Chiral Ward identities for Dirac eigenmodes with staggered fermions

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We study chiral properties of eigenvalue spectrum for staggered quarks. We present a new method to identify would-be zero modes and nonzero modes using their symmetry and chiral properties. Here, we review the traditional method with HYP improved staggered quarks, and extend it to a completely new method which uses the chiral Ward identities and leakage patterns to achieve the goal.

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parameter	value
gluon action	tree level Symanzik [6–8]
tadpole improvement	yes
β	5.0
geometry	20^4
a	0.077 fm
valence quarks	HYP staggered fermions [9–11]
N_f	0 (quenched QCD)

Table 1: Input parameters for the numerical study. For more details, refer to Ref. [12].

1. Introduction

Here, we present recent progress in understanding chiral properties of staggered Dirac eigenmodes based on our previous works in Refs. [1, 2].

2. Eigenvalues of Dirac operators with staggered fermions

Let us consider Dirac operator D_s for staggered fermions. Since D_s is anti-Hermitian, eigenvalues of D_s are purely imaginary or zero:

$$D_s|f_\lambda\rangle = i\lambda\,|f_\lambda\rangle\,, \tag{2.1}$$

where λ is real, and $|f_\lambda\rangle$ is an eigenvector with its eigenvalue $i\lambda$.

D_s also anti-commutes with the operator $\Gamma_\varepsilon = [\gamma_5 \otimes \xi_5]$ which is the generator for U(1)$_A$ symmetry. Here, we adopt the same notation as in Ref. [1]. Since Γ_ε anti-commutes with D_s, one can show that an eigenstate $|f_\lambda\rangle$ with $\lambda \neq 0$ has its partner state $|f_{-\lambda}\rangle$ with eigenvalue $-i\lambda$ [1]. The partner eigenvector $|f_{-\lambda}\rangle$ can be obtained by applying Γ_ε to the eigenvector $|f_{+\lambda}\rangle$ with phase difference:

$$\Gamma_\varepsilon\,|f_{+\lambda}\rangle = e^{+i\theta}\,|f_{-\lambda}\rangle\,, \qquad \Gamma_\varepsilon\,|f_{-\lambda}\rangle = e^{-i\theta}\,|f_{+\lambda}\rangle\,. \tag{2.2}$$

Here, the phase θ is real, and also turns out to be arbitrary [1].

In practice, we do not calculate eigenvalues of D_s directly. Instead, we use a Hermitian and positive semi-definite operator $D_s^\dagger D_s$, which satisfies

$$D_s^\dagger D_s\,|g_{\lambda^2}\rangle = \lambda^2\,|g_{\lambda^2}\rangle\,. \tag{2.3}$$

The corresponding eigenvectors $|f_{\pm\lambda}\rangle$ are obtained by decomposing the eigenvector $|g_{\lambda^2}\rangle$ using the projection operators as in Ref. [1]. Since $D_s^\dagger D_s$ is Hermitian, one can make use of Lanczos algorithm [3] to calculate its eigenvalues and eigenvectors. Here, we use the implicitly restarted Lanczos [4] with acceleration by Chebyshev polynomial [5].

All the numerical calculations are performed on the gauge ensemble described in Table 1. We use HYP staggered fermions as valence quarks which reduce the taste-breaking for staggered fermions, and thus show improved chiral behaviors [12–15].

