PROCEEDINGS OF SCIENCE

$I = 3/2$ $N\pi$ scattering and the $\Delta(1232)$ resonance on $N_f = 2+1$ CLS ensembles using the stochastic LapH method

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Calculations of the elastic $I = \frac{3}{2}$ nucleon-pion scattering phase shifts on two lattice QCD ensembles with $m_\pi = 200$MeV and 280MeV are presented. The ensembles both employ $N_f = 2+1$ Wilson clover fermions. We determine the $\Delta(1232)$ resonance parameters from a finite volume scattering analysis. In one study the single partial wave simplification is employed to compute the p-wave amplitude while in the other we treat the partial wave mixing between s- and p-wave contributions. Fitting our data to a Breit-Wigner resonance model we find $m_\Delta/m_\pi = 7.13(9)$ and $4.75(5)$ on the two ensembles respectively, showing that for a lighter quark mass the resonance mass moves from near the $N\pi$ threshold to near the $N\pi\pi$ threshold, in agreement with experiment.

37th International Symposium on Lattice Field Theory - Lattice2019
16-22 June 2019
Wuhan, China

*Speaker.

PoS(LATTICE2019)039

ID	β	a(fm)	$\left(\frac{L}{a}\right)^3 \times \frac{T}{a}$	m_π, m_K(MeV)	N_{conf}	N_{t_0}	N_{ev}
N401	3.46	0.0763	$48^3 \times 128$	280, 460	275	2	320
D200	3.55	0.0643	$64^3 \times 128$	200, 480	559	2	448

Table 1: The ensembles used in this calculation [14]. The table specifies the gauge coupling β, the lattice constant a [11], the extent of the lattices in lattice units, masses of the pseudoscalar particles, the number of configurations used in the analysis N_{conf}, the number of source times N_{t_0} and the number of eigenvectors used in the smearing N_{ev}.

1. Introduction

In this work we present a determination of elastic $N\pi$ scattering amplitudes. Since the calculations are performed using simulations of QCD on a Euclidean lattice a direct determination of the scattering observables is not possible [1]. A common way to circumvent this problem makes use of the fact that the discrete, interacting energy levels in a finite spatial volume are shifted from their non-interacting values by an amount that can be related to the scattering matrix. This relation was first described for scattering between two identical, spinless particles with total zero momentum by Lüscher [2]. This result has since been extended and generalized with advances relevant for this work found in Refs. [3–10], allowing scattering studies of increasingly impressive precision, in particular in the meson-meson sector. When introducing baryons into the calculations one has to deal with a more severe signal-to-noise problem, increased computational cost and a more involved analysis dealing with particles with differing spin, so comprehensive studies of resonant meson-baryon and baryon-baryon scattering are still lacking.

2. Methods

The two gauge field ensembles used in this work were generated by the CLS consortium [11, 12]. Both simulations employ $N_{\text{f}} = 2+1$ dynamical Wilson clover fermions and both have open boundary conditions in the time direction. All interpolating operators are kept at a minimum distance of t_{bnd} from the boundary, where $t_{\text{bnd}} m_\pi = 2$. The ensemble details are listed in Tab. 1. Results on the N401 ensemble have been published in Ref. [13].

The required finite volume energy levels are extracted from correlation functions of operators with $\Delta(1232)$ quantum numbers including $N\pi$-operators. In order to compute correlation functions of multihadron operators with definite momentum we employ all-to-all quark propagators, which are efficiently handled with the stochastic LapH method [15]. In this framework, the quark propagator is projected into a lower dimensional subspace constructed from N_{ev} eigenvectors of the stout smeared [16] gauge-covariant 3-D lattice Laplace operator. In this way the color and space indices of the quark propagator are converted to eigenvector indices. Converting back to color and space indices results in a spatially smeared quark field that retains all symmetries of the original unsmeared field.

The quark propagator is stochastically estimated in the LapH subspace spanned by time, spin and eigenvectors. The number of eigenvectors N_{ev} used in the smearing can be seen in Tab. 1. We use dilution [17, 18] in time, spin and eigenvector indices to reduce the variance of the stochastic

estimation. The dilution scheme is explained in Ref. [13] for the N401 lattice. Using that notation the dilution scheme for the D200 data is $(\mathrm{TF},\mathrm{SF},\mathrm{LI8})_F$, $(\mathrm{TI8},\mathrm{SF},\mathrm{LI8})_R$, re-using the solutions to the Dirac equation from Ref. [19]. To further increase statistics we average over two source times, all equivalent total momenta $\mathbf{P}$ and all irrep rows λ.

