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Pitch-Angle Diffusion and Bohm-type Approximations in Diffusive Shock Acceleration

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The problem of accelerating cosmic rays is one of fundamental importance, particularly given the uncertainty in the conditions inside the acceleration sites. Here we examine Diffusive Shock Acceleration in arbitrary turbulent magnetic fields, constructing a new model that is capable of bridging the gap between the very weak ($\delta B/B_0 \ll 1$) and the strong turbulence regimes. To describe the diffusion we provide quantitative analytical description of the "Bohm exponent" in each regime. We show that our results converge to the well known quasi-linear theory in the weak turbulence regime. In the strong regime, we quantify the limitations of the Bohm-type models. Furthermore, our results account for the anomalous diffusive behaviour which has been noted previously.

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1. Introduction

Diffusive Shock Acceleration (DSA), also known as first-order Fermi acceleration is a leading model in explaining the acceleration of particles and production of cosmic rays (CRs) in various astronomical objects [1, 2, 3, 4]. In this model particles gain energy by repeatedly crossing a shock wave by elastically reflecting from magnetic turbulence on each side. An E^{-2} energy spectrum consistent with CR particle observations on earth is produced (e.g., [3, 5] and references therein).

Previous studies of DSA can broadly be divided into three categories. The first is the **semi-analytic approach** (e.g. [6, 7, 8]), in which the particles are described in terms of distribution functions, enabling analytic or numerical solution of the transport equations. While this is the fastest method, reliable models only exist in a very limited parameter range (weak turbulence, small-angle scattering, weakly anisotropic, etc.). Furthermore a heuristic prescription for the diffusion is required. The second is the **Monte-Carlo approach** (e.g. [4, 9, 10, 11]), in which the trajectories and properties of representative particles are tracked and the average background magnetic fields are estimated. The advantage of this method is that it enables the study of a large parameter space region, and is very fast and therefore can be used to track the particle trajectories over the entire region where the acceleration is believed to occur [12]. On the other hand this method uses simplifying assumptions about the structure of the magnetic fields and the details of their interaction with the particles. For example, several existing Monte-Carlo codes [13] use scattering models which are either limited to weak turbulence, such as quasilinear theory (QLT; see [14, 15] and further discussion below), or are not well supported theoretically, such as the Bohm type [16]. The third approach is **particle-in-cell (PIC) simulations** [17, 18, 19, 20]. These codes simultaneously solve for particle trajectories and electromagnetic fields in a fully self-consistent way. They therefore provide full treament of particle acceleration, magnetic turbulence and formation of shocks. However, existing codes are prohibitevely expensive computationally and are therefore limited to very small ranges in time and space, typically many orders of magnitude less than the regime in which particles are believed to be accelerated [21].

Of the three approaches the one that currently seems best applicable to astrophysical environments is the Monte-Carlo approach. Analytic techniques quickly become unwieldy when trying to account for e.g. strong turbulence, oblique shocks or plasma instabilites which develop under different conditions [22, 10]. On the other hand, the computational power required for carrying out a PIC simulation over the full dynamical range is not expected to be available for many years. While the Monte-Carlo approach also suffers several weaknesses, some of these weakness can be treated with reasonable computational time.

At the heart of the Monte-Carlo approach lies a description of the particle-field interaction. Various authors use various prescriptions (e.g. [13, 23]), which rely on very different assumptions. In the present work [24] we examine and quantify the validity of the two most frequently used of these assumptions, namely the quasi-linear theory (QLT) and Bohm diffusion approximation.

2. Quantifying particle-field interactions in Monte-Carlo codes

Particle-field interaction occurs continuously along the particle's trajectory. However, in order to carry a numerical computation it needs to be discretized.

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Many monte-Carlo codes (e.g. [4, 9, 10, 11]) typically consider an idealised scenario, where energy changes and local spatial variations are neglected. The wave-particle interaction is determined by a single quantity, the particle's *pitch angle* ϑ, i.e. the angle its velocity vector makes with the direction of the background magnetic field. As the particle undergoes "scattering" from the (turbulent) magnetic field, its pitch angle, $\mu = \cos\vartheta$ changes in a stochastic way. Studies of this type typically treat the pitch angle as undergoing a random walk.

