($\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$), $l, l' = e, \mu, \tau$, are possible. The 3-neutrino oscillation probabilities $P(\nu_l \to \nu_{l'})$ and $P(\bar{\nu}_l \to \bar{\nu}_{l'})$ are functions of the neutrino energy, E, the source-detector distance L, of the elements of U and, for relativistic neutrinos used in all neutrino experiments performed so far, of the two independent neutrino mass squared differences $\Delta m^2_{21} \neq 0$ and $\Delta m^2_{31} \neq 0$, ($\Delta m^2_{jk} \equiv m^2_j - m^2_k$) present in the case of 3-neutrino mixing (see, e.g., ref. [14]). In the widely used convention of numbering the neutrinos with definite mass ν_j we are going to employ, θ_{12}, $\Delta m^2_{\odot} = \Delta m^2_{21} > 0$, and θ_{23}, $\Delta m^2_{\rm atm} = \Delta m^2_{31(23)}$, represent the parameters which drive the solar (ν_e) and the dominant atmospheric ν_μ and $\bar{\nu}_\mu$ oscillations, respectively, and θ_{13} is associated with the oscillations of reactor $\bar{\nu}_e$ observed in the Daya Bay, RENO and Double Chooz experiments [15, 16, 17].

The existing data, accumulated over many years of studies of neutrino oscillations, allow us to determine Δm^2_{21}, θ_{12}, and $|\Delta m^2_{31(32)}|$, θ_{23} and θ_{13}, with an impressively high precision [18, 19].

Since 2013 there are also persistent hints that the Dirac CPV phase δ has a value close to $3\pi/2$ (see [20]). We note that the currently available neutrino oscillation data do not allow to fix the sign of $\Delta m^2_{31(32)}$. In the 3-neutrino mixing scheme under discussion, the two possible signs of $\Delta m^2_{31(32)}$ correspond to two types of neutrino mass spectrum: $\Delta m^2_{31(32)} > 0$ – to spectrum with normal ordering (NO), and $\Delta m^2_{31(32)} < 0$ – to spectrum with inverted ordering (IO) (see further).

The best fit values (b.f.v.) and the 2σ and 3σ allowed ranges of Δm^2_{21}, s^2_{12}, $|\Delta m^2_{31(32)}|$, s^2_{23}, s^2_{13} and δ for the NO and IO spectra, found in the latest analysis of global neutrino oscillation data performed in [18] are given in Table 1. The results quoted in Table 1 imply, in particular, that $\Delta m^2_{21}/|\Delta m^2_{31(32)}| \cong 0.03$. The best fit value of θ_{23} is somewhat larger than $\pi/4$, but the value $\pi/4$ lies within $(1.0-1.5)\sigma$ from the best fit value. The value of $\theta_{12} = \pi/4$, i.e., maximal solar neutrino mixing, is ruled out by the data. One has $\theta_{12} < \pi/4$ and at 99.73% C.L., $\cos 2\theta_{12} \geq 0.30$.

The quoted results imply also that the value of θ_{23} can deviate by approximately ± 0.1 from $\pi/4$, $\theta_{12} \cong \pi/5.4$ and that $\theta_{13} \cong \pi/20$. Thus, the pattern of neutrino mixing differs drastically from the pattern of quark mixing.

In what concerns the Dirac CPV phase δ, the authors of the analysis performed in [18] find that the best fit value of δ is close to $3\pi/2$. More specifically, for NO (IO) spectrum it reads $\delta = 1.38\pi$ (1.52π). According to [18] the CP conserving value $\delta = 0$ [or 2π] is disfavored at 3.0 $(3.6)\sigma$, while the second CP conserving value $\delta = \pi$ is disfavored at 1.8 $(3.6)\sigma$. The CP violating value $\delta = \pi/2$ is strongly disfavored at 4.4 $(5.2)\sigma$ by the data. Finally, at 3σ, δ/π is found in [18] to lie in the interval $0.83-1.99$ (1.07 1.93). Thus, we have an indication from the data, although rather weak, for CP violation in lepton sector due to the Dirac phase δ (i.e., for leptonic Dirac CP violation). Similar results were obtained in the independent global analysis performed in ref. [19].

As we have indicted earlier, the currently available data do not allow to fix the sign of $\Delta m^2_{31(32)}$ and the two possible signs of $\Delta m^2_{31(32)}$ correspond to two types of neutrino mass spectrum - with normal ordering and with inverted ordering. More specifically, in the convention of numbering of the 3 neutrinos ν_j with definite mass employed by us the two spectra read:
i) spectrum with normal ordering (NO): $m_1 < m_2 < m_3$, $\Delta m^2_{j1(32)} > 0$, $m_j = (m_1^2 + \Delta m^2_{j1})^{\frac{1}{2}}, j = 2, 3$;
ii) spectrum with inverted ordering (IO): $m_3 < m_1 < m_2$, $\Delta m^2_{32(31)} < 0$, $\Delta m^2_{21} > 0$, $m_2 = (m_3^2 + \Delta m^2_{23})^{\frac{1}{2}}$, $m_1 = (m_3^2 + \Delta m^2_{23} - \Delta m^2_{21})^{\frac{1}{2}}$.
Depending on the value of the lightest neutrino mass, $\min(m_j)$, the neutrino mass spectrum can be:
a) Normal Hierarchical (NH): $m_1 \ll m_2 < m_3$, $m_{2(3)} \cong (\Delta m^2_{21(31)})^{\frac{1}{2}} \cong 8.6 \times 10^{-3}$ (4.99×10^{-2}) eV;
or *b) Inverted Hierarchical (IH)*: $m_3 \ll m_1 < m_2$, $m_{1,2} \cong |\Delta m^2_{32}|^{\frac{1}{2}} \cong 0.0498$ eV;
or *c) Quasi-Degenerate (QD)*: $m_1 \cong m_2 \cong m_3 \cong m_0$, $m_j^2 \gg |\Delta m^2_{31(32)}|$, $m_0 \gtrsim 0.10$ eV.

Determining the type of neutrino mass spectrum is one of the main goals of the future experiments in the field of neutrino physics [1] (see, e.g., refs. [1, 22]). According to [18], the absolute χ^2 minimum obtained in the analysis of the global neutrino oscillation data takes place for $\Delta m^2_{31(32)} > 0$, i.e., for NO neutrino mass spectrum, the local minimum in the case of IO spectrum $(\Delta m^2_{31(32)} < 0)$ being approximately by 3.1σ higher. Thus, according to [18], the existing data favor

[1] For a review of the experiments which can provide data on the type of neutrino mass spectrum see, e.g., ref. [22]; for some specific proposals see, e.g., ref. [23].

the NO spectrum over the IO spectrum at approximately 3.1σ.

All types of 3-neutrino mass spectrum considered above are compatible with the existing upper limits on the absolute scale of neutrino masses. Information about the absolute neutrino mass scale (or about $\min(m_j)$) can be obtained, e.g., by measuring the spectrum of electrons near the end point in ^{3}H β-decay experiments [24, 25, 26] and from cosmological and astrophysical data. The most stringent upper bound on the $\bar{\nu}_e$ mass was obtained in the Troitzk [27] experiment:

$$m_{\bar{\nu}_e} < 2.05 \text{ eV} \quad \text{at } 95\% \text{ C.L.} \tag{2.3}$$

Similar limit was obtained in the Mainz experiment [25]: $m_{\bar{\nu}_e} < 2.3$ eV (95% C.L.). The limits are in the region of QD spectrum where $m_{\bar{\nu}_e} \cong m_{1,2,3}$. The KATRIN experiment [28] (commissioned on June 11, 2018) is planned to have sensitivity to $m_{\bar{\nu}_e} \sim 0.20$ eV and probe the QD spectrum region.

Constraints on the sum of the neutrino masses can be obtained from cosmological and astrophysical data (see, e.g., ref. [29]). Depending on the model complexity and the input data used one typically obtains [29]: $\sum_j m_j \lesssim (0.3-1.3)$ eV, 95% C.L. Assuming the existence of three light massive neutrinos, the validity of the Λ CDM (Cold Dark Matter) model and using their 2018 data, the Planck Collaboration reported an updated upper limit on the sum of the neutrino masses [30], which depending on the data set used reads:

$$\sum_j m_j \, < \, 0.120 - 0.160 \text{ eV}, \quad 95\% \text{ C.L.} \tag{2.4}$$

One should note that the Planck collaboration analysis is based on the ΛCDM cosmological model. The quoted bounds may not apply in nonstandard cosmological scenarios (see, e.g., [31]).

Apart from the hint that the Dirac phase $\delta \sim 3\pi/2$, no other experimental information on the Dirac and Majorana CPV phases in the neutrino mixing matrix is available at present. Thus, the status of CP symmetry in the lepton sector is essentially undetermined. With $\theta_{13} \cong 0.15 \neq 0$, the Dirac phase δ can generate CP violating effects in neutrino oscillations [32, 13], i.e, a difference between the probabilities of the $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$ oscillations, $l \neq l' = e, \mu, \tau$. The magnitude of CP violation in $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$ oscillations, $l \neq l' = e, \mu, \tau$, is determined by [33] the rephasing invariant J_{CP}, associated with the Dirac CPV phase in U:

$$J_{\rm CP} = {\rm Im}\left(U_{\mu 3}\, U_{e3}^*\, U_{e2}\, U_{\mu 2}^*\right). \tag{2.5}$$

It is analogous to the rephasing invariant of the the CKM quark mixing matrix [34]. In the standard parametrisation of the neutrino mixing matrix (2.2), $J_{\rm CP}$ has the form:

$$J_{CP} \equiv {\rm Im}\,(U_{\mu 3}\, U_{e3}^*\, U_{e2}\, U_{\mu 2}^*) = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\,\sin 2\theta_{23}\,\sin 2\theta_{13}\,\sin\delta\,. \tag{2.6}$$

Thus, given the fact that $\sin 2\theta_{12}$, $\sin 2\theta_{23}$ and $\sin 2\theta_{13}$ have been determined experimentally with a relatively high precision, the size of CP violation effects in neutrino oscillations depends essentially only on the magnitude of the currently not well determined value of the Dirac phase δ. The current data implies $0.029(0.030)|\sin\delta| \lesssim |J_{CP}| \lesssim 0.035|\sin\delta|$, where we have used the 3σ ranges of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ given in Table 1. For the current best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$ and δ we find in the case of NO (IO) spectrum: $J_{CP} \cong 0.033\sin\delta \cong -0.028$ ($J_{CP} \cong 0.033\sin\delta \cong -0.033$). Thus, if the indication that δ has a value close to $3\pi/2$ is confirmed by future more precise data, i) the J_{CP} factor in the lepton sector will be approximately by 3 orders of magnitude larger in absolute value than the corresponding J_{CP} factor in the quark sector, and ii) the CP violation effects in neutrino oscillations would be relatively large and observable.

