

Meson interactions at large N_c from Lattice QCD

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We report on the computation of the scaling of QCD observables with the number of colours, N_c . For this, we use dynamical configurations with four active flavours, $N_f = 4$, and values of $N_c = 3 - 6$. We study the meson masses and decay constants, and compute the leading and subleading contributions to the Low Energy Constants (LECs) of the chiral Lagrangian. We also explore $\pi\pi$ scattering in the $I = 2$ channel, and compute the $K \rightarrow \pi$ weak decay matrix elements. We comment on the relation of the latter to $K \rightarrow \pi\pi$ processes and the $\Delta I = 1/2$ rule.

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1. Introduction

The 't Hooft limit of QCD [1], that is the limit of infinite number of colors, N_c , constitutes a simplification of the theory of the strong interactions that preserves most of its non-trivial properties, such as confinement or spontaneous chiral symmetry breaking. The large N_c limit of QCD is a powerful approximation. For instance, it has been able to predict the hierarchy of the low energy constants (LECs) of the chiral Lagrangian. Therefore, many other phenomenological approaches to QCD often use approximations inspired by the large N_c limit. Still, there are some exceptions where large N_c fails. The most striking being the ratio of isospin amplitudes in the $K \rightarrow (\pi\pi)_I$ weak decay, in which the two pions can be in an isoscalar ($I = 0$) or isotensor ($I = 2$) final state. This ratio, A_0/A_2 , is measured to be ~ 22 , although large N_c predicts it to be $\sqrt{2}$. This is the so-called “ $\Delta I = 1/2$ puzzle”, whose solution remains elusive.

Lattice Field Theory (LFT) offers the possibility of *ab initio* explorations of generic gauge theories, with arbitrary numbers of colours and flavours — see Ref. [2] for a review and Refs. [3, 4] for benchmark calculations. In this context, simulating QCD at a different values of N_c has a considerable amount of yet unexplored potential. In particular, the study of the N_c -scaling of QCD observables can help to improve phenomenological predictions. An illustrative example can be found in Ref. [5], where the systematic error induced by the large N_c limit completely dominates the total uncertainty. In addition, a nonperturbative study of the N_c -scaling of kaon decay amplitudes may reveal hidden features of QCD at the origin of the $\Delta I = 1/2$ rule.¹

Our project consists in the computation of physical observables at different values of N_c [9, 10, 11]. Specifically, we are currently focusing on meson masses and decay constants, kaon decays and meson-meson scattering. In this talk, we aim at summarizing the current status of the project and present several results — some of which are preliminary — for the aforementioned observables.

2. Simulations at large N_c

We are using configurations with $N_f = 4$ and $N_c = 3 - 6$ generated with HiRep [12]. We use the Iwasaki gauge action, and $O(a)$ -improved Wilson fermions. A summary of the values of the simulation parameters can be found in Ref. [11]. We use the gradient flow scale t_0 (see Ref. [13]) for the scale setting [10, 11]. This way, the lattice spacing is kept constant across all values of N_c , and it is $a \sim 0.075$ fm. Furthermore, we use a mixed-action setup, that is, twisted mass fermions at maximal twist in the valence sector. This has several advantages:

- (i) The computation of the meson decay constant requires no renormalization constant. It can be obtained from the bare pseudoscalar correlation function and the value of the bare twisted mass μ_0 , via the PCAC relation:

$$C(t) = \langle P(t)P(0) \rangle \xrightarrow{T \gg t \gg 1} |\langle 0|P|\pi \rangle_0|^2 e^{-M_\pi t}, \quad F_\pi = \frac{2\mu_0 \langle 0|P|\pi \rangle_0}{M_\pi^2}. \quad (2.1)$$

- (ii) The evaluation of weak matrix elements using twisted mass fermions at maximal twist helps alleviating renormalization problems, e.g., mixing with wrong-chirality operators (see Ref. [14]).

¹Lattice QCD attempts have observed the experimental enhancement although with large error bars [6, 7, 8].

- (iii) Several observables of interest with twisted mass in the valence sector show smaller statistical fluctuations. In addition, the twisted mass formulation prevents the appearance of linear cutoff effects in F_π and in the weak matrix elements.