Analytic work has mainly centred on the "quasilinear" family of approximations (**QLT**), originally formulated by [14], in which the deviation from helical orbits is treated perturbatively (e.g., [25, 15]) by averaging out wave contributions over many gyrotimes. The turbulent nature of the magnetic field is quantified by the turbulence ratio $\delta B/B_0$, where B_0 is the strength of the guiding field. A first-order approximation in $\delta B/B_0$ around B_0 has been shown to give an accurate description of particle motion in the weak regime, $\delta B/B_0 \ll 1$. However, no observational evidence exist that this is indeed the regime that operates in many objects in nature. Additionally, QLT approximations result in a resonance condition, in which the particles interact only with a resonant portion of the magnetic turbulent wave spectrum ($k \approx 1/r_g$ for wavenumber k and gyroradius r_g), though this is not necessarily the case [26].

A second approach that is in wide use by many large scale Monte-Carlo simulations (e.g. [22, 12]) is the **Bohm Diffusion** approximation. In this approximation the particle's motion is described as undergoing a series of discrete, isotropic scatterings. In contrast to QLT this approach does not account for pitch-angle dependence of scatterings. Rather, in this model the mean free path $\lambda_{\rm mfp}$ between scattering assumes the form

$$\lambda_{\rm mfp} = \eta r_g^{\alpha} \tag{2.1}$$

where the Bohm exponent α is a free parameter whose value is unknown and is often taken as unity [27], η is a coupling constant and r_g is the particle's gyration radius. There is some numerical support for the validity of this model in the context of DSA, in particular in the intermediate turbulence regime, $\delta B \approx B_0$ [16] and possibly the strong turbulence ($\delta B \gtrsim B_0$) as well [28, 29].

3. New approach to study particle-field interaction: model and methods

In order to provide a better description of particle-field interactions, as well as to quantify the validity and limitations of each of the approximations used, we developed a new code that follow particle trajectories in various turbulent magnetic field background. The magnetic fields are calculated at every timestep at the particle's current spatial location, rather than being evaluated on a grid at the beginning of the simulation (as in previous studies). The advantages of this method are that 1) it makes the spatial resolution effectively continuous and 2) it facilitates modifying the turbulence spectrum during particle motion. The disadvantage is the higher cost of performance.

Initial populations of waves and particles are prescribed and the total magnetic field $\mathbf{B}$ is calculated as a function of position by summing the contribution $\delta\mathbf{B}$ of each wave, along with the background field $\mathbf{B}_0$. The turbulent part of the field is found by summing over a discrete population of waves at each position x^j,

$$\delta\mathbf{B} = \sum_{\rm waves} A_k e^{i\left(k_j x^j + \phi_k\right)} \mathbf{n}. \tag{3.1}$$

Here A_k is the amplitude of the wave with wavenumber k, k_i is its wavevector, $\mathbf{n}$ is its polarisation vector and ϕ_k is its phase. The phases and polarisation angle are chosen randomly from a uniform distribution on $[0, 2\pi]$. We further distinguish waves having $\mathbf{k} = \left(0, 0, k_{\parallel}\right)$ as *slab* waves and $\mathbf{k} = k_{\perp}\left(\cos\vartheta_{\perp}, \sin\vartheta_{\perp}, 0\right)$ for some angle $\vartheta_{\perp}$, as $2d$-waves. Turbulence containing both kinds of wave is said to be *composite*. We denote he ratio of energies in each type of wave by $r_{\rm slab} \equiv \delta B^2_{\rm slab}/\delta B^2_{\rm total}$.

For the spectrum of the waves we choose a general smoothly broken power law form, as proposed by [30]. Such a form naturally incorporates e.g. Kolmogorov and Goldreich-Sridhar turbulence as special cases [31, 32].

Numerical setup. In running the simulations, we assume that the wavenumbers are uniformly distributed in log-space ($\Delta \ln k = \Delta k/k$ is constant) between the minimum and maximum $k_{min} = 10^{-4}$ and $k_{max} = 10^6$ respectively. These values are chosen so as to allow resonant interaction at most values of μ. Particles are initially uniformly distributed in μ-space ensuring they interact with different parts of the spectrum. Their initial velocity is chosen to be $v = 0.1c$. The number of waves and particles per seed is $n_w = 4096$ and $n_p = 256$ respectively. The number of random seeds corresponding to distinct turbulence realisations for ensemble average $n_s = 8$, which was found to be enough to achieve convergence. The integrator used is `bulirsch-stoer` from `odeint` with relative and absolute tolerance $\varepsilon_{rel} = \varepsilon_{abs} = 10^{-9}$. The particle trajectories are tracked and the scattering time t_s and pitch-angle diffusion coefficient $D_{\mu\mu}$ are calculated.