If the neutrinos with definite masses ν_i, $i = 1,2,3$, are Majorana particles, the 3-neutrino mixing matrix contains two additional Majorana CPV phases [13]. However, the flavour neutrino oscillation probabilities $P(\nu_l \to \nu_{l'})$ and $P(\bar{\nu}_l \to \bar{\nu}_{l'})$, $l,l' = e,\mu,\tau$, do not depend on the Majorana phases [13, 35]. The Majorana phases play important role, e.g., in $|\Delta L| = 2$ processes like neutrinoless double beta ($(\beta\beta)_{0\nu}$-) decay $(A,Z) \to (A,Z+2) + e^- + e^-$, L being the total lepton charge, in which the Majorana nature of massive neutrinos manifests itself (see, e.g., refs. [14, 36, 37]).

Our interest in the CPV phases of the neutrino mixing matrix is stimulated also by the intriguing possibility that the Dirac phase and/or the Majorana phases in $U_{\rm PMNS}$ can provide the CP violation necessary for the generation of the observed baryon asymmetry of the Universe (BAU) [38, 39, 41] (see also [42]; for specific models in which this possibility is realised see, e.g., [43, 44]).

Determining the status of CP symmetry in the lepton sector is one of the principal goals of the program of current and future research in neutrino physics. It is part of a very ambitious program of research in neutrino physics, which extends beyond 2030, and which include also [1]:
i) determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos (which is one of the most challenging and pressing problems in present day elementary particle physics);
ii) determination of the spectrum neutrino masses possess, or neutrino mass ordering;
iii) determination of the absolute neutrino mass scale, or $\min(m_j)$.
A successful realisation of this program [2] is of fundamental importance for making progress in understanding the origin of neutrino masses and mixing and its possible relation to new Beyond the Standard Model (BSM) physics.

3. The Simplest Model of Non-Zero Neutrino Masses and Neutrino Mixing

The Standard Model (SM) supplemented by 3 singlet right-handed (RH) neutrino fields $\nu_{lR}(x)$, $l = e,\mu,\tau$, and the assumption of total lepton charge conservation, $L = const.$ [45], is arguably the simplest model of generation of non-zero neutrino masses and non-trivial neutrino mixing. By assumption the theory possess a $U(1)_L$ global symmetry associated with the $L-$conservation. The $SU(2)_L \times U(1)_{Y_W}$ and $U(1)_L$ invariant (and renormalisable) neutrino Yukawa-type term in the Lagrangian [3],

$$\mathscr{L}^{\nu}_{\rm Y}(x) = Y^{\nu}_{l'l}\,\overline{\nu_{l'R}}(x)\,\Phi^T(x)\,(i\tau_2)\,\psi_{lL}(x) + {\rm h.c.}\,, \tag{3.1}$$

where $\psi_{lL}(x)$ and $\Phi(x)$ are the SM lepton and Higgs doublet fields, generates after the spontaneous breaking of the SM gauge symmetry a neutrino Dirac mass term:

$$\mathscr{L}^{\nu}_{\rm D}(x) = -\,M_{Dl'l}\,\overline{\nu_{l'R}}(x)\,\nu_{lL}(x)\,,\;\; M_D = \frac{v}{\sqrt{2}}\,Y^{\nu}\,,\;\; v = 246~{\rm GeV}\,, \tag{3.2}$$

v being the vacuum expectation value of $\Phi(x)$. The neutrino mass matrix M_D is, in general, complex. The diagonalisation of the neutrino mass terms is done in the same way as the diagonalisation

[2]See, e.g., [1] for a rather detailed list of current and planned experiments that are foreseen to contribute to the comprehensive long-term program of research in neutrino physics.

[3]The neutrino Yukawa-type term is written in the basis in which the charged lepton mass term is diagonal.

of the charged lepton and quark mass terms in the SM - by a “bi-unitary” transformation. This leads to 3 neutrinos ν_j with definite and different masses, m_j, $j = 1,2,3$, and to 3-neutrino mixing in the charged current weak interaction Lagrangian (1.1). The massive neutrinos ν_j are predicted to be Dirac particles.

This simple model, however, i) does not provide an explanation of the enormous disparity between the values of the neutrino and charged lepton (m_l) and quark (m_q) masses, $m(\nu_j) <<< m_l, m_q$, ii) does not contain a dark matter candidate, and iii) does not include a mechanism of generation of the observed matter-antimatter (or baryon) asymmetry of the Universe.

The charged lepton flavour violating (ChLFV) processes $\mu^+ \to e^+ + \gamma$ decay, $\mu^- \to e^- + e^+ + e^-$ decay, $\tau^- \to e^- + \gamma$ decay, etc. are allowed in the model since the individual lepton charges L_e, L_μ and L_τ are not conserved. However, they are predicted to proceed with unobservable rates [45]. The $\mu \to e + \gamma$ decay branching ratio, for example, is given by [45]:

$$BR(\mu \to e + \gamma) = \frac{3\alpha}{32\pi} \left| \sum_{j=1,2,3} U_{ej} U^*_{\mu j} \frac{m_j^2}{M_W^2} \right|^2 = \frac{3\alpha}{32\pi} \left| \sum_{k=2,3} U_{ek} U^*_{\mu k} \frac{\Delta m^2_{k1}}{M_W^2} \right|^2 \tag{3.3}$$

$$\cong 3.3 \times 10^{-55}\,, \quad M_W \cong 80 \text{ GeV is the } \mathrm{W}^{\pm} - \text{mass}\,,$$

where we have used the unitarity of the PMNS matrix. The numerical value is obtained for the best fit values of the neutrino oscillation parameters for NO spectrum given in Table 1 [4]. The current experimental upper limit on $BR(\mu \to e + \gamma)$ reads [46]: $BR(\mu \to e + \gamma) < 4.2 \times 10^{-13}$ (90% C.L.).

Thus, the only observable “New Physics” predicted by the discussed model are the nonzero neutrino masses $m_j \neq 0$, $m_j \neq m_k$, $j \neq k = 1,2,3$, and the 3-flavour neutrino and antineutrino oscillations, $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$, $l, l' = e, \mu, \tau$.

An inherent problem of the model considered is the assumption of conservation of the total lepton charge L associated with the global $U(1)_L$ symmetry. To quote E. Witten [47]: “In modern understanding of particle physics global symmetries are approximate.” Similar ideas were expressed by S. Weinberg in [48]. Thus, the global $U(1)_L$ symmetry leading to $L = const.$ is expected to be broken, e.g., by quantum gravity effects. This implies L nonconservation, which in turn leads to massive Majorana neutrinos.

4. Qualitative Understanding of $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$

A natural explanation of the smallness of neutrino masses is provided by i) the seesaw mechanisms of neutrino mass generation [49, 50, 51], and by the ii) mechanisms of radiative generation of neutrino masses and mixing [52, 53] (see also, e.g., [54], the more recent publications [55, 56] and the review article [57]).

4.1 Seesaw Mechanisms of Neutrino Mass Generation

An attractive feature of the seesaw mechanisms is that in addition of providing a natural explanation of the smallness of neutrino masses, they relate via the leptogenesis scenario [58] the generation of neutrino masses to the generation of the baryon asymmetry of the Universe.

There are three types of seesaw mechanisms. Each of the three types is associated with the existence of new degrees of freedom (particles) beyond those present in the SM. In type I seesaw

[4] Similar result is obtained in the case of IO spectrum.

mechanism [49] these are singlet heavy RH neutrino fields ν_{lR}. In type II seesaw this is an $SU(2)_L$ triplet $\mathbf{H}(x)$ of doubly charged, singly charged and neutral scalar fields H^{--}, H^{-}, H^{0}, carrying two units of the weak hypercharge, $Y_W(H(x)) = 2$ [50]. In the type III seesaw mechanism these are $SU(2)_L$ triplets of fermion fields $\mathbf{T}_{jR}(x)$, $j \geq 2$, carrying zero weak hyper-charge [51].

The scale of New Physics is determined by the masses of the new particles. In all three versions of the seesaw mechanism considered the light massive neutrinos ν_j are predicted to be Majorana particles. All three types of seesaw mechanisms have TeV scale versions, predicting rich and observable, in principle, low-energy phenomenology (neutrinoless double beta ($(\beta\beta)_{0\nu}$-)decay, charged lepton flavour violating (ChLFV) processes, etc.) and New Physics at the LHC.

Type I Seesaw Model. In the case of type I seesaw, which is easily incorporated in GUT theories (notably in $SO(10)$ GUTs), it is assumed that the three RH neutrino fields $\nu_{lR}(x)$ have a “high scale” Majorana mass term:

$$\mathscr{L}^{\nu}_{M}(x) = +\,\frac{1}{2}\,\nu^{\mathrm{T}}_{l'R}(x)\,C^{-1}\,(M^{RR})^{*}_{l'l}\,\nu_{lR}(x) + h.c. = -\,\frac{1}{2}\sum_{j}\bar{N}_{j}M_{j}N_{j}\,, \tag{4.1}$$

where C is the charge conjugation matrix ($C^{-1}\gamma_{\mu}C = -\gamma^{T}_{\mu}$, see, e.g., [14]), and $(M^{RR})^{T} = M^{RR}$ is the complex Majorana mass matrix of $\nu_{lR}(x)$, while $N_j = C(\overline{N_j})^T$, $j = 1,2,3$, are heavy Majorana neutrinos having masses M_j. The heavy Majorana neutrinos masses can be at the $\sim$TeV scale or at the much higher scale of $(10^{9} - 10^{13})$ GeV (e.g., in $SO(10)$ GUTs). The mass term (4.1) is $SU(2)_L \times U(1)_{Y_W}$ invariant and can be introduced in the modified SM with RH neutrinos without spoiling any of the attractive features of the SM (renormalisability, unitarity). With the presence of the singlet RH neutrino fields $\nu_{lR}(x)$, the SM gauge symmetries allow also the introduction of the neutrino Yukawa-type term (3.1) in the Lagrangian, which generates a neutrino Dirac mass term with mass matrix M_D after the spontaneous breaking of the SM gauge symmetries:

$$\begin{aligned}
\mathscr{L}^{\nu}_{\mathrm{Y}}(x) &= \bar{Y}^{\nu}_{l'l}\,\overline{\nu_{l'R}}(x)\,\Phi^{T}(x)\,(i\tau_2)\,\psi_{lL}(x) + \text{h.c.}\,, &(4.2)\\
&= Y^{\nu}_{kl}\,\overline{N_{kR}}(x)\,\Phi^{T}(x)\,(i\tau_2)\,\psi_{lL}(x) + \text{h.c.}\,, &(4.3)\\
M_D &= \frac{v}{\sqrt{2}}Y^{\nu}\,,\ \ v = 246\ \text{GeV}\,.
\end{aligned}$$