Two of the practical problems encountered in resonant meson-baryon scattering lattice calculations are the proliferation of Wick contractions that need to be evaluated and the extra annihilation type diagrams in correlation functions between single baryon and meson-baryon interpolating operators. Although not employed in the present work, one way of dealing with the increasing number of Wick contractions is explained in Ref. [20].

In order to determine as many energy levels as possible the fact that our interpolating operators have different overlaps on the lowest lying states in the spectrum is exploited [21, 22]. In practice, we calculate the energy spectrum for various total momenta $\mathbf{P}$ and in various irreducible representations Λ of the little group of $\mathbf{P}$. For each pair of $(\Lambda,\mathbf{P})$ of interest a matrix of correlation functions $C_{ij}(t) = \left\langle \hat{O}_i(t)\hat{O}_j^\dagger(0)\right\rangle$ is computed for a number of operators projected to the given irrep and total momentum. The operator basis consists of 1-2 single-site Δ operators and 2-7 $N\pi$ operators depending on the irrep and total momentum. We then solve the generalized eigenvalue problem (GEVP)

$$C(t_d)v_n(t_d,t_0) = \lambda_n C(t_0)v_n(t_d,t_0), \tag{2.1}$$

and the correlators between "optimal" interpolators for the n'th state can then be found by rotating the correlator matrix by the eigenvectors:

$$\hat{C}_n(t) = (v_n(t_0,t_d), C(t)v_n(t_0,t_d)). \tag{2.2}$$

We make sure that the extracted energies are stable under variation of t_0, t_d and the operator basis.

This optimal correlator is expected to decay exponentially with the energy of the n'th state with overlap on the operator basis. We now form the quantity

$$R_n(t) = \frac{\hat{C}_n(t)}{C_\pi(\mathbf{p}^2_{\pi,n},t)C_N(\mathbf{p}^2_{N,n},t)}, \tag{2.3}$$

where $C_\pi(\mathbf{p}^2_{\pi,n},t)$ and $C_N(\mathbf{p}^2_{N,n},t)$ are the correlators of the interpolating pion and nucleon respectively. For each finite volume level we chose $\mathbf{p}_{\pi,n}$ and $\mathbf{p}_{N,n}$ corresponding to a nearby non-interacting level. $R(t)$ is then expected to decay exponentially with the energy shift from the non-interacting energies, and thus we fit $R(t)$ to the ansatz

$$R_n(t) = Ae^{-\Delta E_n t}, \tag{2.4}$$

and reconstruct the total energy E_n from ΔE_n and the measured values of the pion and nucleon energies.

Given the finite volume energies, elastic 2-to-2 scattering amplitudes can be computed using the determinant equation

$$\det\left(\hat{K}^{-1} - B^{(\mathbf{P})}\right) = 0. \tag{2.5}$$

For a given total momentum $\mathbf{P}$ and irreducible representation Λ of the little group of $\mathbf{P}$, $\hat{K}$ and $B^{(\mathbf{P})}$ are matrices in total angular momentum J, total orbital angular momentum L, total spin S and

occurrence number n. $\hat{K}$ is related to the usual K-matrix by

$$\hat{K}^{-1}_{L'S';LS} = q_{\text{cm}}^{L+L'+1} K^{-1}_{L'S';LS}, \tag{2.6}$$

and is diagonal in J. The box matrix B encodes the reduced symmetries of the finite volume for a given irrep. It is a known matrix of functions of E_{cm} and is diagonal in S and n but dense in the other quantum numbers. All B-matrix elements required here are given in Ref. [10].

In this study we are interested in $N\pi$-scattering with isospin $I = \frac{3}{2}$. The most prominent feature of this system is the p-wave $\Delta(1232)$ resonance ($J^\eta = \frac{3}{2}^+$ where η is the parity), which is the focus of the rest of this work. The first published lattice calculation of the resonant phase shifts of this system appeared in Ref. [13], but other preliminary work can be found in Refs. [23–25]. The Δ-baryons decay almost exclusively to $N\pi$ states [26], so we need only worry about a single open channel.

In order to numerically evaluate Eq. (2.5) the matrices must be truncated at some L_{max}. Since the Δ resonance occurs in $N\pi$ scattering with $L = 1$ setting $L_{\text{max}} \geq 1$ is required. We check the impact of the d-wave explicitly by varying L_{max} from 1 to 2.

