

Hadronic vacuum polarization in finite volume using NNLO ChPT

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We present results for the leading hadronic contribution to the muon $g - 2$ from configurations with 2+1+1 flavors of HISQ quarks. The ensembles have been generated by the MILC collaboration at three lattice spacings. Using the time-momentum representation of the electromagnetic current correlator, we calculate the finite volume effects up to next-to-next-to-leading-order in Chiral Perturbation Theory.

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1. Introduction

One of the more promising quantities with which to test the Standard Model is the anomalous magnetic moment of the muon $a_\mu = (g - 2)/2$, given the strong tension between the theoretical calculation (see, for example, Ref. [1]) and the experimental result [2]. The experiment at Fermilab (E989), running now, is expected to reduce the experimental uncertainty by a factor of four, and thus it is imperative for the theoretical calculation to obtain a similar reduction in uncertainty.

Given the improvements in lattice simulations of the leading hadronic contribution to the muon $g - 2$ over the last decade or so, extracting this quantity with the required precision from a first-principles approach is seeming much more likely. However, in addition to improving statistics on the lattice data, many systematics must be understood and corrected for before a reliable result can be obtained.

We focus here primarily on the systematics that enter when calculating the leading hadronic contribution to the muon $g - 2$ with staggered quarks in a finite volume. To do so, we have calculated these effects to next-to-next-to leading order (NNLO) in chiral perturbation theory (ChPT). A more detailed description of our results can be found in Ref. [3].

2. Simulation details

The leading hadronic contribution to the muon $g - 2$ can be obtained from the expression

$$a_\mu^{\text{HVP}} = 4\alpha^2 \int_0^\infty dq^2 f(q^2) \hat{\Pi}(q^2), \quad (2.1)$$

where $f(q^2)$ is defined in Ref. [4], and $\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$ is the subtracted hadronic vacuum polarization, coming from the Fourier transform of the vector two-point function (we use the conserved vector current here). In this work, we use the time-momentum representation:

$$\Pi(q^2) - \Pi(0) = \sum_t \left(\frac{\cos qt - 1}{q^2} + \frac{1}{2} t^2 \right) C(t), \quad C(t) = \frac{1}{3} \sum_{\vec{x}, i} \langle j^i(\vec{x}, t) j^i(0) \rangle, \quad (2.2)$$

where $C(t)$ is the Euclidean time correlation function, averaged over spatial directions. Eq. (2.1) becomes $a_\mu^{\text{HVP}}(T) = \sum_t w(t) C(t)$, with the weight

$$w(t) = 4\alpha^2 \int_0^\infty d\omega^2 f(\omega^2) \left[\frac{\cos \omega t - 1}{\omega^2} + \frac{t^2}{2} \right]. \quad (2.3)$$

The weight is sometimes modified by replacing the continuum Euclidean momentum-squared with its lattice version $\hat{w}(t)$, where the ω^2 in the denominator of the first term in square brackets is replaced with $[2 \sin(\omega/2)]^2$ [5].

In order to calculate the correlator in Eq. (2.2), we implement the noise reduction techniques developed by RBC/UKQCD [6, 5] including a combination of all-mode and full volume low-mode averaging. The precise details of the implementation of these techniques to the staggered Dirac operator are discussed in Ref. [3].

The ensembles used are 2+1+1 HISQ configurations generated by the MILC collaboration [7], and are listed in Ref. [3]. We simulated at three lattice spacings ($a \approx 0.06, 0.09$, and 0.12 fm) at

approximately physical pion masses. The volumes all have a spatial volume of around $(5.5 \text{ fm})^3$, and $m_\pi L \approx 3.7 - 3.9$. For the two coarser ensembles we generated 3000 eigenvectors for the low-modes, however only 2000 were used on the finest ensemble due to computational limitations.

3. Finite-volume chiral perturbation theory

In order to study the leading finite-volume effects we calculate the vector correlator to two loops (NNLO) in ChPT. There are several strategies we could use for this. One option is to first extrapolate our results to the continuum and then correct for the finite-volume using continuum ChPT. The other option would be to first use staggered ChPT [8] in a finite volume and then extrapolate to the continuum. Given that our pion masses and our physical volumes are not exactly equal, the second approach would be better able to take these differences into account. However, this would require applying staggered ChPT to two loops.