The combination of the neutrino Yukawa-type term (4.2) and the Majorana mass term for the RH neutrinos, eq. (4.1), does not conserve the total lepton charge L. For sufficiently large masses M_k of the heavy Majorana neutrinos N_k, the interplay between the neutrino Dirac mass term and the the RH neutrino Majorana mass term leads to a Majorana mass term for LH flavour neutrino fields $\nu_{lL}(x)$ (flavour neutrinos ν_l):

$$\mathscr{L}^{\nu}_{eff}(x) = \frac{1}{2}\,\nu^{T}_{lL}(x)\,C^{-1}\,(M_{\nu})_{ll'}\,\nu_{l'L}(x) + h.c.\,. \tag{4.4}$$

The mass matrix M_ν, which is diagonalised by the PMNS neutrino mixing matrix, is given by:

$$(M_{\nu})_{ll'} \cong (M^{T}_{D}M^{-1}_{R}M_{D})_{ll'} = v^{2}\,(Y^{\nu})^{T}_{lk}M^{-1}_{k}Y^{\nu}_{kl'} = (U^{*}_{\text{PMNS}}\,M^{\text{diag}}_{\nu}\,U^{\dagger}_{\text{PMNS}})_{ll'}\,, \tag{4.5}$$

where $M^{\text{diag}}_{\nu} = \text{diag}(m_1, m_2, m_3)$, $m_j \geq 0$ being the masses of three light Majorana neutrinos ν_j, $k = 1,2,3$. The diagram leading to the ν_l mass term (4.5) is shown in Fig. 1. Neglecting the ma-

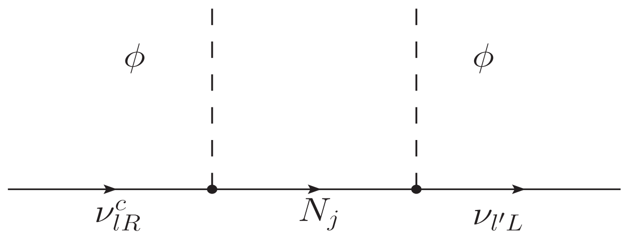

Figure 1: The diagram generating the Majorana mass term (4.4) for flavour neutrinos ν_l, $l = e, \mu, \tau$, in the case of type I seesaw mechanism. See text for further details.

trix structure for simplicity and choosing $vY^\nu = M_D \sim 1$ GeV and $M_k = 10^{10}$ GeV (suggested by GUTs), we find $M_\nu \sim 0.1$ eV, which is the correct magnitude of the light neutrino masses. Thus, the smallness of the neutrino masses is a consequence of the smallness of the ratio v/M_k, i.e., of the electroweak symmetry breaking scale v and the scale of masses of the heavy Majorana neutrinos M_k, which in GUTs is by few to several orders below the unification scale of electroweak and strong interactions of $\sim 10^{16}$ GeV.

Type II Seesaw Model. The diagram generating an effective Majorana mass term (4.4) for the flavour neutrinos ν_l, $l = e, \mu, \tau$, in type II seesaw scenario is shown in Fig. 2. It is a result of the interplay of the $SU(2)_L \times U(1)_{Y_W}$ invariant couplings i) of two lepton doublets to a Higgs triplet field $\mathbf{H}(x)$, and ii) of the triple scalar coupling of $\mathbf{H}(x)$ with two Higgs doublet fields:

$$\begin{aligned} \mathscr{L}_{\rm II}(x) \;=&\; h_{l'l}\,\overline{\psi_{l'L}}(x)\frac{\tau}{2}\,i\tau_2\,\psi^C_{lR}(x)\,\mathbf{H}(x) - \mu_H\,\Phi^\dagger(x)\frac{\tau}{2}\,i\tau_2(\Phi^\dagger(x))^T\,\mathbf{H}^\dagger(x) \\ &-\; M_H^2\mathbf{H}^\dagger(x)\mathbf{H}(x) + h.c.\,, \end{aligned} \tag{4.6}$$

where $(\psi^C_{lR}(x))^T = ((\nu^C_{lR}(x))^T\,(l^C_R(x))^T)$, $l^C_R(x) \equiv C(\overline{l_L}(x))^T$, $l_L(x)$ being the LH component of the charged lepton field $l(x)$, $l = e, \mu, \tau$, $h_{l'l}$ are coupling constants μ_H is a constant with dimension of mass and M_H is the mass of $\mathbf{H}(x)$. The model does not include RH neutrino fields $\nu_{lR}(x)$. The Lagrangian $\mathscr{L}_{\rm II}(x)$ does not conserve the lepton charge L. The flavour neutrino Majorana mass matrix thus generated has the form

$$M_\nu \cong h\,v^2\,\mu_H M_H^{-2} = U^*_{\rm PMNS}\,M_\nu^{\rm diag}\,U^\dagger_{\rm PMNS}\,. \tag{4.7}$$

For getting an idea of the magnitude of of h, μ_H and M_H for which we get the correct scale of light neutrino masses we neglect the matrix structure of h and set $\mu_H \sim M_H$. With this simplifications and recalling that $v = 246$ GeV, we find that $M_\nu \sim 0.1$ eV for, e.g., $h = 10^{-2}$ and $M_H \sim 6\times 10^{12}$ GeV. The smallness of neutrino masses is a consequence of the smallness of the ratio $v\,\mu_H M_H^{-2}$.

The Higgs Triplet Model (HTM). The TeV scale version of the type II seesaw model is usually referred to as the "Higgs Triplet Model (HTM)". In this version the Higgs particles H^{--}, H^-, H^0,

$$\frac{\tau}{2}\mathbf{H}(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} H^-/\sqrt{2} & H^0 \\ H^{--} & -H^-/\sqrt{2} \end{pmatrix}, \tag{4.8}$$

have masses at the TeV scale. A Majorana mass term of the flavour neutrinos ν_l can be generated just by the first term in the expression for $\mathscr{L}_{\rm II}(x)$, eq. (4.6), if the neutral component $H^0(x)$ of the

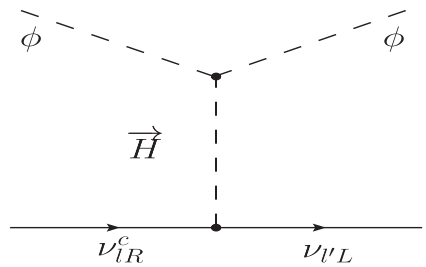

Figure 2: The diagram generating the Majorana mass term (4.4) for the flavour neutrinos ν_l, $l = e, \mu, \tau$, in the case of type II seesaw mechanism. See text for further details.

Higgs triplet field, develops a non-zero vacuum expectation value (vev), $< H^0 > /\sqrt{2} \equiv v_T \neq 0$ [5]. This can be achieved by using a rather elaborate scalar potential in the theory [59] (see also, e.g., [60, 61]). The flavour neutrino Majorana mass matrix generated by the vev of $H^0(x)$ is given by:

$$(M_\nu)_{\ell\ell'} \cong 2 h_{\ell\ell'} v_T \, . \tag{4.9}$$

Thus, the smallness of the neutrino masses is related in the HTM to the smallness of the vacuum expectation value of H^0. An upper limit on v_T can be obtained from considering its effect on the parameter $\rho = M_W^2 / M_Z^2 \cos^2\theta_W$. In the SM, $\rho = 1$ at tree-level, while in the HTM one has

$$\rho \equiv 1 + \delta\rho = \frac{1 + 2x^2}{1 + 4x^2}\, , \;\; x \equiv \sqrt{2} v_H / v \, . \tag{4.10}$$

The measurement $\rho = 1.0003 \pm 0.00023$ [62] leads (at 3σ) to the bound $v_T \lesssim 2.43$ GeV. A lower limit on v_T follows from the magnitude of $|(M_\nu)_{\ell\ell'}|$ and the requirement of perturbative values of the couplings $h_{\ell\ell'}$, $|h_{\ell\ell'}|^2 \leq 4\pi$. Taking for simplicity $|(M_\nu)_{\ell\ell'}| = 0.1$ eV, one finds $v_T \gtrsim 10^{-2}$ eV.

In the phenomenologically interesting case under discussion of masses of H^0, H^- and H^{--}, M_{H^0}, M_{H^-} and $M_{H^{--}}$, which we will denote generically as M_H, satisfying $M_H \sim (100 - 1000)$ GeV, the model predicts a plethora of beyond the SM physics phenomena (see, e.g., [60, 63, 64, 65, 66, 67]), most of which can be probed at the LHC and in the experiments on charged lepton flavour violation, if the Higgs triplet vacuum expectation value v_H is relatively small, roughly $v_H \sim (1 - 100)$ eV, so that the couplings $h_{ll'}$ are sufficiently large in magnitude. Under the indicated conditions one can have testable predictions of the model in low energy experiments, and in particular, in the planned future experiments on the lepton flavour violating processes $\mu \to e\gamma$, $\mu \to 3e$ and $\mu + \mathcal{N} \to e + \mathcal{N}$ (see, e.g., [65, 67]) [6]. The TeV scale HTM predicts the existence of rich new physics at LHC as well [68], associated with the presence of the singly and doubly charged Higgs particles H^- and H^{--} in the theory (see, e.g., [60, 61, 64]). The existing data imply combined limits on v_T and M_T. Assuming that $M_{H^-} > M_{H^{--}}$, for $v_T < 10^4$ eV the decays of $H^{--} \to l^- + l'^-$, $l, l' = e, \mu, \tau$ are dominant. At LHC, the pair $H^{--} + H^{++}$ is produced via virtual

[5] The second term in the expression for $\mathscr{L}_{\rm II}(x)$, eq. (4.6), breaks the lepton charge conservation explicitly. As a consequence, a massless Goldstone boson is not present in the theory.

[6] The physical singly-charged Higgs scalar field (particle) practically coincides with the triplet scalar field H^+, the admixture of the charged component Φ^+ of the SM Higgs doublet Φ being suppressed by the factor v_T/v. The singly- and doubly- charged Higgs scalars H^- and H^{--} have, in general, different masses [59]: $M_{H^-} \neq M_{H^{--}}$. Both cases $M_{H^-} > M_{H^{--}}$ and $M_{H^-} < M_{H^{--}}$ are possible.