3. Large N_c scaling of meson masses and decay constants

The computation of the large N_c scaling of meson masses and decay constants was recently published in Ref. [11]. In this section, we aim to summarize the approach and highlight some relevant results. We start by pointing out that the chiral perturbation theory (ChPT) description of pseudoscalar mesons is valid for an arbitrary number of colors². The prediction of next-to-leading order (NLO) ChPT with N_f degenerate flavours for the non-singlet meson mass and decay constant are [15]:

$$F_\pi = F \left[1 - \frac{N_f}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \frac{M_\pi^2}{\mu^2} + 4 \frac{M_\pi^2}{F_\pi^2} (L_5^r + N_f L_4^r) \right], \quad (3.1)$$

$$M_\pi^2 = 2Bm \left[1 + \frac{1}{N_f} \frac{M_\pi^2}{(4\pi F_\pi)^2} \log \frac{M_\pi^2}{\mu^2} + 8 \frac{M_\pi^2}{F_\pi^2} (N_f(2L_6^r - L_4^r) + 2L_8^r - L_5^r) \right], \quad (3.2)$$

in terms of the leading order (LO) couplings, B, F , and the NLO Gasser-Leutwyler coefficients, $L_{4,5,6,8}^r(\mu)$, defined at the renormalization scale μ . Even though there is no explicit N_c dependence in Eqs. (3.1-3.2), the LECs scale with N_c as:

$$O(N_c) : F^2, L_5, L_8; \quad O(1) : B, L_4, L_6. \quad (3.3)$$

Loop corrections are suppressed in $1/F_\pi^2 = O(1/N_c)$, and hence the loop expansion is expected to converge better at larger N_c . Keeping only leading and subleading dependence on N_c for the LECs, convenient parametrizations are:

$$F = \sqrt{N_c} \left(F_0 + \frac{F_1}{N_c} \right), \quad B = B_0 + \frac{B_1}{N_c}, \quad (3.4)$$

$$L_5 + N_f L_4 \equiv L_F = N_c L_F^{(0)} + L_F^{(1)}, \quad 2L_8 - L_5 + N_f(2L_6 - L_4) \equiv L_M = N_c L_M^{(0)} + L_M^{(1)}. \quad (3.5)$$

Note that according to the scaling of Eq. (3.3) and the definition of Eq. (3.5):

$$L_F^{(0)} = \frac{L_5}{N_c} + \mathcal{O}\left(\frac{1}{N_c}\right), \quad L_M^{(0)} = \frac{2L_8 - L_5}{N_c} + \mathcal{O}\left(\frac{1}{N_c}\right). \quad (3.6)$$

We have performed fits to the expressions in Eqs. (3.2) and (3.1) at fixed values of N_c . The results are presented in Fig. 1. As can be seen, Eqs. (3.4) and (3.5) seem to describe the scaling properly, with the exception of the decay constant at $N_c = 3$, where subleading effects seem larger.

Once we have checked the ansätze from Eqs. (3.4) and (3.5), it is better to perform a simultaneous chiral and N_c fit. In this fit, the number of parameters is small in comparison to the number of degrees of freedom. For the decay constant, this yields:

$$\frac{F}{\sqrt{N_c}} = \left(67(3) - 26(4) \frac{N_f}{N_c} \right) (3\%)^a \text{ MeV}, \quad \frac{L_F(\mu)}{N_c} \cdot 10^3 = 0.1(4) + 0.6(3) \frac{N_f}{N_c}, \quad (3.7)$$

²Two-coloured QCD is the exception as mesons and baryons have the same quark content, see Ref. [16] for recent calculations in this theory.

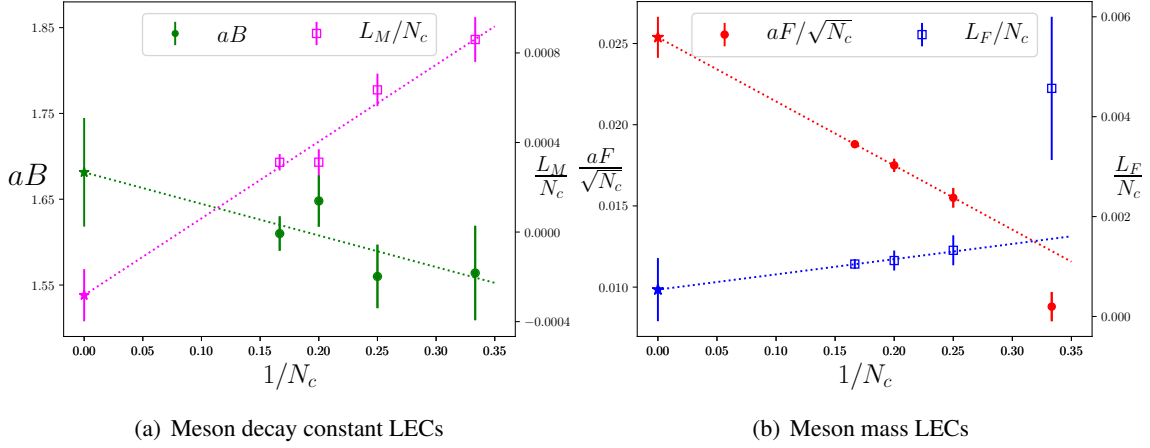


Figure 1: N_c -scaling of the LECs of the chiral Lagrangian with the number of colors for the non-singlet pseudoscalar meson decay constant (left) and mass (right).

with a 3% of scale setting error, and where we have assumed the N_f dependence, as argued in Ref. [11]. From the previous fit results, we can infer results for QCD with $N_f = 3$:

$$F^{N_c=3, N_f=3} = 71(3) \text{ MeV}, \quad L_M^{N_c=3, N_f=3} = 2.1(3) \cdot 10^{-3}, \quad (3.8)$$

which are in agreement with other phenomenological and lattice determinations — See Ref. [17] for a summary of current results.

