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Strong Kähler with Torsion as Generalised Geometry

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Strong Kähler with Torsion is the target space geometry of $(2,1)$ and $(2,0)$ supersymmetric non-linear sigma models. We discuss how it can be represented in terms of Generalised Complex Geometry in analogy to the Gualtieri map from the geometry of $(2,2)$ supersymmetric nonlinear sigma models to Generalised Kähler Geometry.

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1. Introduction

The target space geometry of $(2,2)$ supersymmetric nonlinear sigma models was shown a long time ago to be bi-hermitean, a generalisation of Kähler geomtery with torsion [2]. More recently this geometry was reformulated as Generalised Kähler Geometry (GKG) [4], a class of Generalised Complex Geometry [3]. The relation between bi-hermitean geometry and Generalised Kähler Geometry is encoded in the Gualtieri map that relates the metric and two complex structures of the bi-hermitan geometry to the two commuting Generalised Complex structures of GKG.

In this presentation we will discuss the extension of this to the target space geometry of $(2,0)$ supersymmetric nonlinear sigma models formulated in [5] and the geometry of $(2,1)$ models formulated in [6]. The geometry of general (p,q) models was given in [9] and further discussed in [11], and recent progress includes [18] and [19]. The Strong Kähler with Torsion (SKT) geometry of the $(2,0)$ and $(2,1)$ target spaces has a complex structure that is covariantly constant with respect to a connection with a torsion given by a closed 3-form and a hermitian metric. We shall map these onto an (integrable) "Half Generalised Complex Structure" [1].

The content of this lecture has been extended and formalised in a paper [1] to which the interested reader is referred for more details, including the generalisation to (p,q) geometries.

2. Definitions

We shall consider GCG as defined on the generalised target space

$$\mathbb{T} := T\mathscr{M} \oplus T^*\mathscr{M} \tag{2.1}$$

The natural pairing $\mathscr{P}$ defines a product $<,>$ defined for[1] $\varphi_1, \varphi_2 \in \mathbb{T}$

$$< \varphi_1, \varphi_2 >= \varphi_1^t \mathscr{P} \varphi_2 \ . \tag{2.2}$$

Writing

$$\varphi = \begin{pmatrix} v_i \\ \xi_i \end{pmatrix} , \tag{2.3}$$

with $v_i \in T$ and $x_i, \in T*$, it follows that

$$< \varphi_1, \varphi_2 >= \xi_1(v_2) + \xi_2(v_1) \tag{2.4}$$

defines a metric of signature (d,d).

We shall further assume the existence of a generalised metric which, in matrix notation, can be arranged to read

$$\mathscr{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{2.5}$$

[1] See below for the matrix. notation

corresponding to a metric splitting[2] of $\mathbb{T}$. Introducing the projection operators

$$\pi^g_\pm := \tfrac{1}{2}\left(1 \pm \mathscr{G}\right) \tag{2.6}$$

the splitting reads

$$\mathbb{T} = \mathbb{T}_+ \oplus \mathbb{T}_- \;, \tag{2.7}$$

with

$$\mathbb{T}_\pm := \pi^g_\pm \mathbb{T} \tag{2.8}$$

being the ± 1 eigenspaces of $\mathbb{T}$. The description of the SKT geometry will focus on the $+1$ eigenspace $\mathbb{T}_+$.

3. A (half) generalised complex structure on $\mathbb{T}_+$

A generalised almost complex structure $\mathscr{J}$ is an endomorphism of $\mathbb{T}$ which squares to minus the identity and preserves the natural pairing $\mathscr{P}$:

$$\begin{aligned}
&\mathscr{J}^2 = -1 \\
&\mathscr{J}^t \mathscr{P} = -\mathscr{P} \mathscr{J} \;.
\end{aligned}$$

Here we consider instead a half generalised structure; a map $\mathscr{J}_+$ which acts on $\mathbb{T}_+$, vanishes on $\mathbb{T}_-$, satisfying

$$\begin{aligned}
&\mathscr{J}_+^2 = -\pi^g_+ \\
&[\mathscr{J}_+, \mathscr{G}] = 0 \\
&(\pi^g_+)^t \mathscr{J}_+^t \mathscr{P} \mathscr{J}_+ \pi^g_+ = (\pi^g_+)^t \mathscr{P} \pi^g_+ \;.
\end{aligned} \tag{3.1}$$

Since $\mathscr{J}_+ \pi^g_- = 0$, we write it as

$$\mathscr{J}_+ = \mathscr{J} \pi^g_+ \;, \tag{3.2}$$

where $\mathscr{J}$ is an almost complex structure on $\mathbb{T}$ as in (3.1). The second condition in (3.1) then implies the generalised Hermiticity condition

$$[\mathscr{J}, \mathscr{G}] = 0 \;, \tag{3.3}$$

and the first and third conditions in (3.1) then also hold.