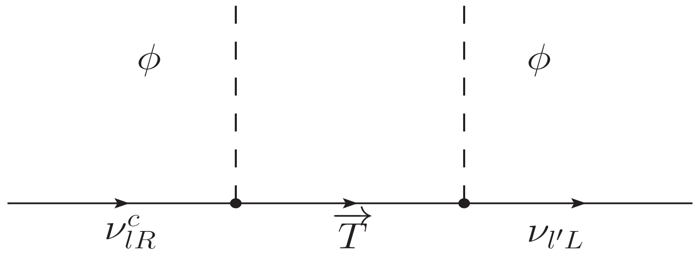

Figure 3: The diagram generating the Majorana mass term (4.4) for flavour neutrinos ν_l, $l = e, \mu, \tau$, in the case of type III seesaw mechanism. See text for further details.

γ and Z^0. The search for pairs of same sign charged leptons $\mu^\pm + \mu^\pm$, $e^\pm + e^\pm$ and $e^\pm + \mu^\pm$ at LHC leads for $v_T \sim (10 - 10^4)$ eV to the following lower limit (at 95% C.L.) on the mass of H^{--} [61]: $M_{H^{--}} > 620$ GeV (for further details and other constraints on the model see, e.g., [61]).

Type III Seesaw Model. The diagram generating an effective Majorana mass term (4.4) for the flavour neutrinos ν_l in type III seesaw model is shown schematically in Fig. 3. The model includes at least two $SU(2)_L$ triplets of RH fermion fields [7] $\mathbf{T}_{jR}(x)$, $j \geq 2$, $(\mathbf{T}_{jR}(x))^T = \mathrm{diag}(T_{1jR}, T_{2jR}, T_{3jR})$, carrying zero weak hyper-charge, which possesses Yukawa-type couplings to the SM lepton and Higgs doublets $\psi_{lL}(x)$ and $\Phi(x)$ and a Majorana mass term:

$$\mathscr{L}_{\mathrm{T}}(x) = -\lambda_{Tlj}\overline{\psi_{lL}}(x)\frac{\tau}{2}\, i\tau_2\, (\Phi^\dagger(x))^T\, \mathbf{T}_{jR}(x) + \frac{1}{2}M_{T\,jk}(\mathbf{T}_{jR}(x))^T C^{-1}\mathbf{T}_{kR}(x) + h.c., \tag{4.11}$$

where λ_{Tlk} are constants and $M_T = (M_T)^T$ is the complex, in general, $\mathbf{T}_{jR}(x)$ mass matrix. The states with definite electric charge are $T^\pm_{jR} = T_{1jR} \pm i\, T_{2jR}$ and $T^0_{jR} = T_{3jR}$. The states correspond to electrically charged Dirac and neutral Majorana fermions $E^\pm_j$ and E^0_j read:

$$E^-_J = T^-_{jR} + C(\overline{T^+_{jR}})^T\,,\;\; E^0_j = T^0_{jR} + C(\overline{T^0_{jR}})^T\,. \tag{4.12}$$

The effective Majorana mass matrix of the flavour neutrinos ν_l is given by:
$(M_\nu)_{l'l} \cong v^2\, (\lambda_{Tl'j})M^{-1}_{T\,jk}\lambda^T_{Tkl} = (U^*_{\mathrm{PMNS}}\, M^{\mathrm{diag}}_\nu\, U^\dagger_{\mathrm{PMNS}})_{l'l}$. For $v\,\lambda_T \sim 1$ GeV, $M_T \sim 10^{10}$ GeV one obtains $M_\nu \sim 0.1$ eV. The model has TeV scale version, $M_T \sim (100 - 10^3)$ GeV, with rich low-energy phenomenology (see, e.g., [65] and references quoted therein).

In the effective operator formalism all three seesaw mechanisms generate non-zero neutrino masses and neutrino mixing after the spontaneous electroweak symmetry breaking via dimension 5 Weinberg operators [69]. In the case of type I seesaw the Weinberg operator has the form:

$$\mathscr{L}_{\mathrm{dim}\;5}(x) = \frac{\lambda_{ll'}}{\Lambda}\,\left(\psi^T_{lL}(x)C^{-1}\, i\tau_2\,\Phi(x)\right)\left(\Phi^T(x)\, i\tau_2\,\psi_{l'L|}(x)\right) + h.c., \tag{4.13}$$

where $\lambda_{ll'}$ are constants and Λ is the scale of New Physics.

5. Radiative Generation of Neutrino Masses and Mixing

Within the radiative mechanism, no-zero neutrino masses arise as higher order – one, two, three, etc. loop – corrections to the Lagrangian of the theory. The models employing the radiative

[7] In the case of only one triplet the seesaw generated flavour neutrino Majorana mass matrix, as can be easily shown, has two zero eigenvalues, which is ruled out by the data.

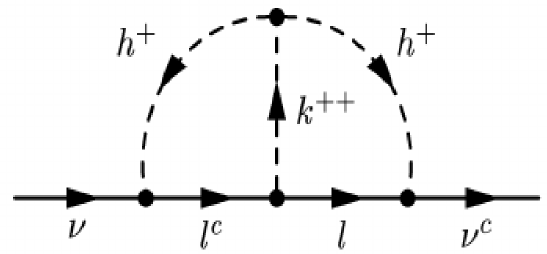

Figure 4: Diagram generating a Majorana mass term (see, e.g., 4.4) for flavour neutrinos ν_l, $l = e, \mu, \tau$, at two loop level in the model [53]. See text for further details.

mechanism of generation of neutrino masses possess certain generic features.
— The loop suppression factors help explaining the smallness of neutrino masses.
— The new particles involved in the respective specific models need not be super heavy, they can be at the TeV scale. Models with New Physics at the TeV scale are testable experimentally.
— In the models with radiative mechanism there is no need to introduced RH neutrino fields $\nu_R(x)$.
— The models typically include extended scalar sectors.

We will consider briefly two examples of models of radiative neutrino mass generation. The first [53] involves a doubly and singly charge scalar particles k^{++} and h^+, which carry i) weak hypercharges $Y_W(k^{++}) = 4$ and $Y_W(h^+) = 2$, and ii) total lepton charges $L(k^{++}) = -2$ and $L(h^+) = -2$, and are $SU(2)_L$ singlets. The scalars h^+ and k^{++} couple to two lepton doublets and two RH charged leptons (lepton fields). These couplings are $SU(2)_L \times U(1)_{Y_W}$ invariant and conserve the total lepton charge L. The $L-$nonconservation originates from a triple $k^{++} - h^+ - h^+$ coupling. The model does not include RH neutrinos. Non-zero neutrino masses and neutrino mixing arise at two loop level via the diagram shown in Fig. 4.

The neutrinos with definite mass ν_j in this model are Majorana particles. The scalars k^{++} and h^+ can have masses at the TeV scale: $M(k^{++}) \sim$ TeV, $M(h^+) \sim$ TeV. In this case the model has very rich lepton flavour and total lepton charge violating low-energy phenomenology: i) ChLFV processes, $\mu \to e + \gamma$, $\mu \to 3e$, $\mu^- + (A,Z) \to e^- + (A,Z)$, mediated by k^{++} and h^+, and ii) $(\beta\beta)_{0\nu}-$decay, which can have rates close to the existing stringent experimental upper bounds. The doubly charged scalar k^{++} can be produced at LHC and be observed via its characteristic same sign di-lepton decay mode, $k^{++} \to l^+ + l'^+$, $l, l' = e, \mu, \tau$.

The second model is based on $SU(2)_L \times U(1)_{Y_W} \times Z_2$ symmetry [70], where the Z_2 symmetry is assumed to be exact. The model includes i) singlet RH neutrinos ν_{lR} (neutrino fields $\nu_{lR}(x)$) which possess a Majorana mass term, and ii) additional $SU(2)_L$ doublet scalar field $\eta(x)$, $\eta^T(x) = (\eta^+ \eta^0)$. Due to the choice of the Z_2 charges, $Z_2(\eta) = Z_2(\nu_R) = -1$, $Z_2(\Phi) = Z_2(\psi_{lL}) = Z_2(l_R) = +1$, $l = e, \mu, \tau$, and the requirement of Z_2 invariance of the scalar potential of the theory, η does not develop non-zero vev. The Z_2 symmetry forbids the Yukawa type coupling involving the SM Higgs doublet, $\overline{\psi_{lL}}\, i\tau_2\, \Phi^*\, \nu_{l'R}$, but allows the analogous coupling involving η, $\overline{\psi_{lL}}\, i\tau_2\, \eta^*\, \nu_{l'R}$. The interplay of this Yukawa coupling with the quartic scalar coupling $(\Phi^\dagger \eta)^2$ and the ν_{lR} Majorana mass term generates a Majorana mass for the LH flavour neutrinos ν_l via the diagram shown in

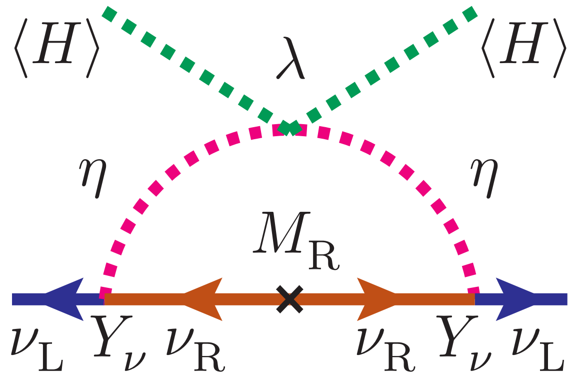

Figure 5: Diagram generating a Majorana mass term for the flavour neutrinos ν_l, $l = e, \mu, \tau$, at one loop level in the model [70] (the figure is taken from [71]). In the diagram H denotes the SM Higgs doublet (which is denoted as Φ in the text of this article). See text for further details.

Fig. 5. Thus, the massive neutrinos ν_j are predicted to be Majorana fermions. The model contains a dark matter candidate: it could be either the lightest of the three heavy Majorana neutrinos $N_{1,2,3}$ having masses $M_{1,2,3}$, e.g., N_1, $\min(M_j) = M_1$, or $\sqrt{2}\mathrm{Re}\eta^0$.

The masses of $\eta^\pm$, $M(\eta^\pm)$, and of $N_{1,2,3}$ can be at the TeV scale: $M(\eta^\pm) \sim$ TeV, $M_{1,2,3} \sim$ TeV. This possibility is associated with rich low-energy phenomenology. If $M(\eta^\pm) > M_{1,2,3}$, the following characteristic chain of decays with multilepton final state are possible: $\eta^\pm \to l^\pm N_{1,2,3}$, $N_2 \to l^\pm l^\mp N_1$, $N_3 \to l^\pm l^\mp N_{1,2}$. For $M(\eta^\pm) < M_{1,2,3}$, one has: $N_{1,2,3} \to l^\pm \eta^\mp$. The ChLFV processes and the $(\beta\beta)_{0\nu}-$ decay can proceed with rates close to the existing upper bounds (for further details see [70] and, e.g., the review [71]).