We conclude this section by mentioning that in the large N_c limit, the flavour singlet (η') becomes a Goldstone boson. Thus, one should also repeat the analysis using $U(N_f)$ ChPT. We refer the reader to our publication for more details [11]. In addition, the mass and decay constants of the flavour singlet are currently under exploration.

4. $I=2$ $\pi\pi$ scattering in the large N_c limit

As mentioned above, the large N_c limit preserves quark confinement and chiral symmetry breaking. However, the interactions between mesons become weaker as N_c increases. Thus, QCD at large N_c is made up of infinitely narrow non-interacting resonances. The goal of this section is the study of the least complicated two-hadron interaction in the large N_c limit: $\pi\pi$ scattering at maximal isospin, $I = 2$.

Within lattice QCD, the study of two-particle interactions is performed using the Lüscher method [18]. It is an indirect way of obtaining two-particle scattering amplitudes from the spectrum of a theory in finite volume. In its simplest form, the Lüscher formalism relates the s -wave scattering length, a_0 , of two identical (pseudo)scalar particles to the energy shift of the two-particle ground state in finite volume. This expression, valid only for $L \gg a_0$, is called the threshold expansion and takes the form:

$$\Delta E = E_{\pi\pi} - 2M_\pi = -\frac{4\pi a_0}{mL^3} \left[1 + c_1 \frac{a_0}{L} + c_2 \left(\frac{a_0}{L} \right)^2 \right] + O(L^{-6}), \quad (4.1)$$

with c_1, c_2 numerical constants (see Ref. [18]).

The low-energy pion interactions are in general well-described by ChPT. The leading order prediction for the $I = 2$ channel is given by:

$$M_\pi a_0^{I=2} = -\frac{M_\pi^2}{16\pi F_\pi^2} + O\left[\frac{M_\pi^4}{(4\pi F_\pi)^4}\right], \quad (4.2)$$

and thus, by knowing that $F_\pi^2 \sim O(N_c)$ and $M_\pi \sim O(1)$, we can easily extract the leading N_c -scaling:

$$M_\pi a_0 \propto \frac{1}{N_c} + O(N_c^{-2}). \quad (4.3)$$

The results for $a_0^{I=2}$ in our 16 ensembles can be found in Fig. 2. We display our results by multiplying both the x - and y -axis with the leading N_c dependence, so that points at different values of N_c should lie on a universal line up to subleading $1/N_c$ corrections. As it can be seen, our results show excellent agreement with leading order ChPT even at pion masses of $M_\pi = 560$ MeV, corresponding to the heaviest point in Fig. 2.

The NLO prediction of ChPT for this scattering channel exists and includes the NLO LECs of the chiral Lagrangian. In a next step of our work, the data of Fig. 2 will be used to try to extract the N_c -scaling of the LECs in a similar manner to the one explained in Section 3.

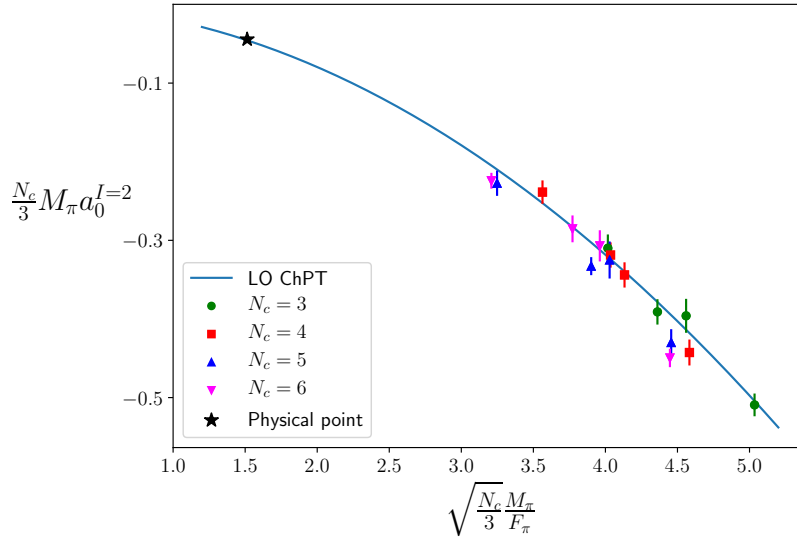


Figure 2: Preliminary results on the s -wave $I=2$ $\pi\pi$ scattering length for different values of N_c . We rescale the x - and y -axis by the leading N_c -dependence. The line indicates the prediction of leading order chiral perturbation theory, and the black star marks the value for physical pion masses.