Instead we choose a hybrid approach. First we calculate the corrections using staggered ChPT at one loop for each ensemble, and then extrapolate to the continuum. At this point, we calculate the NNLO *continuum* finite volume corrections, without the use of (or need for) the staggered taste-breaking corrections. There will still be a small systematic effect coming from the slight mistunings of the pion masses and volumes, however it will be much smaller than if we were to extrapolate to the continuum first, and then apply the complete NLO+NNLO continuum ChPT to correct for finite-volume effects.

In Euclidean space we have performed a relatively straightforward calculation in the time-momentum representation to obtain $C(t)$ to NNLO:

$$C(t) = \frac{10}{9} \frac{1}{3} \left\{ \frac{1}{L^d} \sum_{\vec{p}} \frac{\vec{p}^2}{E_p^2} e^{-2E_p t} \left[1 - \frac{2}{F^2} \frac{1}{L^d} \sum_{\vec{k}} \left(\frac{1}{2E_k} \right) - \frac{8(\vec{p}^2 + m_\pi^2)}{F^2} \ell_6 \right] \right. \\ \left. + \frac{1}{2dF^2} \frac{1}{L^{2d}} \sum_{\vec{p}, \vec{k}} \frac{\vec{p}^2 \vec{k}^2}{E_p^2 E_k^2} \frac{E_k e^{-2E_p t} - E_p e^{-2E_k t}}{\vec{k}^2 - \vec{p}^2} \right\}, \quad (3.1)$$

where we have defined $E_p = \sqrt{m_\pi^2 + \vec{p}^2}$. The sums over \vec{p} and \vec{k} are over the momenta $2\pi\vec{n}/L$ with \vec{n} a three-vector of integers in a box with periodic boundary conditions. We define the renormalized ℓ_6^r by

$$\ell_6 = \ell_6^r(\mu) - \frac{1}{3} \frac{1}{16\pi^2} \left(\frac{1}{\varepsilon} - \log \mu - \frac{1}{2} (\log(4\pi) - \gamma + 1) \right), \quad (3.2)$$

to take the limit $d = 3 + \varepsilon \rightarrow 3$ in Eq. (3.1) to obtain a finite result for $C(t)$.

Looking at the expression in Eq. (3.1) term-by-term, we can obtain expressions for the NLO and NNLO finite volume corrections [3], defined by

$$\Delta a_\mu^{\text{HVP}} = \left[\lim_{L \rightarrow \infty} a_\mu^{\text{HVP}}(L) \right] - a_\mu^{\text{HVP}}(L). \quad (3.3)$$

Using the parameter values from Ref. [3], we obtain the results for $\Delta a_\mu^{\text{HVP}}$ at NLO (coming from the “1” in square brackets on the first line of Eq. (3.1)), shown in the second column in Table 1. From the same expression used for the NLO corrections, we can use the staggered pion spectrum to include the effects of the different taste masses in finite volume (third column) and we can calculate

a (fm)	NLO	taste (lattice)	taste (cont)	NNLO	total
0.12121(64)	18.08	2.1	51.6	7.40	25.5 ± 3.0
0.08787(46)	21.60	6.9	34.2	9.01	30.6 ± 3.8
0.05684(30)	20.59	15.6	9.5	9.13	29.7 ± 4.0

Table 1: ChPT FV corrections to the muon a_μ , $\Delta a_\mu^{\text{HVP}}$, in units of 10^{-10} . The columns are discussed in the text.

the effect of taste breaking in the pion masses in the infinite volume limit to NLO in ChPT, shown in the fourth column of Table 1.

The NNLO continuum finite-volume corrections come from an application of the Poisson summation formula to the remaining terms in Eq. (3.1), and the results are listed in the fifth column of Table 1. Finally, we obtain the total NLO+NNLO corrections and show these in the final column of Table 1. The errors are determined by assuming the omitted corrections are smaller than the NNLO terms by the same factor (~ 0.4 – 0.45) as the NNLO corrections are compared to the NLO contributions. Finally, the effects of taste-splittings in the staggered pion spectrum have been taken into account, and the details can be seen in Ref. [3].