General cases of neutrino masses generated at 1-loop, 2-loop, 3-loop, etc. level are discussed in [55]. In [57] the authors review the effective operator approach to the problem of radiative generation of neutrino masses. All $SU(2)_L \times U(1)_{Y_W}$ invariant dimension 7 operators, which generate non-zero neutrino masses at one or two loop level are listed and their ultraviolet (UV) completions are given. In [57] it was noticed, in particular, that the neutrino masses can be generated at loop level by two non-renormalisable dimension 7 operators (Q_3 and Q_8), whose UV completions involve lepto-quarks which are well known candidates to explain the flavour anomalies (the indications for breaking of lepton universality) in B-meson decays (see, e.g., [72]).

6. Understanding the Pattern of Neutrino Mixing. The Quest for Nature's Message

The observed pattern of 3-neutrino mixing is characterised, as we have seen, by two large mixing angles θ_{12} and θ_{23}, and one small mixing angle θ_{13}: $\theta_{12} \cong 33^\circ$, $\theta_{23} \cong 45^\circ \pm 6^\circ$ and $\theta_{13} \cong 8.4^\circ$. Understanding the origin of this pattern is part of the fundamental flavour problem in particle physics and represents a remarkable theoretical challenge.

I believe, and I am not alone in holding this view, that with the observed pattern of neutrino mixing Nature is "sending" us a Message. The Message is encoded in the values of the neutrino mixing angles, leptonic CP violation (CPV) phases in the PMNS neutrino mixing matrix and neutrino masses. We do not know at present what is the content of Nature's Message. However, on the basis of the current ideas about the possible origins of the observed pattern of neutrino mixing, the

Nature's Message can have two completely different contents, each of which can be characterised by one word: ANARCHY or SYMMETRY.

6.1 Anarchy

In the "Anarchy" approach to neutrino mixing [73, 74, 75, 76] it is assumed that the PMNS neutrino mixing matrix $U_{\rm PMNS}$ can be described as the result from a random draw of unbiased distribution of 3×3 unitary matrices. Thus, the neutrino mixing angles θ_{12}, θ_{23} θ_{13} and CP violation (CPV) phases δ, α_{21} and α_{31}, are supposed to be random quantities. In other words, it is assumed that Nature "threw dice" when Nature was "choosing" the values of θ_{ij}, δ, α_{21} and α_{31}. Within the Anarchy approach one obtains predictions for distributions (and not for values) of the three neutrino mixing angles and the leptonic Dirac and Majorana CPV phases. According to these distributions values of $\theta_{ij}=\pi/4$, $ij=12,23,13$, and $\delta=\pi/2,3\pi/2$ are the most probable. Thus, three large mixing angles is the most natural pattern for the Anarchy approach to neutrino mixing. However, $\sin^2\theta_{13}\cong 0.0214$ is rather small, so the value of $\theta_{13}\cong 0.15$ is explained as arising from the "tail" of the corresponding distribution. It is claimed in [75] that within the Anarchy approach one has $\sin^2\theta_{13}>0.011$ at the 2σ C.L. In what concerns the neutrino masses m_j and neutrino mass squared differences Δm^2_{ij}, they are not predicted within the Anarchy approach.

One of the characteristic prediction of the Anarchy approach to neutrino mixing is the absence of whatsoever correlations between the values of some of the neutrino mixing angles and/or between the values of the neutrino mixing angles and the CPV phases.

6.2 The Family Symmetry Approach

Since the pioneering work [77] the continuous flavour (or horizontal) symmetries approach to the quark and lepton flavour problems has been extensively exploited (see, e.g., [78, 79, 80] which contain comprehensive lists of references). The well known Frogatt-Nielsen (FN) [77] simplest and effective mechanism of explaining the hierarchical patterns of quark and charged lepton masses is based on $U(1)_{\rm FN}$ local flavour symmetry assumed to be valid at some high energy scale $M_{\rm FN}$. This symmetry is spontaneously broken when a scalar "flavon" field $\varphi(x)$, carrying a $U(1)_{\rm FN}$ charge chosen to be (-1), acquires a non-zero vacuum expectation value $<\varphi>$ below the scale $M_{\rm FN}$. The ratio $<\varphi>/M_{\rm FN}$ plays the role of a small parameter in the theory. It is universally assumed to be $<\varphi>/M_{\rm FN}\sim\sin\theta_C\cong 0.22$, where θ_C is the Cabibbo angle. The charged lepton mass matrix M_e, for example, which in the SM originates from the $SU(2)_L\times U(1)_{Y_W}$ invariant charged lepton Yukawa coupling after the spontaneous breaking of the SM symmetry, in the FN approach is generated by $SU(2)_L\times U(1)_{Y_W}$ and $U(1)_{\rm FN}$ invariant non-renormalisable terms in the low-energy effective Lagrangian of the theory after the breaking of the SM and the $U(1)_{\rm FN}$ symmetries:

$$Y^{\ell}_{\tilde{l}\tilde{l}'}\,\overline{\psi_{\tilde{l}L}}(x)\Phi(x)\,\tilde{l}'_R(x)\to A_{\tilde{l}\tilde{l}'}\left(\frac{\varphi}{M}\right)^{n_{\tilde{l}\tilde{l}'}}\overline{\psi_{\tilde{l}L}}(x)\,\Phi(x)\,\tilde{l}'_R(x)\to A_{\tilde{l}\tilde{l}'}\left(\frac{<\varphi>}{M}\right)^{n_{\tilde{l}\tilde{l}'}}\overline{\tilde{l}_L}(x)\,\frac{v}{\sqrt{2}}\,\tilde{l}'_R(x)\,. \quad (6.1)$$

Here $\tilde{l}_L(x)$ and $\tilde{l}'_R(x)$, $\tilde{l},\tilde{l}'=\tilde{e},\tilde{\mu},\tilde{\tau}$, are respectively the $SU(2)$ doublet and singlet left-handed (LH) and right-handed (RH) components of the charged lepton fields in the basis in which the charged lepton mass term $\mathscr{L}_\ell(x)$ is not diagonal, and $n_{\tilde{l}\tilde{l}'}=n(\overline{\psi_{\tilde{l}L}})+n(\Phi)+n(\tilde{l}'_R)$, $n(\overline{\psi_{\tilde{l}L}})$, $n(\Phi)$ and $n(\tilde{l}'_R)$ being the $U(1)_{\rm FN}$ charges of $\overline{\psi_{\tilde{l}L}}$, Φ and $\tilde{l}'_R$. The coefficients $A_{\tilde{l}\tilde{l}'}$ are assumed to be numbers of order 1. There exist choices of the $U(1)_{\rm FN}$ charges $n(\overline{\psi_{\tilde{l}L}})$, $n(\Phi)$ and $n(\tilde{l}'_R)$ (see, e.g., [81]), that ensure the correct hierarchical structure of the elements of M_e, which in turn leads to the known hierarchies of the charged lepton masses. The ratios of, e.g., $e^{\pm}$ and $\mu^{\pm}$, and $\mu^{\pm}$ and $\tau^{\pm}$ masses, are understood as powers of the small parameter $< \varphi > /M_{\rm FN} \equiv \varepsilon \cong 0.22$: $m_e/m_\mu \sim \varepsilon^4$, $m_\mu/m_\tau \sim \varepsilon^2$. The angles in the unitary matrix U_e which diagonalises the product $M_e M_e^\dagger$ and enters into the expression for the PMNS matrix, $U_{\rm PMNS} = U_e^\dagger U_\nu$, where U_ν results from the diagonalisation of the neutrino mass term, are determined by the charged lepton mass ratios and are typically predicted to be small. All predictions have uncertainties related to the unknown order 1 coefficients $A_{\bar{l}l'}$. Similar considerations apply to the FN approach to the quark flavour problem. Thus, the quark mass hierarchies and the small values of the quark mixing angles have a natural explanation within the FN approach.

Explaining the observed pattern of neutrino mixing with two large and one small maxing angles represents a challenge for the FN approach. As was shown, e.g., in [78, 79, 81], this can be done for quasi-degenerate neutrino mass spectrum or mass spectrum with mild hierarchy with the help of certain amount of fine tuning. The results and predictions of interest always involve uncertainties related to the presence of unknown order 1 factors [8]. These uncertainties can be avoided in much more complex models with larger continuous flavour symmetries ($U(2)$, $SU(3)$, etc., see, e.g., [83] and [80] which includes also an extensive list of relevant references).

6.3 Towards Quantitative Understanding of the Pattern of Neutrino Mixing

The observed pattern of 3-neutrino mixing and the specific values of the three neutrino mixing angles can most naturally be explained by extending the SM with a flavour symmetry corresponding to a non-Abelian discrete (finite) group G_f. This symmetry is supposed to exist at some high-energy scale and to be broken at lower energies to residual symmetries of the charged lepton and neutrino sectors, described respectively by subgroups G_e and G_ν of G_f. Thus, within the SYMMETRY approach, the observed pattern of neutrino mixing can be naturally understood on the basis of specific class of symmetries - the class of non-Abelian discrete flavour symmetries (see, e.g., [84, 85, 86, 87]). Accordingly, the specific form of the neutrino mixing can have its origin in the existence of new fundamental symmetry in the lepton sector. We will consider the discussed approach to neutrino mixing is some detail.

The most distinctive feature of the approach to neutrino mixing based on non-Abelian discrete flavour symmetries is the predictions of the values of some of the neutrino mixing angles and leptonic CPV phases, and/or of existence of correlations between the values of at least some the neutrino mixing angles and/or between the values of the neutrino mixing angles and the Dirac CPV phase in the PMNS matrix, etc. (see, e.g., [86, 88, 89, 90, 91, 92]). This is in stark contrast with the predicted "randomness" of, and the absence of whatsoever correlations between, the values of the neutrino mixing angles within the ANARCHY approach. Most importantly, the predictions and predicted correlations of the discrete symmetry approach, and thus the approach itself, can be tested experimentally (see, e.g., [88] and [89, 93, 94, 95, 96, 97, 98]).

Flavour symmetry groups G_f that have been used in the considered symmetry approach to neutrino mixing and lepton flavour include A_4 [99], S_4 [100], T' [101], A_5 [102], D_n (with $n =$

[8] For applications of the FN mechanism for obtaining, for example, highly hierarchical spectrum of the heavy Majorana neutrinos of the seesaw mechanism see, e.g., [82].