Meanwhile, the index theorem [16] states that

$$Q_t = n_- - n_+\,, \qquad Q_t = \frac{1}{32\pi^2}\int_E d^4x\,\varepsilon_{\alpha\beta\mu\nu}\,\mathrm{Tr}\left(F_{\alpha\beta}\,F_{\mu\nu}\right) \tag{2.4}$$
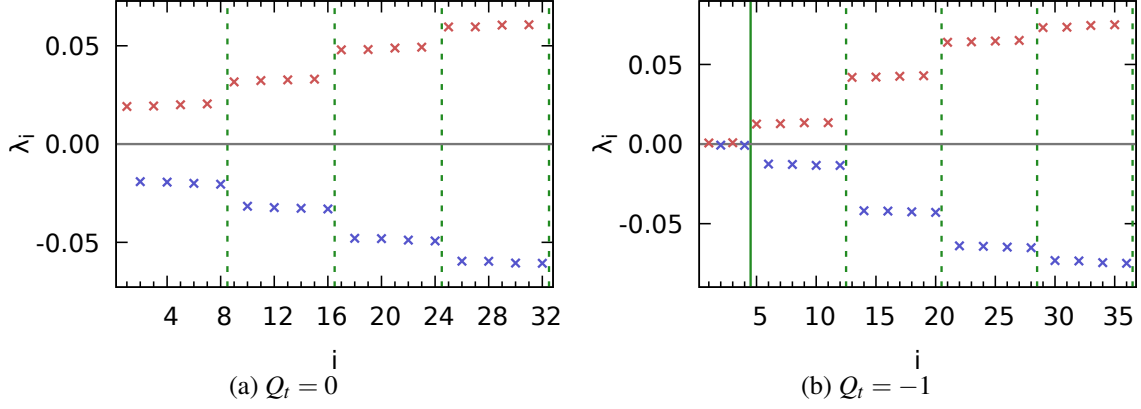
Figure 1: Eigenvalue spectra of staggered Dirac operator on gauge configurations with $Q_t = 0$ and $Q_t = -1$. Here, i represents an index of eigenvalue λ_i.

where Q_t is the topological charge [17], the subscript $_E$ represents the Euclidean space, and n_+ (n_-) is the number of zero modes with right-handed (left-handed) helicity. In the continuum, Eq. (2.4) indeed comes from the axial Ward identity [2]. For staggered fermions, a similar relation holds but four-fold degeneracy which comes from the approximate SU(4) taste symmetry should be counted [1]:

$$Q_t = \frac{1}{4}(n_-^s - n_+^s), \tag{2.5}$$

where $n_\pm^s$ represent the number of zero modes with right-handed ($+$) and left-handed ($-$) helicities for staggered quarks. Here, $n_\pm^s$ must be multiples of four due to the taste symmetry.

3. Eigenvalue spectrum

In Fig. 1, we present tens of low-lying eigenvalues of the Dirac operator with HYP staggered quarks on gauge configurations with topological charges $Q_t = 0, -1$. Here, we measure Q_t using the $Q(5\text{Li})$ operator defined in Ref. [18, 19] after $10 \sim 30$ iterations of the APE smearing with $\alpha = 0.45$ [20–22]. In the plot, eigenvalues are sorted in ascending order of their absolute values $|\lambda_i|$. Here, we assign the index $(2n)$ of the eigenvalue such that it satisfies $\lambda_{2n} = -\lambda_{2n-1}$. Even the would-be zero modes have tiny but nonzero values of λ_i at finite lattice spacing $a \neq 0$. Hence, each eigenvalue has its parity partner with opposite sign even though it belongs to the would-be zero modes. The solid green line for $Q_t = -1$ is drawn at the boarder between the would-be zero modes and the nonzero modes.