[2]See [8] for the conditions for this to be possible in general.

4. A coordinate description

To describe the geometry, it is sometimes convenient to choose a coordinate basis $(\partial/\partial X^\mu, dX^\mu)$ for $\mathbb{T}$ and to write the elements of sections as

$$\varphi = \begin{pmatrix} v^\mu \\ \xi_\mu \end{pmatrix} \tag{4.1}$$

We then have a matrix description of our operators as

$$\mathscr{J} = \begin{pmatrix} I & P \\ L & K \end{pmatrix} , \quad \pi_+^g = \tfrac{1}{2}\begin{pmatrix} 1 & g^{-1} \\ g & 1 \end{pmatrix} , \quad \mathscr{P} = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2}$$

where

$$\begin{aligned} I = (I^\mu_{\ \nu}) :\ T\mathscr{M} \to T\mathscr{M} \ , &\qquad P = (P^{\mu\nu}) :\ T\mathscr{M} \to T^*\mathscr{M} \\ L = (L_{\mu\nu}) :\ T^*\mathscr{M} \to T\mathscr{M} \ , &\qquad K = (K_\mu^{\ \nu}) :\ T^*\mathscr{M} \to T^*\mathscr{M} \\ g^{-1} = (g^{\mu\nu}) :\ T\mathscr{M} \to T^*\mathscr{M} \ , &\qquad g = (g_{\mu\nu}) :\ T^*\mathscr{M} \to T\mathscr{M} \ . \end{aligned} \tag{4.3}$$

Using these, we find from (3.1) that $\mathscr{J}^2 = -1$ implies

$$\begin{aligned} I^2 + PL &= -1 \\ IP + PK &= 0 \\ LI + KL &= 0 \\ LP + K^2 &= -1 \ , \end{aligned} \tag{4.4}$$

and that $\mathscr{J}^t \mathscr{P} = -\mathscr{P}\mathscr{J}$ implies

$$\begin{aligned} P^t + P &= 0 \\ L^t + L &= 0 \\ I + K^t &= 0 \ . \end{aligned} \tag{4.5}$$

Finally, the condition (3.3) that $[\mathscr{J}, \mathscr{G}] = 0$ gives

$$\begin{aligned} Pg - g^{-1}L &= 0 \\ Kg - gI &= 0 \ . \end{aligned} \tag{4.6}$$

The conditions (4.4)-(4.6) may be summarised by saying that $\mathscr{J}$ may be written

$$\mathscr{J} = \begin{pmatrix} \hat{I} \mp Pg & P \\ gPg & -\hat{I}^t \pm gP \end{pmatrix} \tag{4.7}$$

where P is antisymmetric and $\hat{I}$ is an almost complex structure on $\mathscr{M}$ that preserves the metric:

$$P = -P^t \ , \quad \hat{I} := I \pm Pg \ , \quad (\hat{I})^2 = -1 \ , \quad (\hat{I})^t g \hat{I} = g \ . \tag{4.8}$$

It follows that the complex structure $\mathscr{J}_+$ on $\mathbb{T}_+$ is

$$\mathscr{J}_+ = \mathscr{J}\pi_+^g = \pi_+^g \mathscr{J}\pi_+^g = \tfrac{1}{2}\begin{pmatrix} \hat{I} & -(\hat{\omega})^{-1} \\ \hat{\omega} & -\hat{I}^t \end{pmatrix} \tag{4.9}$$

where $\hat{\omega} := g\hat{I}$.

5. Integrability

Using $\mathscr{J}_+$ from (4.9), we define another two projection operators

$$\pi^j_\pm := \tfrac{1}{2}(1 \pm i\,\mathscr{J}_+)\ . \tag{5.1}$$

This allows a further split

$$\mathbb{T}_+ = \mathbb{T}_+^{(1,0)} \oplus \mathbb{T}_+^{(0,1)} \tag{5.2}$$

where

$$\mathbb{T}_+^{(1,0)} = \pi^j_+ \mathbb{T}_+\ , \quad \mathbb{T}_+^{(0,1)} = \pi^j_- \mathbb{T}_+ \tag{5.3}$$

are $+i$ and $-i$ eigenspaces, respectively. We then require $\mathbb{T}_+^{(1,0)}$ to be involutive with respect to the $H-$twisted Courant bracket, which reads