10,12) [103, 104], $\Delta(27)$ [105], the series $\Delta(6n^2)$ [106], to name several [9] (see, e.g., ref. [85] for definitions of these groups and discussion of their properties [10]). The choice of the non-Abelian discrete groups A_4, S_4, T', A_5, etc. is related, in particular, to the fact that they describe symmetries with respect to rotations on fixed large mixing angles and, correspondingly, lead to values of the neutrino mixing angles θ_{12} and θ_{23}, which can differ from the measured values at most by sub-leading perturbative corrections, with θ_{13} typically (but not universally) predicted to be zero. The PMNS matrix, as we have already indicated and is well known, has the form:

$$U_{\rm PMNS} = U_e^{\dagger}\, U_{\nu}\,,$$

where the unitary matrix $U_e^{\dagger}$ (U_{ν}) originates from the diagonalisation of the charged lepton (neutrino) mass term. The angles in the matrix U_{ν} are typically assumed to have values dictated by the symmetry. The requisite corrections can most naturally be provided by the matrix $U_e^{\dagger}$ (see, e.g., [111, 112, 88, 89, 90] and references quoted therein). As we have noticed already, U_e diagonalises the product $M_e M_e^{\dagger}$, M_e being the charged lepton mass matrix in the charged lepton mass term $\mathscr{L}_{\ell}(x)$:

$$\mathscr{L}_{\ell}(x) = -\,\overline{\tilde{l}_L(x)}\,(M_e)_{\tilde{l}\tilde{l}'}\,\tilde{l}'_R(x) + \text{ h.c.}\,, \tag{6.2}$$

$$U_e^{\dagger}\, M_e M_e^{\dagger}\, U_e = \text{diag}(m_e^2, m_{\mu}^2, m_{\tau}^2)\,, \tag{6.3}$$

where $\tilde{l}_L(x)$ and $\tilde{l}'_R(x)$, $\tilde{l}, \tilde{l}' = \tilde{e}, \tilde{\mu}, \tilde{\tau}$, have been defined after eq. (6.1) and m_e, m_{μ} and m_{τ} are the masses of the charged leptons [11]. In certain classes of models, however, U_e coincides with the unit 3×3 matrix and the requisite corrections are incorporated in a factor contained in the matrix U_{ν} (see, e.g., [86, 87] and references quoted therein). Assuming that the weak-eigenstate neutrino fields (in the basis in which charged lepton mass term is not diagonal), $\nu_{\tilde{e}}(x)$, $\nu_{\tilde{\mu}}(x)$ and $\nu_{\tilde{\tau}}(x)$, possess a Majorana mass term, $\mathscr{L}_M^{\nu}(x)$, and thus the neutrinos with definite mass $\nu_{1,2,3}$ are Majorana particles [12], then U_{ν} is the matrix diagonalising the neutrino Majorana mass matrix M_{ν}:

$$\mathscr{L}_M^{\nu}(x) = \tfrac{1}{2}\, \nu^{\rm T}_{\tilde{l}' L}(x)\, C^{-1}\, M_{\nu \tilde{l}'\tilde{l}}\;\, \nu_{\tilde{l}L}(x) \;+\; h.c.\,, \tag{6.4}$$

$$U_{\nu}^{\rm T}\, M_{\nu}\, U_{\nu} = \text{diag}(m_1, m_2, m_3)\,. \tag{6.5}$$

In the approach under discussion it is standardly assumed that the LH neutrino fields, $\nu_{\tilde{l}L}(x)$, and the LH components of the charged lepton fields $\tilde{l}_L(x)$, which form an $SU(2)_L$ doublet in the SM, are assigned to the same r-dimensional irreducible representation $\rho_r(g_f)$ of the Group G_f, g_f being an element of G_f. In the cases of $G_f = A_4$, S_4, T' and A_5, $\rho(g_f)$ is typically taken to be

[9] Some of the groups T', A_5, etc. can be and have been used also for a unified description of the quark and lepton flavours, see, e.g., refs. [101, 107, 108, 109, 110] and references quoted therein.

[10] A_4 is the group of even permutations of 4 objects and the symmetry group of the regular tetrahedron. S_4 is the group of permutations of 4 objects and the symmetry group of the cube. T' is the double covering group of A_4. A_5 is the icosahedron symmetry group of even permutations of five objects, etc. All these groups are subgroups of $SU(3)$.

[11] The LH components of the fields of the electron, muon, and tauon, $l_L(x)$, $l = e, \mu, \tau$, are related to the fields $\tilde{l}_L(x)$ via the matrix U_e: $l_L(x) = (U_e^{\dagger})_{l\tilde{l}}\tilde{l}_L(x)$.

[12] It should be noted, however, that the approach to neutrino mixing we are discussing can be employed also if $\nu_{1,2,3}$ are Dirac fermions (see, e.g., [90]), e.g., when the theory contains right-handed neutrino fields $\nu_{\tilde{l}R}(x)$ which form a Dirac mass term with the LH neutrino fields $\nu_{\tilde{l}'L}(x)$ $\tilde{l}, \tilde{l}' = \tilde{e}, \tilde{\mu}, \tilde{\tau}$, and and the total lepton charge $L = L_e + L_{\mu} + L_{\tau}$ is conserved [45], as discussed in Section 3.

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a 3-dimensional irreducible unitary representation **3**, $\rho_r(g_f) = \rho_3(g_f)$. This is equivalent to the assumption of unification of the three lepton families at some high energy scale. We are going to consider this choice in what follows [13].

At low energies the flavour symmetry G_f has necessarily to be broken so that the electron, muon and tauon as well as the three neutrinos with definite mass ν_1, ν_2 and ν_3, can get different masses. The breaking of G_f is realised in specific models by scalar "flavon" fields, which are singlets with respect to the Standard Theory gauge group but transform under certain irreducible representations of G_f and acquire non-zero vacuum expectation values, thus breaking G_f spontaneously. The breaking of the flavour symmetry G_f can leave certain subgroups of G_f, G_e and G_ν, unbroken in the charged lepton and neutrino sectors. The unbroken symmetries $G_e \in G_f$ and $G_\nu \in G_f$ are *residual symmetries* of the charged lepton and neutrino mass matrices, M_e and M_ν. The residual symmetry G_e restricts the forms of M_e and $M_e M_e^\dagger$ and, thus the form of U_e, while the symmetry G_ν restrict the form of M_ν and therefore of U_ν. As a consequence, the neutrino mixing matrix U_{PMNS} is either completely determined or is restricted to have a specific form. The form of U_{PMNS} one obtains depends on G_f, $\rho_r(g_f)$, G_e and G_ν. If the residual symmetries are "sufficiently large" (see, e.g., [86, 87]), the form of U_{PMNS} will be completely fixed.

As we have already indicated, one of the main characteristics of the discussed approach to neutrino mixing based on discrete flavour symmetries is that it leads to certain specific predictions for the values of, and/or correlations between, the low-energy neutrino mixing parameters, which can be tested experimentally. We give a few examples [86, 88, 89, 90, 93, 94, 113, 114, 115, 116].

I. In a large class of models one gets $\sin^2\theta_{23} = 0.5$.

II. In different class of models one finds that the values of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ are correlated: $\sin^2\theta_{23} = 0.5(1 \mp \sin^2\theta_{13} + O(\sin^4\theta_{13}))$.

III. In certain models $\sin^2\theta_{23}$ is predicted to have specific values which differ significantly from those in cases **I** and **II** [89]: $\sin^2\theta_{23} = 0.455$; or 0.463; or 0.537; or 0.545, the uncertainties in these predictions being insignificant.

IV. Certain class of models predict a correlation between the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$: $\sin^2\theta_{12} = 1/(3\cos^2\theta_{13}) = (1+\sin^2\theta_{13} + O(\sin^4\theta_{13}))/3 \cong 0.340$, where we have used the b.f.v. of $\sin^2\theta_{13}$.

V. In another class of models one still finds a correlation between the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$, which, however, differs from that in Case **IV**: $\sin^2\theta_{12} = (1-3\sin^2\theta_{13})/(3\cos^2\theta_{13}) = (1-2\sin^2\theta_{13} + O(\sin^4\theta_{13}))/3 \cong 0.319$, where we have used again the b.f.v. of $\sin^2\theta_{13}$.

VI. In large classes of models in which the elements of the PMNS matrix are predicted to be functions of just one real continuous free parameter ("one-parameter models"), the Dirac and the Majorana CPV phases have "trivial" CP conserving values 0 or π. In certain one-parameter schemes, however, the Dirac phase $\delta = \pi/2$ or $3\pi/2$.

VII. In theories/models in which the elements of the PMNS matrix are functions of two (angle and a phase) or three (two angles and one phase) parameters, the Dirac phase δ satisfies a sum rule by which $\cos\delta$ is expressed in terms of the three neutrino mixing angles θ_{12}, θ_{23}, θ_{13} and one (or more) fixed (known) parameters θ^ν which depend on the discrete symmetry G_f employed and on

[13] In specific models the choice $\rho_r(g_f) = \rho_3(g_f)$ is usually accompanied by the assumption that $\tilde{e}_R(x)$, $\tilde{\mu}_R(x)$ and $\tilde{\tau}_R(x)$ transform as singlet irreducible representations of G_f (see, e.g., [86]).

the residual symmetries G_e and G_ν [88, 89, 90]:

$$\cos\delta = \cos\delta(\theta_{12},\theta_{23},\theta_{13};\theta^{\nu}). \tag{6.6}$$

In these cases the J_{CP} factor which determines the magnitude of CP violation effects in neutrino oscillations, is also completely determined by the values of the three neutrino mixing angles and the symmetry parameter(s) θ_ν:

$$J_{CP} = J_{CP}(\theta_{12},\theta_{23},\theta_{13},\delta(\theta_{12},\theta_{23},\theta_{13})) == J_{CP}(\theta_{12},\theta_{23},\theta_{13};\theta^{\nu})\,. \tag{6.7}$$

The predictions listed above, and therefore the respective models, can be and will be tested in the currently running (T2K [117] and NOνA [118]) and planned future (JUNO [119], T2HK [120], T2HKK [121] and DUNE [122]) experiments.