4. Results and discussion

4.1 On the validity of the diffusion approximation and limits of the QLT model

A key unerlying assumption of both the QLT and the Bohm approximations is that the evolution of the particles pitch angle is well described by a diffusion equation in μ-space, with diffusion coefficient $D_{\mu\mu} \equiv \left\langle \frac{(\Delta\mu)^2}{\Delta t} \right\rangle$, where $\Delta\mu = \mu(t) - \mu(0)$ [see Ref. [25] for discussion on $D_{\mu\mu}$].

Of particular importance here is the time scale Δt over which $\Delta\mu$ is measured. In the limit $\Delta t \to 0$ the diffusion coefficient approaches zero, regardless of the details of the diffusion, because the numerator in $D_{\mu\mu}$ is second order in Δt, while the denominator is only first-order. On the other hand the value of Δt cannot be too long. Since $\Delta\mu$ can be at most 2 an arbitrarily large value of Δt causes $D_{\mu\mu}$ to vanish. We thus conclude that in order for the QLT approximation to be valid, one must consider time step $t_g \ll \Delta t \ll t_D$, and Bohm requires $t_g \lesssim \Delta t \ll t_D$. Here, t_g is the gyrotime, and $t_D = 1/\left\langle D_{\mu\mu} \right\rangle$ is the diffusion time. This is demonstrated in Figure 1, where we plot $D_{\mu\mu}$ as a function of Δt for various turbulence levels. Only in the green (shaded) region is the diffusion approximation valid.

In Figure 2, we show the diffusion coefficient as a function of turbulence level. It initially increases as δB^2 (QLT regime), gradually flattens around $\delta B/B_0 = 10^{-1}$ and stays roughly constant thereafter. We argue that this flattening is not physical but an artifact of the fact that we are measuring a bounded quantity $\Delta\mu$ over a time period longer than its dynamical time $1/D_{\mu\mu}$. We thus find that the QLT approxmation is valid as long as the turbulence level is $\delta B/B_0 \leq 10^{-1}$.

4.2 Validity of the Bohm approximation

One can model the diffusion of charged particles as a power law relationship between mean

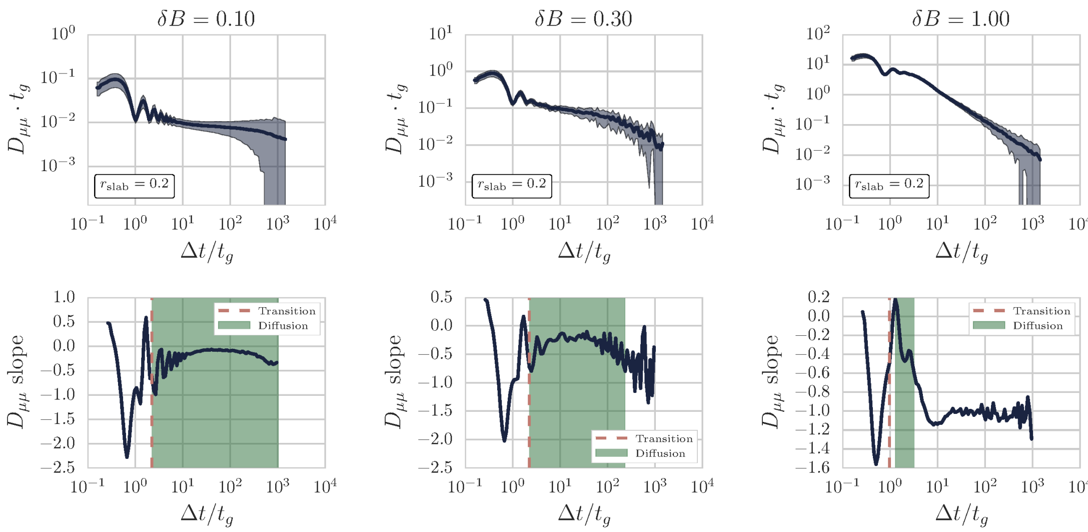

Figure 1: The upper plots show $D_{\mu\mu}$ (pitch-angle averaged) as a function of Δt for $r_{\mathrm{slab}} = 0.2$, with the solid line for the particle average and the shaded area representing one standard deviation. The lower plots show the average slope of $D_{\mu\mu}$ as a function of Δt, highlighting the transition from ballistic to subdiffusive behaviour ($\frac{d\ln D_{\mu\mu}}{d\ln \Delta t} < 0$, dashed red line), and the diffusive regime ($\frac{d\ln D_{\mu\mu}}{d\ln \Delta t} \approx 0$, green shading). The diffusive range shrinks as the turbulence strength increases. These plots are normalized to $B_0 = 1$.