$$\begin{aligned}&[\![\varphi_1, \varphi_2]\!]_H \\ &:= \begin{pmatrix} [v_1, v_2] \\ \mathscr{L}_{v_1} d\xi_2 - i_{v_2} d\xi_1 + i_{v_1} i_{v_2} H \end{pmatrix} = \begin{pmatrix} [v_1, v_2] \\ 2i_{v_{[1}} d\xi_{2]} + d(i_{v_1}\xi_2) + i_{v_1} i_{v_2} H \end{pmatrix}\ ,\end{aligned} \tag{5.4}$$

with $dH = 0$. When $\varphi_i \in \mathbb{T}_+^{(1,0)}$, they may be written

$$\varphi_i = \pi^j_+ \pi^g_+ \varphi_i\ , \quad \Longleftrightarrow \quad \varphi_i = \tfrac{1}{2}\begin{pmatrix} p_+ \hat{v}_i \\ g p_+ \hat{v}_i \end{pmatrix}\ . \tag{5.5}$$

where

$$\begin{aligned} p_+ &:= \tfrac{1}{2}(1 + i\hat{I}) \\ \hat{v}_i &:= v_i + g^{-1}\xi_i \end{aligned} \tag{5.6}$$

The involution conditions translate into

$$\pi^g_- [\![\varphi_1, \varphi_2]\!]_H = 0\ , \Longleftrightarrow \quad < \tilde{\varphi}_3, [\![\varphi_1, \varphi_2]\!]_H >= 0\ , \quad \tilde{\varphi}_3 \in \mathbb{T}_- \tag{5.7}$$

and

$$\pi^j_- [\![\varphi_1, \varphi_2]\!]_H = 0 \quad \Longleftrightarrow \quad < \tilde{\varphi}_3, [\![\varphi_1, \varphi_2]\!]_H >= 0\ , \quad \tilde{\varphi}_3 \in \mathbb{T}_+^{(0,1)} \tag{5.8}$$

to stay in $\mathbb{T}_+^{(1,0)}$. (Note that $[\pi^g, \pi^j] = 0$). The first condition leads to $\mathscr{J}_+$ being parallel, the second to the vanishing of the Nijenhuis tensor for I.

Letting $p_+\hat{v}_i =: w_i$, we learn from (5.7) that

$$[w_1, w_2]^\mu - g^{\mu\nu}\Big(2i_{w_{[1}} d(gw)_{2]} + d(i_{w_1} g w_2) + i_{w_1} i_{w_2} H\Big)_\nu = 0\ . \tag{5.9}$$

For $\varphi_i \in \mathbb{T}_+ : \xi_i = g v_i$ we have

$$\Big(2i_{v_{[1}} d(gv)_{2]} + d(i_{v_1} g v_2) + i_{v_1} i_{v_2} H\Big)_\mu = g_{\mu\kappa}[v_1, v_2]^\kappa + 2 v_{2\nu} \nabla^{(+)}_\mu v_1^\nu \tag{5.10}$$

where $\nabla^{(+)}_\mu$ is the covariant derivative with torsion

$$\nabla^{(+)}_\mu v_1^\nu = \nabla^{(0)}_\mu v_1^\nu + \tfrac{1}{2} H_{\mu\rho}{}^\nu v_1^\rho \ , \tag{5.11}$$

with $\nabla^{(0)}_\mu$ the Levi-Civita connection for g. Using this in (5.7) yields

$$w_{2\nu} \nabla^{(+)}_\mu w_1^\nu = 0 \ . \tag{5.12}$$

Since $p_+ g p_+ = 0$, we can peel off the $\hat{v}_i$s to conclude that

$$\nabla^{(+)}_\mu \hat{I}^\tau_\nu = 0 \ . \tag{5.13}$$

On $T\mathscr{M}$ the complex structure $\hat{I}$ is thus parallel with respect to this torsionful connection. From (5.8) we find

$$(\tfrac{1}{2} + p_-)[w_1, w_2] + \tfrac{i}{2}(\hat{\omega})^{-1}\Big(2 i_{w_{[1}} d(gw)_{_{2]}} + d(i_{w_1} g w_2) + i_{w_1} i_{w_2} H\Big) = 0 \ , \tag{5.14}$$

$$-\tfrac{i}{2}\hat{\omega}[w_1, w_2] + (\tfrac{1}{2} + p_+^t)\Big(2 i_{w_{[1}} d(gw)_{_{2]}} + d(i_{w_1} g w_2) + i_{w_1} i_{w_2} H\Big) = 0 \ . \tag{5.15}$$