5. Nonleptonic kaon decays at large N_c

The study of the $\Delta I = 1/2$ rule in the large N_c limit can be performed via an indirect method: the study of the $K \rightarrow \pi$ matrix elements in the GIM limit [9, 19]. We refer to our previous work for technical details [9, 10]. The key point is that the ratio of isospin amplitudes in the GIM limit can be related to leading order in ChPT to the $K \rightarrow \pi$ matrix elements:

$$\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left(1 + 3\frac{g^-}{g^+}\right), \quad \langle K | C_\pm \mathcal{O}^\pm | \pi \rangle = g^\pm, \quad (5.1)$$

where \mathcal{O}^\pm are the operators in the effective weak Hamiltonian, C_\pm are the Wilson coefficients and g^\pm are the LECs of the chiral weak Hamiltonian.

On the lattice, the effective couplings g^\pm can be determined via a three-point function properly normalized:

$$g^\pm \sim R^\pm \equiv \frac{\langle \pi | \mathcal{O}^\pm | K \rangle}{f_K f_\pi m_K m_\pi} = Z_R^\pm(\mu) R_{bare}^\pm, \quad (5.2)$$

and the Wilson coefficients can be calculated as explained in Ref. [9].

We show our preliminary results in Fig. 3 for the dynamical ensembles of this work, together with the quenched results of Ref. [9]. We also fit these numbers to the expected N_c dependence. We obtain that the $1/N_c$ terms agree in both calculations, and that quenching effects are sizeable and enter at $O(N_c^{-2})$. With these results, we can calculate the ratio of isospin amplitudes for $N_c = 3$ using Eq. (5.1):

$$\frac{A_0}{A_2} \Big|_{N_c=3, N_f=4} = 6.3(3), \quad \frac{A_0}{A_2} \Big|_{N_c=3, N_f=0} = 3.7(1), \quad (5.3)$$

where the error is purely statistical. In addition, an extra 10 – 20% effect can be explained from a contribution to NLO ChPT (See Ref. [10]). Still, a factor three remains to be explained.

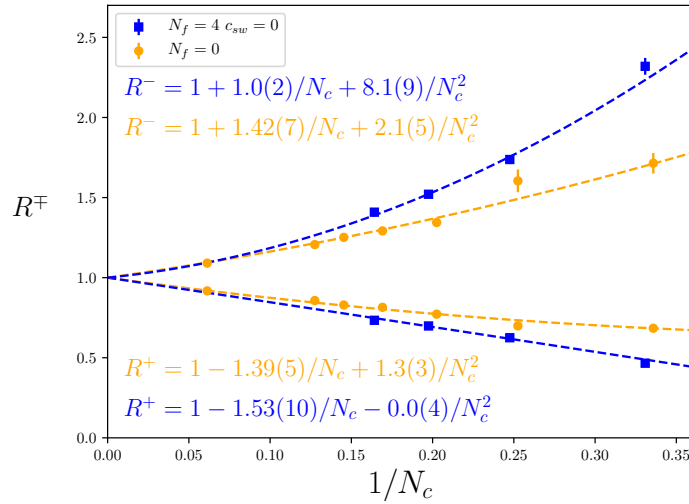


Figure 3: $K \rightarrow \pi$ matrix elements as a function of the number of colors for dynamical and quenched configurations. The label $c_{sw} = 0$ in the legend refers to the value of c_{sw} in the valence action.

6. Conclusion and Outlook

We have presented the current status of the determination of the N_c scaling of QCD observables from first principles. We have achieved significant progress in the computation of meson masses and decay constant, the $I = 2$ $\pi\pi$ scattering length and $K \rightarrow \pi$ matrix elements. First, we have been able to extract the leading and subleading contributions to the LECs of the chiral Lagrangian. Second, we have seen the expected scaling in the $I = 2$ $\pi\pi$ scattering length. Finally, the study of the $K \rightarrow \pi$ matrix elements has shown that $1/N_c^2$ effects are relevant and significantly increase the ratio of couplings. This is related to the $\Delta I = 1/2$ rule, and constitutes a source of enhancement.

In future work, we intend to address the scaling of QCD resonances. In particular, $I = 0$ $\pi\pi$ scattering length represents a very interesting topic, as it couples to the sigma resonance and it could have implications for the $\Delta I = 1/2$ rule. Other relevant topics are the ρ resonance, the flavour singlet meson and the existence of tetraquarks in the large N_c limit.

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