For would-be zero modes, their eigenvalues are exactly zero in the continuum limit, and so they are their own parity partners by themselves. The number of the would be zero modes must be multiple of four since the SU(4) taste symmetry is exactly conserved in the continuum. In Fig. 1b, one can see the would-be zero modes appear with four-fold degeneracy. For nonzero modes, one eigenvalue must have four-fold degeneracy due to the SU(4) taste symmetry in the continuum, and its U(1)$_A$ parity partner should have the same four-fold degeneracy. Hence, for each nonzero eigenvalue, it has a set of eight-fold degeneracy due to the exact U(1)$_A$ symmetry on the lattice and the SU(4) taste symmetry in the continuum. In Fig. 1, one can see the nonzero modes show up with eight-fold degeneracy.
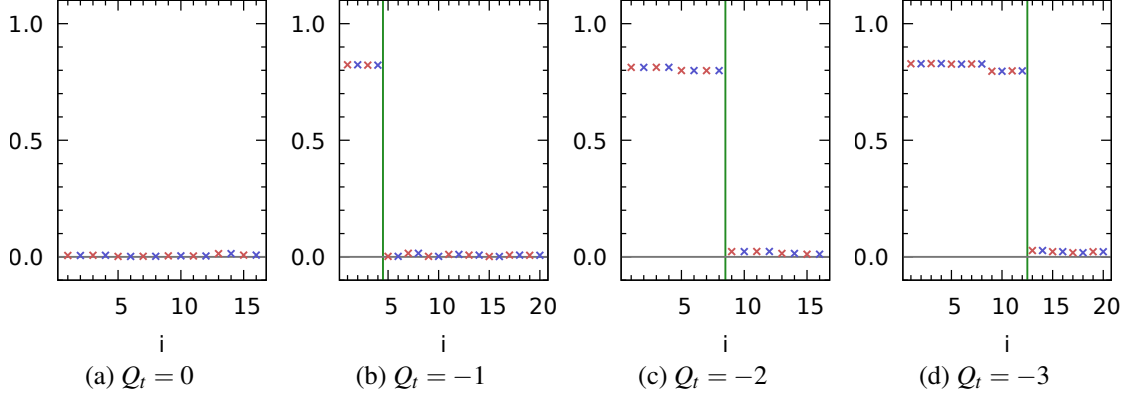
Figure 2: $\Gamma_5(\lambda_i)$ for various topological charges. Here, i represents an index of eigenvalue λ_i.

4. Chirality for staggered fermions

Let us consider three chirality operators: Γ_ε, Γ_5, and Ξ_5 defined as

$$\Gamma_\varepsilon \equiv [\gamma_5 \otimes \xi_5], \qquad \Gamma_5 \equiv [\gamma_5 \otimes \mathbb{1}], \qquad \Xi_5 \equiv [\mathbb{1} \otimes \xi_5]. \tag{4.1}$$

Γ_ε represents a chirality of the conserved $U(1)_A$ symmetry for staggered fermions. A taste singlet operator Γ_5 corresponds to the generator for the anomalous $U(1)_A^{\text{anom}}$ symmetry in the continuum. Similarly, Ξ_5 represents the parity partner for the chirality operator Γ_5.

The Γ_ε, Γ_5, and Ξ_5 operators satisfy the same relations as the continuum chirality operator γ_5 as follows,

$$(\Gamma)^{2n+1} = \Gamma, \qquad (\Gamma)^{2n} = \mathbb{1}, \tag{4.2}$$

where $\Gamma \in \{\Gamma_\varepsilon, \Gamma_5, \Xi_5\}$. Furthermore, they are related to each other by

$$\Gamma_\varepsilon = \Gamma_5 \, \Xi_5 \, ; \quad \Gamma_5 = \Xi_5 \, \Gamma_\varepsilon \, ; \quad \Xi_5 = \Gamma_5 \, \Gamma_\varepsilon \, . \tag{4.3}$$

These properties insure that they are the best choice to examine the chiral symmetry for staggered fermions.