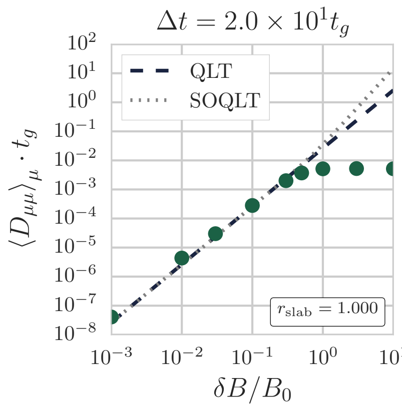

Figure 2: $D_{\mu\mu}$ (pitch-angle averaged) measured at $\Delta t = 20t_g$ for various (slab-only) turbulence levels. Good agreement with the (second order) QLT models for weak turbulence is found. For turbulence levels $\delta B \gtrsim 1$ the measurement time Δt of $20t_g$ is no longer within the diffusion regime (see Figure 1) and so the values are no longer meaningful.

free path λ_{mfp} and gyroradius [33] (see Equation 2.1). Here, λ_{mfp} refers to the expected value of the distance travelled by a particle in the time it takes for ϑ to change by $\pi/2$ [4, 10].

Considering the "Bohm approximation" to mean $\alpha \approx 1$ in Equation 2.1, implies a single scattering every gyration time (as long as η is of the order unity). This ratio is presented in Figure 3 (left). From the figure we see that this form of Bohm approximation is valid only around $\delta B/B_0 \approx 1$ for the unmodified gyrotime t_g. In [24] we provide analytical fits to this function, for various types of turbulence.

It is clear from Figure 3 (left), that the value of α is in fact very different than unity, except around $\delta B/B_0 \approx 1$. Hence, the "classical" Bohm diffusion model has a very limited valdity. In order to extend the valdity range of the "Bohm"-type models, we defined the "auxiliary Bohm exponent", $\tilde{\alpha}$, via

$$\alpha = \tilde{\alpha}\left(1 + \left(\frac{\delta B}{B_0}\right)^{-2}\right). \tag{4.1}$$

With this definition, $\tilde{\alpha}$ is the slope of the data presented in the left side of Figure 3. The values of $\tilde{\alpha}$ are shown in Figure 3 (right), for various values of the turbulence levels.