Underlying Symmetry Forms of $U_{\rm PMNS}$ and Predictions for Dirac CP Violation. As example of cases leading to the correlations (6.6) and (6.7), we consider a sum rule for $\cos\delta$ corresponding to five widely discussed symmetry forms of U_ν, with the requisite corrections provided by two possible forms of U_e. The symmetry forms in question are: i) tribimaximal (TBM) mixing [123], ii) bimaximal (BM) mixing [14] [125], iii) the golden ratio type A (GRA) mixing [102, 126], iv) the golden ratio type B (GRB) mixing [127], and v) hexagonal (HG) mixing [104]. The matrix U_ν for all five forms has the following simple form:

$$U_\nu = R_{23}\left(\theta^{\nu}_{23} = -\pi/4\right) R_{13}\left(\theta^{\nu}_{13} = 0\right) R_{12}\left(\theta^{\nu}_{12}\right) P^{\circ} = \begin{pmatrix} \cos\theta^{\nu}_{12} & \sin\theta^{\nu}_{12} & 0 \\ -\frac{\sin\theta^{\nu}_{12}}{\sqrt{2}} & \frac{\cos\theta^{\nu}_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta^{\nu}_{12}}{\sqrt{2}} & \frac{\cos\theta^{\nu}_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P^{\circ}\,, \tag{6.8}$$

where $P^{\circ} = {\rm diag}(1, e^{i\frac{\xi_{21}}{2}}, e^{i\frac{\xi_{31}}{2}})$ and $R_{ij}(\theta^{\nu}_{ij})$ is a 3×3 orthogonal matrix of rotation in the $i-j$ plane. The phases in the matrix P° contribute to the Majorana phases of the PMNS matrix. The value of the angle θ^{ν}_{12}, and thus of $\sin^2\theta^{\nu}_{12}$, depends on the symmetry form of U_ν. For the TBM, BM, GRA, GRB and HG forms [15] we have: i) $\sin^2\theta^{\nu}_{12} = 1/3$ (TBM), ii) $\sin^2\theta^{\nu}_{12} = 1/2$ (BM), iii) $\sin^2\theta^{\nu}_{12} = (2+\tilde r)^{-1} \cong 0.276$ (GRA), $\tilde r$ being the golden ratio, $\tilde r = (1+\sqrt{5})/2$, iv) $\sin^2\theta^{\nu}_{12} = (3-\tilde r)/4 \cong 0.345$ (GRB), and v) $\sin^2\theta^{\nu}_{12} = 1/4$ (HG).

It follows from eq. (6.8) that for the five discussed symmetry forms of U_ν we have: i) $\theta^{\nu}_{13} = 0$, which should be corrected to the measured value of $\theta_{13} \cong 0.15$, and ii) $\sin^2\theta^{\nu}_{23} = 0.5$, which might also need to be corrected if it is firmly established that $\sin^2\theta_{23}$ deviates significantly from 0.5. In the case of the BM form $\sin^2\theta_{12} = 0.5$, which is ruled out by the existing data and should be corrected. Finally, the value of $\sin^2\theta^{\nu}_{12}$ for the HG form lies outside the current 3σ allowed range of $\sin^2\theta_{12}$ and needs also to be corrected. However, all necessary corrections are perturbative and sub-leading, i.e., small.

[14]Bimaximal mixing can also be a consequence of the conservation of the lepton charge $L' = L_e - L_\mu - L_\tau$ (LC) [124], supplemented by $\mu-\tau$ symmetry.

[15]For a detailed discussion of how the symmetry forms TBM, BM, GRA, GRB and HG are obtained from the symmetries A_4 (or T' or S_4), S_4, A_5, D_{10} and D_{12}, respectively, see, e.g., [86, 87] and references quoted therein.

Table 2: Predicted values of $\cos\delta$ and δ for the five symmetry forms, TBM, BM, GRA, GRB and HG, and $\tilde{U}_e$ given by the form **A** in eq. (6.9), obtained using eq. (6.11) and the best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ for NO and IO neutrino mass spectra from ref. [18].

Scheme	$\cos\delta$ (NO)	δ (NO)	$\cos\delta$ (IO)	δ (IO)
TBM	-0.23	$\pm 103^\circ$	-0.24	$\pm 104^\circ$
BM (LC)	-1.81	$\cos\delta-$unphysical	-1.82	$\cos\delta-$unphysical
GRA	0.31	$\pm 72^\circ$	0.30	$\pm 72^\circ$
GRB	-0.34	$\pm 110^\circ$	-0.35	$\pm 111^\circ$
HG	0.56	$\pm 56^\circ$	0.55	$\pm 56^\circ$

The requisite corrections can be provided by the matrix $U_e^\dagger$ (see, e.g., [88, 89, 90, 96, 115]). For the two possible forms of $U_e^\dagger$,

$$\mathbf{A}:\; U_e^\dagger = R_{12}(\theta^e_{12})\,R_{23}(\theta^e_{23})\,\Psi_{\rm A}\,,\;\; \Psi_{\rm A} = {\rm diag}\left(1, e^{-i\psi}, e^{-i\omega}\right)\,, \tag{6.9}$$

$$\mathbf{B}:\; U_e^\dagger = R_{12}(\theta^e_{12})\,\Psi_{\rm B}\,,\;\; \Psi_{\rm B} = {\rm diag}\left(1, e^{-i\psi}, 1\right)\,, \tag{6.10}$$

where θ^e_{12} and θ^e_{23} are free real angle parameters and ψ and ω are two phases [16], $\cos\delta$ was shown to satisfy the following sum rule [88]:

$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\,\left[\cos 2\theta^\nu_{12} + \left(\sin^2\theta_{12} - \cos^2\theta^\nu_{12}\right)\,\left(1 - \cot^2\theta_{23}\,\sin^2\theta_{13}\right)\right]\,. \tag{6.11}$$

Within the approach employed this sum rule is exact [17] and is valid for any value of the angle θ^ν_{23} [89] (and not only for $\theta^\nu_{23} = -\pi/4$ of the five considered symmetry forms of U_ν). For the form **B** of U_e we also have [88]:

$$\sin^2\theta_{23} = \frac{1}{2}\,\frac{1 - 2\sin^2\theta_{13}}{1 - \sin^2\theta_{13}} \cong \frac{1}{2}\,(1 - \sin^2\theta_{13})\,. \tag{6.12}$$

Thus, in contrast to the case **A** in which $\sin^2\theta_{23}$ is not constrained and is allowed to differ significantly from 0.5, in case **B** the values of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ are correlated and $\sin^2\theta_{23}$ can deviate from 0.5 only by $0.5\sin^2\theta_{13} \cong 0.011$. The fact that the value of the Dirac CPV phase δ is determined (up to an ambiguity of the sign of $\sin\delta$) by the values of the mixing angles θ_{12}, θ_{23} and θ_{13} of the PMNS matrix and of θ^ν_{12} of matrix U_ν, eq. (6.8), is the most striking prediction of the models considered. This result implies that in the discussed models the sum rule (6.7) for the $J_{\rm CP}$ factor also holds with $\theta^\nu = \theta^\nu_{12}$.

Using the sum rule (6.11) predictions for $\cos\delta$ and the $J_{\rm CP}$ factor in the cases of the TBM, BM, GRA, GRB and HG underlying symmetry forms of $U_{\rm PMNS}$ were derived first in [88] using the results on $\sin^2\theta_{ij}$ available in 2013 from [20]. In Table 2 we present updated predictions of

[16] Cases **A** and **B** correspond to G_f completely broken and broken to $G_e = Z_2$ by the charged lepton mass term.

[17] The renormalisation group corrections to the sum rule for $\cos\delta$, eq. (6.11), in the cases of neutrino Majorana mass term generated by the Weinberg (dimension 5) operator added to i) the Standard Model, and ii) the minimal SUSY extension of the Standard Model, have been investigated in [128, 129]. They were found in [128] to be negligible, e.g., when the Weinberg operator was added to the Standard Model.

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$\cos\delta$ and δ for the b.f.v. of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ given in Table 1. The results in Table 2 show that the predictions of $\cos\delta$ vary significantly with the symmetry form of U_ν [18]. Thus, a measurement of $\cos\delta$ can allow to distinguish between at least some of the different symmetry forms of U_ν, provided θ_{12}, θ_{13} and θ_{23} are known, and $\cos\delta$ is measured with sufficiently high precision [88]. This conclusion was confirmed by the statistical analyses performed in [96, 97, 98].

The results of the statistical analysis [96] of the predictions for the $J_{\rm CP}$ factor for the five symmetry forms considered are shown in Fig. 6. They were obtained using the "data" (b.f.v. and χ^2-distributions) on $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and δ from [20] for NO spectrum; similar results are valid for the IO spectrum. These results show, in particular, that the CP-conserving value of $J_{\rm CP} = 0$ is excluded in the cases of the TBM, GRA, GRB and HG symmetry forms, respectively, at approximately 5σ, 4σ, 4σ and 3σ C.L. with respect to the C.L. of the corresponding best fit values which all lie in the interval $J_{\rm CP} = (-0.034)-(-0.031)$. The b.f.v. for the BM (LC) form is much smaller and close to zero: $J_{\rm CP} = (-5\times10^{-3})$. For the TBM, GRA, GRB and HG forms at 3σ we have $0.020 \leq |J_{\rm CP}| \leq 0.039$. Thus, for these four forms the CP violating effects in neutrino oscillations are predicted to be relatively large and observable in the T2HK and DUNE experiments [120, 122]. These conclusions hold if one uses in the analysis the results on the neutrino mixing parameters and δ, obtained in the more recent global analyses [18, 19].

In Fig. 7 we present results of statistical analysis of the predictions for $\cos\delta$, namely the likelihood function versus $\cos\delta$ within the Gaussian approximation (see [96] for details) performed using the b.f.v. of the mixing angles given in Table 1 and the prospective 1σ uncertainties i) of 0.7% on $\sin^2\theta_{12}$, planned to be reached in JUNO experiment [119], ii) of 3% on $\sin^2\theta_{13}$, foreseen to be obtained in the Daya Bay experiment [131], and iii) of 3% on $\sin^2\theta_{23}$, expected to be reached in the currently running and future planned long baseline neutrino oscillation experiments [117, 120, 122]. The BM (LC) case is very sensitive to the b.f.v. of $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ and is strongly disfavored for the b.f.v. given in Table 1 [19]. The measurement of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ with the quoted precision will open up the possibility to distinguish between the BM (LC), TBM/GRB, GRA and HG forms of U_ν. More specifically, it was shown in [97] that a combined analysis of the data from the DUNE and T2HK experiments would allow to distinguish between TBM and HG (GRA) symmetry forms of the PMNS matrix at approximately 3σ (2σ) confidence level; and the same data would allow to distinguish between GRB and HG (GRA) forms at more than 3σ (at approximately 2σ) confidence level [20]. In what concerns the BM (LC) form, as was shown in [97], it can be distinguished from the other four symmetry forms – TBM, GRB, GRA and HG – at more than 5σ using only the data from the DUNE experiment. Using the T2HK, T2HKK and DUNE combined data is expected to lead to a better discrimination between the different symmetry forms of $U_{\rm PMNS}$ owing to the better prospective sensitivity to δ of the combined data from the T2HK and

[18]The unphysical value of $\cos\delta$ in the BM (LC) case is a reflection of the fact that the discussed scheme with BM (LC) form of the matrix U_ν does not provide a good description of the data on θ_{12}, θ_{23} and θ_{13} [130].