Let us define the chirality as

$$\Gamma_5(\alpha, \beta) \equiv \langle f_\alpha | [\gamma_5 \otimes \mathbb{1}] | f_\beta \rangle, \qquad \Gamma_5(\lambda_i) \equiv \Gamma_5(\lambda_i, \lambda_i). \tag{4.4}$$

In Fig. 2, we measure the chirality $\Gamma_5(\lambda_i)$ for topological charges $Q_t = 0, -1, -2, -3$, respectively. Comparing with Fig. 1, the would-be zero modes has a non-trivial chirality around 0.8 in magnitude, while nonzero modes have values of $\Gamma_5(\lambda_i)$ close to zero. Consequently, would-be zero modes are manifestly distinguishable from nonzero modes by the Γ_5 chirality as shown in Ref. [12–14]. The magnitudes of the chirality for would-be zero modes are somewhat smaller than one, the continuum expectation value. It is because the Γ_5 operator is not conserved at $a \neq 0$ and receives a finite renormalization on the lattice.

parameter	value	parameter	value	parameter	value
$\lvert\Gamma_5(\lambda_1,\lambda_1)\rvert$	0.8238257	$\lvert\Gamma_5(\lambda_{75},\lambda_{70})\rvert$	0.5008622	$\lvert\Gamma_5(\lambda_{70},\lambda_{75})\rvert$	0.5008622
$\lvert\Xi_5(\lambda_2,\lambda_1)\rvert$	0.8238257	$\lvert\Xi_5(\lambda_{69},\lambda_{75})\rvert$	0.5008622	$\lvert\Xi_5(\lambda_{75},\lambda_{69})\rvert$	0.5008622
$\lvert\Xi_5(\lambda_1,\lambda_2)\rvert$	0.8238257	$\lvert\Xi_5(\lambda_{70},\lambda_{76})\rvert$	0.5008622	$\lvert\Xi_5(\lambda_{76},\lambda_{70})\rvert$	0.5008622
$\lvert\Gamma_5(\lambda_2,\lambda_2)\rvert$	0.8238257	$\lvert\Gamma_5(\lambda_{69},\lambda_{76})\rvert$	0.5008622	$\lvert\Gamma_5(\lambda_{76},\lambda_{69})\rvert$	0.5008622

        (a) Diagonal WI   (b) Off-diagonal WI

Table 2: Numerical demonstration of chiral Ward identity (WI) in Eq. (5.8). Here, $\lambda_2 = -\lambda_1$, $\lambda_{70} = -\lambda_{69}$, $\lambda_{76} = -\lambda_{75}$.

5. Chiral Ward identity

Rewriting Eqs. (2.2) by implementing Eq. (4.2) and Eq. (4.3), we obtain the following chiral Ward identities for staggered fermions:

$$\Gamma_5\lvert f_{+\lambda}\rangle = e^{+i\theta}\,\Xi_5\lvert f_{-\lambda}\rangle, \qquad \Gamma_5\lvert f_{-\lambda}\rangle = e^{-i\theta}\,\Xi_5\lvert f_{+\lambda}\rangle. \tag{5.1}$$

Let us define the chirality matrix elements sandwiched between the two eigenvectors as

$$\Gamma_\varepsilon(\alpha,\beta) \equiv \langle f_\alpha\lvert\Gamma_\varepsilon\rvert f_\beta\rangle = \langle f_\alpha\lvert[\gamma_5\otimes\xi_5]\rvert f_\beta\rangle, \tag{5.2}$$

$$\Gamma_5(\alpha,\beta) \equiv \langle f_\alpha\lvert\Gamma_5\rvert f_\beta\rangle = \langle f_\alpha\lvert[\gamma_5\otimes\mathbb{1}]\rvert f_\beta\rangle, \tag{5.3}$$

$$\Xi_5(\alpha,\beta) \equiv \langle f_\alpha\lvert\Xi_5\rvert f_\beta\rangle = \langle f_\alpha\lvert[\mathbb{1}\otimes\xi_5]\rvert f_\beta\rangle. \tag{5.4}$$

Using the Ward identity of Eqs. (5.1), we rewrite the chirality matrix elements as follows,

$$\Gamma_5(\alpha,+\beta) = e^{+i\theta_\beta}\,\Xi_5(\alpha,-\beta), \qquad \Gamma_5(\alpha,-\beta) = e^{-i\theta_\beta}\,\Xi_5(\alpha,+\beta), \tag{5.5}$$

$$\Gamma_5(+\alpha,\beta) = e^{-i\theta_\alpha}\,\Xi_5(-\alpha,\beta), \qquad \Gamma_5(-\alpha,\beta) = e^{+i\theta_\alpha}\,\Xi_5(+\alpha,\beta). \tag{5.6}$$