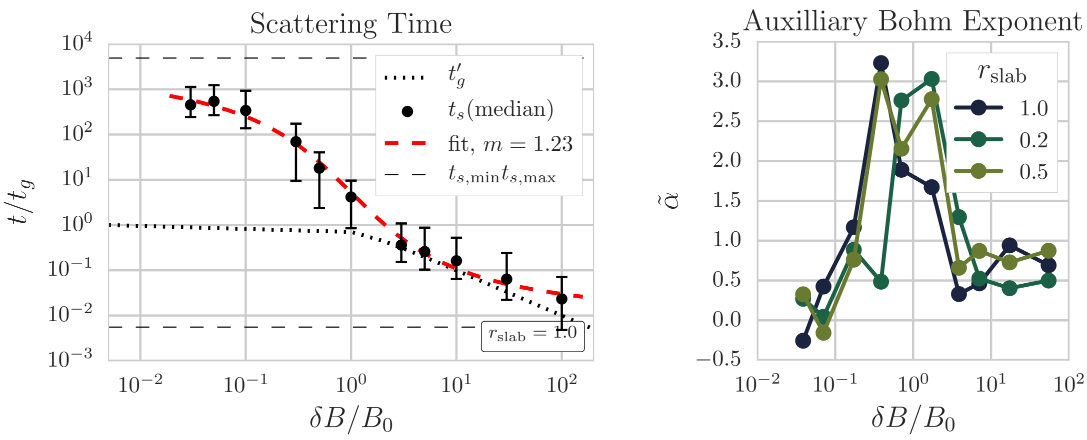

Figure 3: Left: scattering time t_s as a function of turbulence level $\delta B/B_0$ with $r_{slab} = 1.0$. Here, t'_g represents the "effective" gyrotime determined by the effective field, $B_{eff} = \sqrt{B_0^2 + \delta B^2}$. Points represent the ensemble and pitch-angle median of the measured scattering times for each turbulence level, and the error bars the first and third quartile. The red line is an analytical fit, whose parameters can be found in [24]. **Right:** Auxillary Bohm exponent as a function of turbulence level for several values of r_{slab}. It shows how a Bohm-type model can interpolate from weak up to intermediate and strong turbulence, by choosing the approximate values $\tilde{\alpha} = 0$, 2.5 and 0.7 for these regions respectively.

4.3 Concluding remarks

While a comprehensive model of CR transport in accelerators is necessary for understanding the origins of high-energy CRs, existing diffusion models are limited and may not cover some relevant ranges of parameters. This is because current analytic approaches (QLT, Bohm) rely on approximations that are invalid in important turbulence regimes.

The applicability of the diffusion model depends on the time step / measuring time Δt, the choice of which depends on several factors. It is bounded below by the wave crossing time, the gyrotime, and also the dynamical timescale for other relevant phenomena (e.g., plasma instabilities) and is bounded above by the diffusion time. For strong turbulence, therefore, there may be no region in which a valid Δt exists. In the absence of such a Δt, it is not meaningful to treat the problem as diffusive, and more sophisticated models, e.g., anomalous diffusion, must be used.

The Bohm approximation, while generally applied for its convenience, has been shown to be generally inapplicable to the case of diffusion in collisionless plasmas of the type described here. We propose a modification to this model, namely anomalous diffusion, and measured the anomalous diffusion exponent $\tilde{\alpha}(\delta B)$ (Figure 2). The prescription we provide is useful for next generation MC codes of particle acceleration.

References

[1] E. Fermi, *On the Origin of the Cosmic Radiation*, *Physical Review* **75** (1949) 1169.

[2] A. R. Bell, *The acceleration of cosmic rays in shock fronts. I*, *Mon. Not. R. Astron. Soc.* **182** (1978) 147.

[3] R. Blandford and D. Eichler, *Particle acceleration at astrophysical shocks: A theory of cosmic ray origin*, *Phys. Rep.* **154** (1987) 1.

[4] D. C. Ellison, S. P. Reynolds and F. C. Jones, *First-order Fermi particle acceleration by relativistic shocks*, *Astrophys. J.* **360** (1990) 702.

[5] C. D. Dermer and G. Menon, *High Energy Radiation from Black Holes: Gamma Rays, Cosmic Rays, and Neutrinos*. 2009.

[6] J. G. Kirk and A. F. Heavens, *Particle acceleration at oblique shock fronts*, *Mon. Not. R. Astron. Soc.* **239** (1989) 995.

[7] E. Amato and P. Blasi, *A general solution to non-linear particle acceleration at non-relativistic shock waves*, *Mon. Not. R. Astron. Soc.* **364** (2005) L76 [astro-ph/0509673].

[8] D. Caprioli, E. Amato and P. Blasi, *Non-linear diffusive shock acceleration with free-escape boundary*, *Astroparticle Physics* **33** (2010) 307 [0912.2714].

[9] D. C. Ellison and G. P. Double, *Nonlinear particle acceleration in relativistic shocks*, *Astroparticle Physics* **18** (2002) 213 [astro-ph/0202106].

[10] E. J. Summerlin and M. G. Baring, *Diffusive Acceleration of Particles at Oblique, Relativistic, Magnetohydrodynamic Shocks*, *Astrophys. J.* **745** (2012) 63 [1110.5968].

[11] A. M. Bykov, D. C. Ellison and S. M. Osipov, *Nonlinear Monte Carlo model of superdiffusive shock acceleration with magnetic field amplification*, *Physical Review E* **95** (2017) 033207 [1703.01160].

[12] D. C. Ellison, D. C. Warren and A. M. Bykov, *Monte Carlo Simulations of Nonlinear Particle Acceleration in Parallel Trans-relativistic Shocks*, *Astrophys. J.* **776** (2013) 46 [1308.5114].

[13] A. Vladimirov, D. C. Ellison and A. Bykov, *Nonlinear Diffusive Shock Acceleration with Magnetic Field Amplification*, *Astrophys. J.* **652** (2006) 1246 [arXiv:astro-ph/0606433].