[19]This case might turn out to be compatible with the data for larger (smaller) measured b.f.v. of $\sin^2\theta_{12}$ ($\sin^2\theta_{23}$).

[20]Distinguishing between the TBM and GRB forms seems to require unrealistically high precision measurement of $\cos\delta$. However, elf-consistent models or theories of (lepton) flavour which lead to the GRB form of U_ν might still be possible to distinguish from those leading to the TBM form using the specific predictions of the two types of models for the neutrino mixing angles and other observables. The same observation applies to models which lead to the GRA and HG forms of U_ν.

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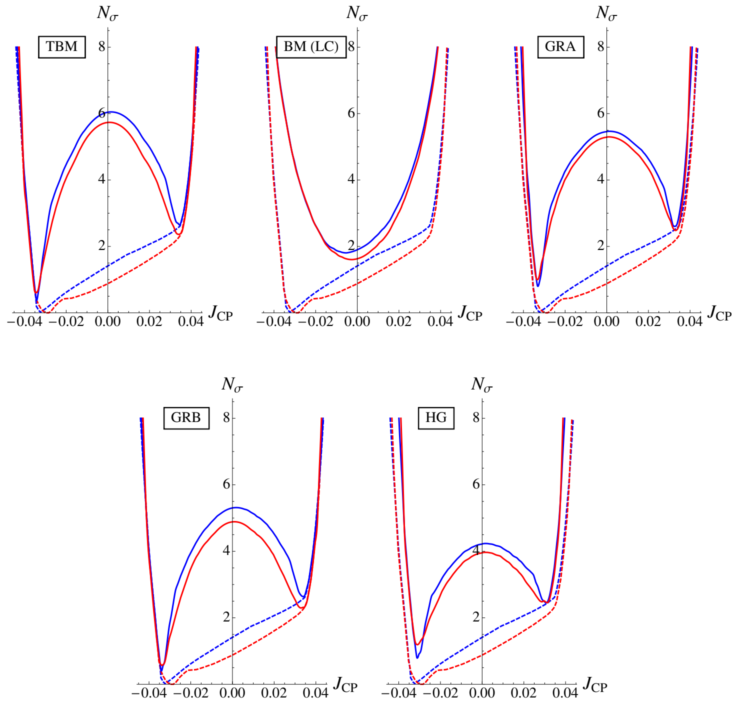

Figure 6: $N_\sigma \equiv \sqrt{\chi^2}$ as a function of $J_{\rm CP}$. The dashed lines represent the results of the global fit [20], while the solid lines represent the results we obtain for the TBM, BM (LC), GRA (upper left, central, right panels), GRB and HG (lower left and right panels) neutrino mixing symmetry forms. The blue (red) lines are for NO (IO) neutrino mass spectrum. (From ref. [96].)

T2HKK experiments.

Predictions for correlations between neutrino mixing angle values and/or sum rules for $\cos\delta$, which can be tested experimentally, were further derived in [90] (see also [89]) for a large number of models based on $G_f = S_4$, A_4, T' and A_5 and all symmetry breaking patterns, i.e., all possible combinations of residual symmetries, which could lead to the correlations and sum rules of interest. Remarkably, of the extremely large number of possible cases only a very limited number – only fourteen – turned out to be phenomenologically viable, i.e., to be compatible with the existing data on the neutrino mixing angles [98] (see also [90]). Statistical analysis of these 14 cases was performed in [98] using the b.f.v. of the three neutrino mixing parameters $\sin^2\theta_{ij}$ from [19] and taking into account the prospective (1σ) uncertainties in the determination of the mixing angles, planned to be achieved in currently running (Daya Bay [131]) and the next generation (JUNO [119], T2HK [120], DUNE [122]) of neutrino oscillation experiments: 3% on $\sin^2\theta_{13}$ [131], 0.7% on $\sin^2\theta_{12}$ [119] and 3% on $\sin^2\theta_{23}$ [122, 120]. This analysis revealed that only six cases would be compatible with the indicated prospective data from the Daya Bay, JUNO, T2HK, DUNE experiments. This number can be further reduced by a precision measurement of the Dirac phase δ.

The results of the studies [90, 98] summarised above lead to the important conclusion that although the number of cases of non-Abelian discrete symmetry groups and their subgroups that can be used for description of lepton mixing is extremely large, only a very limited number survive when confronted with the existing data on the three neutrino mixing angles. This limited number

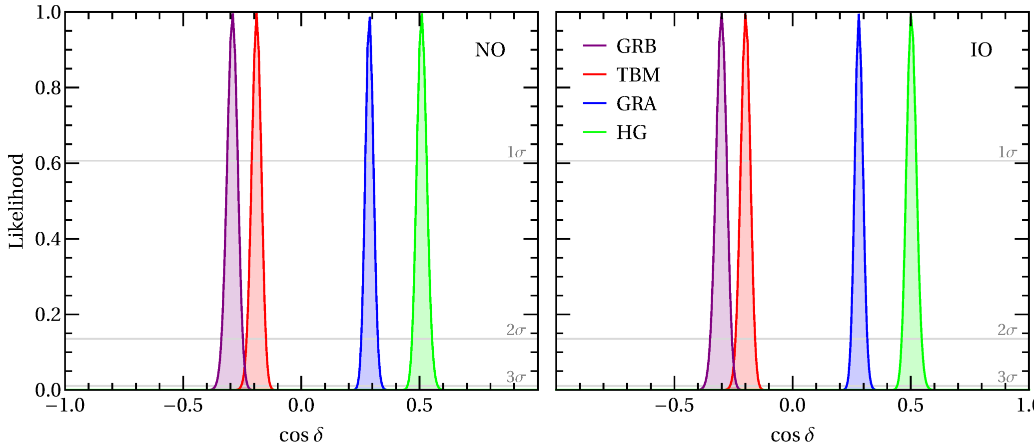

Figure 7: The likelihood function versus $\cos\delta$ for NO and IO neutrino mass spectra after marginalising over $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, for the TBM, BM (LC), GRA, GRB and HG symmetry forms of the mixing matrix U_ν. The figure is obtained by using the prospective 1σ uncertainties in the determination of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ within the Gaussian approximation. The neutrino mixing angles being fixed to their NO and IO b.f.v. taken from [18] (Table 1). See text for further details. (Updated by A.V. Titov of a figure from ref. [96].)

of presently phenomenologically viable cases will be further considerably reduced by the precision measurements of the three neutrino mixing angles and the Dirac phase δ in the currently running (Daya Bay) and future planned (JUNO, T2HK, T2HKK, DUNE) neutrino oscillation experiments.

7. Outlook

The results obtained in refs. [88, 89, 90, 93, 94, 96, 97, 98, 113, 114, 115, 116, 132, 133] and in many other studies (quoted in the present and the cited articles) show that a sufficiently precise measurement of the Dirac phase δ of the PMNS neutrino mixing matrix in the current and future neutrino oscillation experiments, combined with planned improvements of the precision on the neutrino mixing angles, can provide unique information about the possible discrete symmetry origin of the observed pattern of neutrino mixing and, correspondingly, about the existence of new fundamental symmetry in the lepton sector. Thus, these experiments will not simply provide a high precision data on the neutrino mixing and Dirac CPV parameters, but will probe at fundamental level the origin of the observed form of neutrino mixing. In order to test critically the discrete symmetry approach to neutrino mixing, the three neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$ and the Dirac phase δ should be measured with 1σ uncertainties not larger than those quoted below (see, e.g., [98]):

$$\delta(\sin^2\theta_{12}) = 0.7\%\ (\text{JUNO})\,, \tag{7.1}$$
$$\delta(\sin^2\theta_{13}) = 3\%\ (\text{DayaBay})\,, \tag{7.2}$$
$$\delta(\sin^2\theta_{23}) = 3\%\ (\text{T2HK, DUNE; T2K} + \text{NO}\nu\text{A}(?))\,, \tag{7.3}$$
$$\delta(\delta) = 12^\circ \text{ at } 270^\circ\ (\text{T2HK} + \text{THKK} + \text{DUNE}(?))\,. \tag{7.4}$$

The measurement of δ with 1σ uncertainty of 10° at the central value of δ of 270° is highly desirable but is extremely challenging. These future data will show, in particular, whether Nature followed the discrete symmetry approach for fixing the values of the three neutrino mixing angles and of the Dirac and Majorana CP violation phases of the PMNS matrix. We are looking forward to these data and to the future exciting developments in neutrino physics.

The program of research in neutrino physics aims at shedding light on some of the fundamental aspects of neutrino mixing:
i) the status of CP symmetry in the lepton sector;
ii) the nature of massive neutrinos ν_j, which can be Dirac fermions possessing distinct antiparticles, or Majorana fermions, i.e., spin 1/2 particles that are identical with their antiparticles;
iii) the type of spectrum the neutrino masses obey;
iv) the absolute scale of neutrino masses.

The program extends beyond the year 2030 (see, e.g., refs. [1, 117, 118, 119, 120, 121, 122, 134, 135, 136]). Our ultimate goal is to understand at a fundamental level the mechanism giving rise to neutrino masses and mixing and to non-conservation of the lepton charges L_l, $l = e, \mu, \tau$. This includes understanding the origin of the patterns of neutrino mixing and of neutrino masses suggested by the data. The remarkable experimental program of research in neutrino physics and the related theoretical efforts are stimulated by the fact that the existence of nonzero neutrino masses and the smallness of the neutrino masses suggest the existence of new fundamental mass scale in particle physics, i.e., the existence of New Physics beyond that predicted by the Standard Theory. It is hoped that progress in the theory of neutrino mixing will also lead, in particular, to progress in the theory of flavour and to a better understanding of the mechanism of generation of the baryon asymmetry of the Universe.

Acknowledgements. I would like to thank Prof. Jihn E. Kim and his colleagues for the excellent organisation of ICHEP 2018 and for partial support of my participation. This work was supported in part by the INFN program on Theoretical Astroparticle Physics (TASP), by the European Union Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grants 674896 and 690575, and by the World Premier International Research Center Initiative (WPI Initiative, MEXT), Japan.

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