If we take the norm of them, then

$$\lvert\Gamma_5(\alpha,\beta)\rvert = \lvert\Xi_5(\alpha,-\beta)\rvert = \lvert\Xi_5(-\alpha,\beta)\rvert = \lvert\Gamma_5(-\alpha,-\beta)\rvert. \tag{5.7}$$
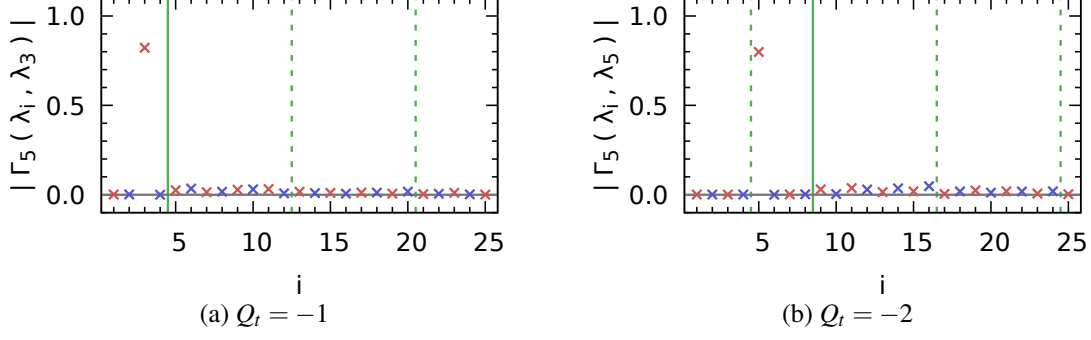
In addition, the Hermiticity insures interchanging α and β, which provides the final form of the Ward identities:

$$\lvert\Gamma_5(\alpha,\beta)\rvert = \lvert\Xi_5(\alpha,-\beta)\rvert = \lvert\Xi_5(-\alpha,\beta)\rvert = \lvert\Gamma_5(-\alpha,-\beta)\rvert$$
$$= \lvert\Gamma_5(\beta,\alpha)\rvert = \lvert\Xi_5(\beta,-\alpha)\rvert = \lvert\Xi_5(-\beta,\alpha)\rvert = \lvert\Gamma_5(-\beta,-\alpha)\rvert. \tag{5.8}$$

Table 2 shows how the chiral Ward identities of Eq. (5.8) works in our numerical study. Here, it confirms that they are valid within our numerical precision.
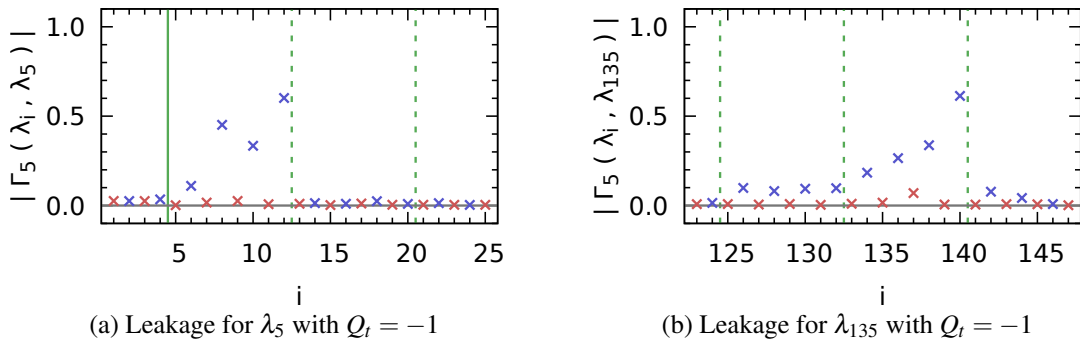
6. Leakage of chirality