From these and (5.9) it follows that

$$\begin{aligned} & p_-[w_1, w_2] = p_-[p_+\hat{v}_1, p_+\hat{v}_2] = 0 \ , \\ & p_- g^{-1}\Big(2 i_{w_{[1}} d(gw)_{_{2]}} + d(i_{w_1} g w_2) + i_{w_1} i_{w_2} H\Big) \qquad (5.16) \\ & \qquad = p_-[w_1, w_2] + 2 p_- i_{gw_2} g^{-1} \nabla^{(+)} w_1 = 0 \ . \qquad (5.17) \end{aligned}$$

The first of these is the integrability condition for $\hat{I}$ on $\mathscr{M}$:

$$\mathscr{N}(\hat{I}) = 0 \ , \tag{5.18}$$

where $\mathscr{N}$ is the Nijenhuis tensor. The second relation follows from the first and (5.12).

From the vanishing of the Nijenhuistensor, (5.18) in conjunction with the parallel condition (5.13), one derives the torsion relation

$$T_{\mu\nu\rho} = T_{\sigma\tau[\rho} \hat{I}^\sigma{}_\mu \hat{I}^\tau{}_{\nu]} \ , \tag{5.19}$$

which is equivalent to the final condition

$$d^c \hat{\omega} = H \ , \tag{5.20}$$

for $T = \frac{1}{2}H$. Note tha H is closed by assumption which means that

$$dd^c \hat{\omega} = 0 \ . \tag{5.21}$$

6. Generalised Kähler Geometry

Generalised Kähler Geometry (GKG) is the target space geometry of $(2,2)$ sigma models. It is determined by two complex structures $\hat{I}^{(\pm)}$, a metric g which is hermitian with respect to both of these and a closed three form H which enters the conditions for integrability. The relation to GCG is given by the Gualtieri map [4]

$$\mathscr{J}^{(1,2)} = \frac{1}{2}\begin{pmatrix} \hat{I}^{(+)} \pm \hat{I}^{(-)} & -(\omega_{(+)}^{-1} \mp \omega_{(-)}^{-1}) \\ \omega_{(+)} \mp \omega_{(-)} & -(\hat{I}^{t(+)} \pm \hat{I}^{t(-)}) \end{pmatrix} \tag{6.1}$$

where $\mathscr{J}^{(1)}$ and $\mathscr{J}^{(2)}$ are two commuting Generalised Complex Structures with integrability defined with respect to the H-twisted Courant bracket (5.4). GKG has been extensively studied in the context of sigma models [12]-[17], but here we just want to elucidate the relation to the half generalised complex structures discussed above.

Half generalised complex structures can be defined on $\mathbb{T}_-$ in the same way as described for $\mathbb{T}_+$. Assume that $\mathscr{J}_+$ is defined on $\mathbb{T}_+$ and takes the form (4.9)

$$\mathscr{J}_+ = \mathscr{J}\pi_+^g = \pi_+^g\mathscr{J}\pi_+^g = \frac{1}{2}\begin{pmatrix} \hat{I}^{(+)} & -(\hat{\omega}_{(+)})^{-1} \\ \hat{\omega}_{(+)} & -\hat{I}^{(+)t} \end{pmatrix}, \tag{6.2}$$

and $\mathscr{J}_-$ is defined on $\mathbb{T}_-$

$$\mathscr{J}_- = \tilde{\mathscr{J}}\pi_-^g = \pi_-^g\tilde{\mathscr{J}}\pi_-^g = \frac{1}{2}\begin{pmatrix} \hat{I}^{(-)} & -(\hat{\omega}_{(-)})^{-1} \\ \hat{\omega}_{(-)} & -\hat{I}^{(-)t} \end{pmatrix}, \tag{6.3}$$

both integrable with respect to the same H-twisted Courant bracket. Their sum and difference then yield precisely (6.1)

$$\mathscr{J}^{(1,2)} = \pi_+^g\mathscr{J}\pi_+^g \pm \pi_-^g\tilde{\mathscr{J}}\pi_-^g\ . \tag{6.4}$$

At the sigma model level this is mirrored by the fact that a $(2,2)$ model can be thought of as the sum of a $(2,0)$ and a $(0,2)$ model.

7. Conclusions

We have briefly described how SKT geometry fits into Generalised Complex Geometry as half generalised structures, related it to $(2,0)$ and $(2,1)$ sigma model target space geometry and shown how two half generalised complex structures can give rise to the Generalised Kähler Geometry of $(2,2)$ sigma models. These considerations generalise to (p,q) supersymmetric models and their target space geometries as shown in [1].

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