In the irreps $(\Lambda, \mathbf{P}^2) = \{(H_g, 0), (G_2, 1), (F_1, 3), (F_2, 3), (G_2, 4)\}$ the B-matrix elements corresponding to $J = \frac{1}{2}$ and/or $L = 0$ are identical to 0. This means that if $L_{\text{max}} = 1$ there is a 1-to-1 correspondence between a measured energy level and a p-wave scattering phase shift. We also measure the energy spectrum in the irreps $(\Lambda, \mathbf{P}^2) = \{(G_{1g}, 0), (G_{1u}, 0), (H_u, 0), (G_1, 1), (G, 2), (G, 3), (G_1, 4)\}$. In these irreps we have to take partial wave mixing into account. However for $N\pi$ scattering the $\hat{K}$-matrix is fully diagonal in J and L, so we can write

$$\hat{K}^{-1} = \text{diag}\left((\hat{K}^{-1})_{\frac{1}{2},0}, (\hat{K}^{-1})_{\frac{1}{2},1}, (\hat{K}^{-1})_{\frac{3}{2},1} \right), \tag{2.7}$$

with subscripts denoting (J, L), and so only three elements of the $\hat{K}$ matrix need to be parameterized.

An important limitation of this formalism is that Eq. (2.5) only holds below any relevant three-particle thresholds. For this system the first such three-particle state is $N\pi\pi$, meaning that any energy above $m_N + 2m_\pi$ is excluded from the scattering analysis.

3. Results

The energies are determined by fitting the optimized correlators to the ansatz in Eq. (2.4) from some minimum time separation t_{min} to a fixed maximum time separation of $t_{\text{max}} = 25a$. The value of t_{min} is chosen large enough that the statistical error on the fitted energy is larger than the systematic error from the excited state contamination. We determine 6 and 26 energy levels in the elastic region on the N401 and D200 ensembles respectively.

The measured energies can now be inserted into Eq. (2.5) to constrain the $\hat{K}$-matrix elements. We do this by parameterizing the elements of the $\hat{K}$-matrix and then fitting those parameters using a correlated χ^2 fit. The residuals of the fitting procedure are taken to be [10]

$$\Omega(\mu, A) = \frac{\det(A)}{\det\left([\mu^2 + AA^\dagger]^{1/2} \right)} \tag{3.1}$$

with $A = \hat{K}^{-1} - B^{(\mathbf{P})}$ and $\mu = 5$. Since a Breit-Wigner resonance in the $J = \frac{3}{2}^+$ p-wave is expected we fit

$$\left(\hat{K}^{-1}\right)_{\frac{3}{2},1} = \left(\frac{m_\Delta^2}{m_\pi^2} - \frac{E_{\text{cm}}^2}{m_\pi^2}\right) \frac{6\pi E_{\text{cm}}}{g^2 m_\pi}, \tag{3.2}$$

with the fit parameters m_Δ/m_π being the resonance mass in units of the pion mass and g which is related to the resonance width. For assessing s- and d-wave contributions we use a truncated effective range expansion with just one fit parameter per $\hat{K}$-matrix element:

$$\left(\hat{K}^{-1}\right)_{J,L} = \frac{1}{m_\pi^{2L+1} a_{J,L}}, \qquad (J,L) \neq \left(\tfrac{3}{2},1\right). \tag{3.3}$$

The results of the fits are shown in Tab. 2. For the N401 ensemble we only measure energies in the irreps where the p-wave is the lowest contributing partial wave and so make no attempt to include an s-wave parameterization of the $\hat{K}$-matrix. We additionally perform a fit including a d-wave parameterization and an extra energy level in the $(\Lambda, \mathbf{P}^2) = (H_u, 0)$, where the d-wave is the lowest contributing partial wave. Doing this had no significant impact on the p-wave scattering parameters. On the D200 we also study the impact of including irreps where the s-wave is present. Tab. 2 shows that this reduces the statistical error on the resonance mass by a factor of 2 compared to the fit using only p-wave levels and improves the quality of the fit significantly. We also see that including a d-wave parameterization actually shifts the resonance mass slightly outside of its statistical error. However, no extra d-wave energy levels were included here since they were all above the inelastic threshold, and it should also be noted that the χ^2/d.o.f. is rather low, suggesting some over fitting.