Here, we focus on off-diagonal elements ($\alpha \neq \beta$) of chirality $\Gamma_5(\alpha,\beta)$ and $\Xi_5(\alpha,\beta)$. We are interested in how much of the chirality of an eigenmode leaks into other eigenmodes. Traditionally, the diagonal elements of chirality $\Gamma_5(\lambda_i)$ were measured and studied as in Ref. [12–14]. Here,

(a) $Q_t = -1$ (b) $Q_t = -2$

Figure 3: Leakage patterns of would-be zero modes for the Γ_5 operator.

we study on the off-diagonal elements of chirality $\Gamma_5(\alpha, \beta)$, and $\Xi_5(\alpha, \beta)$ with $\alpha \neq \beta$. In the continuum, the SU(4) taste symmetry is respected. The net consequence of the conserved taste symmetry is that each nonzero eigenvalue has eight-fold degeneracy, and these eight degenerate eigenmodes will mix with one another within the eight-fold degenerate members. In other words, if $|\alpha| \neq |\beta|$, then $\Gamma_5(\alpha, \beta) = \Xi_5(\alpha, \beta) = 0$ in the continuum ($a = 0$) thanks to the SU(4) taste symmetry. However, at finite lattice ($a \neq 0$), the SU(4) taste symmetry is not exact, but mostly respected near the continuum. Hence, the leakage from one set of the eight-fold degeneracy to other set of eight-fold degeneracy will be very small near the continuum ($a \approx 0$). Therefore, it will be very interesting to study this leakage pattern in the chirality measurement. Fig. 3 shows leakage patterns of Γ_5 for would-be zero modes. Here, we observe that for Γ_5 there is only one non-trivial signal at the would-be zero mode itself and there is almost no leakage to nearest zero and nonzero eigenmodes.

For nonzero modes, leakage of the Γ_5 operator for an eigenvalue λ_i is supposed to go into the four-fold parity partners with eigenvalue $-\lambda_i$ in the continuum. Near the continuum ($a \approx 0$), the SU(4) taste symmetry is almost respected and the leakage to eigenmodes outside the eight-fold degeneracy is also almost prohibited. This kind of leakage patterns for nonzero eigenmodes are presented in Fig. 4. In Fig. 4a, the leakage of the Γ_5 chirality for the eigenmode $|f_{\lambda_5}\rangle$ goes into the eigenmodes with eigenvalue $-\lambda_5$: $|f_{\lambda_j}\rangle$ with $j = 6, 8, 10, 12$, as the theory predicts. In Fig. 4b, the leakage of the Γ_5 chirality for the eigenmode $|f_{\lambda_{135}}\rangle$ goes into the eigenmodes with eigenvalue $-\lambda_{135}$: $|f_{\lambda_j}\rangle$ with $j = 134, 136, 138, 140$. Here, we also observe a small effect of the SU(4) taste symmetry breaking in that a small amount of leakage of the Γ_5 chirality for the eigenmode $|f_{\lambda_{135}}\rangle$ goes into eigenmodes outside of the eight-fold degeneracy such as $|f_{\lambda_j}\rangle$ with $j = 126, 128, 130, 132, 142, 144$.

(a) Leakage for λ_5 with $Q_t = -1$ (b) Leakage for λ_{135} with $Q_t = -1$

Figure 4: Leakage patterns of Γ_5 for nonzero eigenmodes.

7. Conclusion

We have studied Γ_5 and Ξ_5 chirality for eigenmodes of staggered fermions. Thanks to the Ward identities, Γ_5 chirality is completely correlated with Ξ_5 chirality. We demonstrate how the leakage patterns of Γ_5 chirality can be used to distinguish zero eigenmodes and nonzero eigenmodes.

Acknowledgments

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