[14] J. R. Jokipii, *Cosmic-Ray Propagation. I. Charged Particles in a Random Magnetic Field*, *Astrophys. J.* **146** (1966) 480.

[15] *Nonlinear Cosmic Ray Diffusion Theories*, vol. 362, Jan, 2009. 10.1007/978-3-642-00309-7.

[16] F. Casse, M. Lemoine and G. Pelletier, *Transport of cosmic rays in chaotic magnetic fields*, *Physical Review D* **65** (2002) 023002 [astro-ph/0109223].

[17] C. K. Birdsall and A. B. Langdon, *Plasma Physics via Computer Simulation*. 1991.

[18] L. O. Silva, R. A. Fonseca, J. W. Tonge, J. M. Dawson, W. B. Mori and M. V. Medvedev, *Interpenetrating Plasma Shells: Near-equipartition Magnetic Field Generation and Nonthermal Particle Acceleration*, *Astrophys. J.* **596** (2003) L121 [arXiv:astro-ph/0307500].

[19] A. Spitkovsky, *Particle Acceleration in Relativistic Collisionless Shocks: Fermi Process at Last?*, *Astrophys. J.* **682** (2008) L5 [0802.3216].

[20] X.-N. Bai, D. Caprioli, L. Sironi and A. Spitkovsky, *Magnetohydrodynamic-particle-in-cell Method for Coupling Cosmic Rays with a Thermal Plasma: Application to Non-relativistic Shocks*, *Astrophys. J.* **809** (2015) 55 [1412.1087].

[21] A. Vladimirov, *Modeling magnetic field amplification in nonlinear diffusive shock acceleration*, Ph.D. thesis, North Carolina State University, Feb, 2009.

[22] D. Caprioli, H. Kang, A. E. Vladimirov and T. W. Jones, *Comparison of different methods for non-linear diffusive shock acceleration*, *Mon. Not. R. Astron. Soc.* **407** (2010) 1773 [1005.2127].

[23] R. C. Tautz, A. Dosch, F. Effenberger, H. Fichtner and A. Kopp, *Pitch-angle scattering in magnetostatic turbulence. I. Test-particle simulations and the validity of analytical results*, *Astron. Astrophys.* **558** (2013) A147 [1309.7836].

[24] J. D. Riordan and A. Pe'er, *Pitch-angle Diffusion and Bohm-type Approximations in Diffusive Shock Acceleration*, *Astrophys. J.* **873** (2019) 13 [1810.11817].

[25] R. Schlickeiser, *Cosmic Ray Astrophysics*. 2002.

[26] Y. Li, P. H. Yoon, C. S. Wu, A. T. Weatherwax, J. K. Chao and B. H. Wu, *Ion pitch-angle scattering by Alfvén waves*, *Physics of Plasmas* **4** (1997) 4103.

[27] M. G. Baring, *Probes of Diffusive Shock Acceleration using Gamma-Ray Burst Prompt Emission*, in *American Institute of Physics Conference Series*, C. Meegan, C. Kouveliotou and N. Gehrels, eds., vol. 1133 of *American Institute of Physics Conference Series*, pp. 294–299, May, 2009, 0901.2535, DOI.

[28] A. Shalchi, *Diffusive shock acceleration in supernova remnants: On the validity of the Bohm limit*, *Astroparticle Physics* **31** (2009) 237.

[29] M. Hussein and A. Shalchi, *Detailed Numerical Investigation of the Bohm Limit in Cosmic Ray Diffusion Theory*, *Astrophys. J.* **785** (2014) 31.

[30] A. Shalchi and B. Weinhorst, *Random walk of magnetic field lines: Subdiffusive, diffusive, and superdiffusive regimes*, *Advances in Space Research* **43** (2009) 1429.

[31] R. C. Tautz and A. Shalchi, *Numerical Test of Improved Nonlinear Guiding Center Theories*, *Astrophys. J.* **735** (2011) 92.

[32] M. Hussein, R. C. Tautz and A. Shalchi, *The influence of different turbulence models on the diffusion coefficients of energetic particles*, *Journal of Geophysical Research (Space Physics)* **120** (2015) 4095 [1505.05099].

[33] P. Pommois, G. Zimbardo and P. Veltri, *Anomalous, non-Gaussian transport of charged particles in anisotropic magnetic turbulence*, *Physics of Plasmas* **14** (2007) 012311.

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