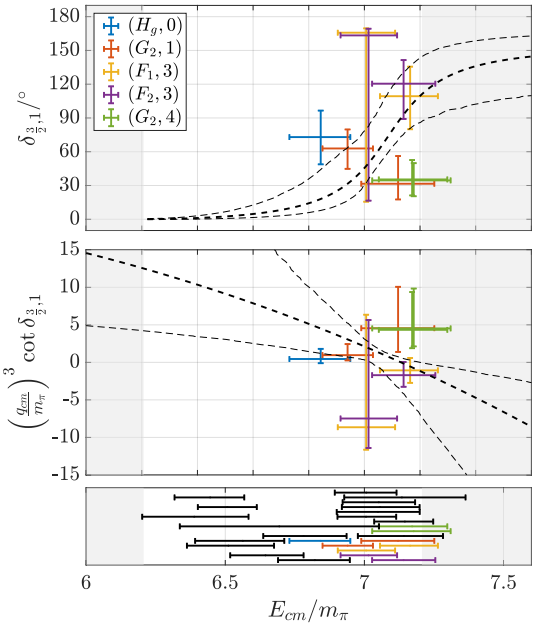

Figure 1: *Bottom:* The energies of all 26 states included in the scattering analysis. Colored errorbars show states in irreps where the p-wave is the lowest contributing partial wave with one color per irrep (see legend). *Middle and top:* In irreps where we apply the single-partial wave approximation the value of the $\hat{K}$-matrix element and $\delta_{\frac{3}{2},1}$ respectively is shown with errorbars from bootstrapping. The fit (including all 26 energies) is shown with dotted lines.

Fig. 1 shows the result of the calculation on the D200 lattice. All energies determined and included in the analysis are shown in the bottom panel while the upper panels show the value of $\left(\hat{K}^{-1}\right)_{\frac{3}{2},1} = \left(\frac{q_{\text{cm}}}{m_\pi}\right)^3 \cot \delta_{\frac{3}{2},1}$ and $\delta_{\frac{3}{2},1}$ for the irreps with a trivial determinant condition. The dotted lines show the fit to s- and p-wave including all 26 energy levels.

ID	L's	N_E	$\frac{m_\Delta}{m_\pi}$	g	χ^2/d.o.f.
N401	1	6	4.75(5)	19(5)	1.1
N401	1,2	7	4.73(6)	19(7)	4.2
D200	1	9	7.2(2)	18(11)	1.7
	0,1	26	7.13(9)	11(6)	0.8
	0,1,2	26	6.88(19)	25(17)	0.4

Table 2: Result of the scattering analyses. L's indicates the partial waves parameterized in the $\hat{K}$-matrix, N_E is the number of energy levels included and the fitting parameters are explained in the text.

4. Conclusion

We have presented a preliminary calculation of $N\pi$ scattering phase shifts complementing already published results [13]. The two ensembles have different m_π, lattice spacing and physical volume. Assuming a Breit-Wigner fit-form, we get a good estimate of the $J = \frac{3}{2}$ p-wave resonance mass although the width is still not determined precisely. Including energy levels in irreps which mix partial waves helps to constrain the p-wave scattering parameters despite the mixing. While we expect the systematic errors from the finite volume and lattice spacing to be small we postpone a quantitative assessment of these effects. In the future we plan to further increase statistics on both ensembles using improved estimators with more dilution projectors and more gauge configurations to obtain more precise constraints on the scattering amplitude parameters.

References

[1] L. Maiani and M. Testa, *Final state interactions from Euclidean correlation functions*, *Phys. Lett.* **B245** (1990) 585.

[2] M. Lüscher, *Two particle states on a torus and their relation to the scattering matrix*, *Nucl. Phys.* **B354** (1991) 531.

[3] S. A. Gottlieb and K. Rummukainen, *Resonance scattering on the lattice with nonzero total momentum*, *Nucl. Phys. Proc. Suppl.* **47** (1996) 819 [hep-lat/9509088].

[4] Z. Fu, *Rummukainen-Gottlieb's formula on two-particle system with different mass*, *Phys. Rev.* **D85** (2012) 014506 [1110.0319].

[5] T. Luu and M. J. Savage, *Extracting Scattering Phase-Shifts in Higher Partial-Waves from Lattice QCD Calculations*, *Phys. Rev.* **D83** (2011) 114508 [1101.3347].

[6] L. Leskovec and S. Prelovsek, *Scattering phase shifts for two particles of different mass and non-zero total momentum in lattice QCD*, *Phys. Rev.* **D85** (2012) 114507 [1202.2145].

[7] M. Göckeler, R. Horsley, M. Lage, U. G. Meissner, P. E. L. Rakow, A. Rusetsky et al., *Scattering phases for meson and baryon resonances on general moving-frame lattices*, *Phys. Rev.* **D86** (2012) 094513 [1206.4141].

[8] M. T. Hansen and S. R. Sharpe, *Multiple-channel generalization of Lellouch-Luscher formula*, *Phys. Rev.* **D86** (2012) 016007 [1204.0826].

[9] R. A. Briceño, *Two-particle multichannel systems in a finite volume with arbitrary spin*, *Phys. Rev.* **D89** (2014) 074507 [1401.3312].

[10] C. Morningstar, J. Bulava, B. Singha, R. Brett, J. Fallica, A. Hanlon et al., *Estimating the two-particle K-matrix for multiple partial waves and decay channels from finite-volume energies*, *Nucl. Phys.* **B924** (2017) 477 [1707.05817].

[11] M. Bruno, T. Korzec and S. Schaefer, *Setting the scale for the CLS* $2+1$ *flavor ensembles*, *Phys. Rev.* **D95** (2017) 074504 [1608.08900].

[12] RQCD collaboration, *Lattice simulations with* $N_f = 2+1$ *improved Wilson fermions at a fixed strange quark mass*, *Phys. Rev.* **D94** (2016) 074501 [1606.09039].

[13] C. W. Andersen, J. Bulava, B. Hörz and C. Morningstar, *Elastic* $I = 3/2$*p-wave nucleon-pion scattering amplitude and the* $\Delta(1232)$ *resonance from* N_f*=2+1 lattice QCD*, *Phys. Rev.* **D97** (2018) 014506 [1710.01557].

[14] M. Bruno et al., *Simulation of QCD with* $N_f = 2 + 1$ *flavors of non-perturbatively improved Wilson fermions*, *JHEP* **02** (2015) 043 [1411.3982].

[15] C. Morningstar, J. Bulava, J. Foley, K. J. Juge, D. Lenkner, M. Peardon et al., *Improved stochastic estimation of quark propagation with Laplacian Heaviside smearing in lattice QCD*, *Phys. Rev.* **D83** (2011) 114505 [1104.3870].

[16] C. Morningstar and M. J. Peardon, *Analytic smearing of SU(3) link variables in lattice QCD*, *Phys. Rev.* **D69** (2004) 054501 [hep-lat/0311018].

[17] W. Wilcox, *Noise methods for flavor singlet quantities*, in *Numerical challenges in lattice quantum chromodynamics. Proceedings, Joint Interdisciplinary Workshop, Wuppertal, Germany, August 22-24, 1999*, pp. 127–141, 1999, hep-lat/9911013.

[18] J. Foley, K. Jimmy Juge, A. O'Cais, M. Peardon, S. M. Ryan and J.-I. Skullerud, *Practical all-to-all propagators for lattice QCD*, *Comput. Phys. Commun.* **172** (2005) 145 [hep-lat/0505023].

[19] C. W. Andersen, J. Bulava, B. Hörz and C. Morningstar, *The* $I = 1$ *pion-pion scattering amplitude and timelike pion form factor from* $N_f = 2+1$ *lattice QCD*, *Nucl. Phys.* **B939** (2019) 145 [1808.05007].

[20] B. Hörz and A. Hanlon, *Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD*, *Phys. Rev. Lett.* **123** (2019) 142002 [1905.04277].

[21] C. Michael, *Adjoint Sources in Lattice Gauge Theory*, *Nucl. Phys.* **B259** (1985) 58.

[22] M. Luscher and U. Wolff, *How to Calculate the Elastic Scattering Matrix in Two-dimensional Quantum Field Theories by Numerical Simulation*, *Nucl. Phys.* **B339** (1990) 222.

[23] S. Paul et al., *Towards the P-wave nucleon-pion scattering amplitude in the* $\Delta(1232)$ *channel*, *PoS* **LATTICE2018** (2018) 089 [1812.01059].

[24] D. Mohler, *Review of lattice studies of resonances*, *PoS* **LATTICE2012** (2012) 003 [1211.6163].

[25] V. Verduci, *Pion-nucleon scattering in lattice QCD*, Ph.D. thesis, Graz U., 2014.

[26] PARTICLE DATA GROUP collaboration, *Review of particle physics*, *Phys. Rev. D* **98** (2018) 030001.