4. Results & Conclusions

In addition to using the techniques discussed above to reduce statistical noise, we additionally apply the bounding method of Refs. [5, 9] where we set upper and lower bounds on the correlator for $t > T$: $C(t) = 0$ and $C(t) = C(T)e^{-E_0(t-T)}$ respectively, where the lowest energy state in the vector channel is $E_0 = 2\sqrt{m_\pi^2 + (2\pi/L)^2}$. At sufficiently large T the bounds overlap, and an estimate for a_μ can be made which may be more precise than simply summing over the noisy long-distance tail.

In Fig. 1 results for the bounding method are shown for each ensemble. We calculate central values for a_μ by averaging over a suitable range where T is large enough for the bounds to overlap but not so large that statistical errors blow up. The ranges used were 2.7-3.2 fm for the 48^3 and 64^3 ensembles, and 2.6-2.8 fm for 96^3 ensemble, with statistical errors computed using the jackknife method.

a (fm)	lattice value	FV corr.	FV + taste corr.	FV+taste+ m_π corr.
0.12121(64)	562.1(8.4)	564.2(8.4)	615.8(8.4)	613.6(8.4)
0.08787(46)	594.8(10.4)	601.7(10.4)	635.9(10.4)	630.2(10.4)
0.05684(30)	623.1(27.5)	638.7(27.5)	648.2(27.5)	647.1(27.5)
0		648.3(20.0)	657.9(20.0)	651.1(20.1)

Table 2: HVP contributions to the muon a_μ , in units of 10^{-10} , including ChPT corrections. The columns are discussed in the text.

The corrected results are tabulated in Table 2 and shown with the continuum limits in Fig. 2 (left figure). In Table 2, the second column includes the results from the bounding method, the third

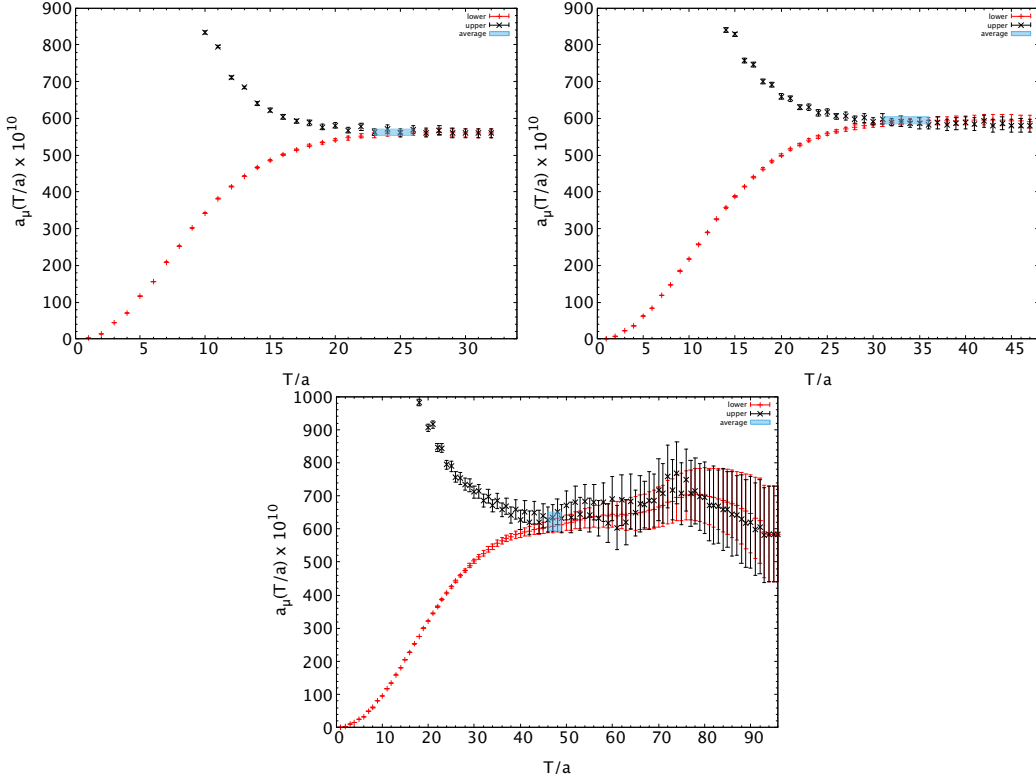


Figure 1: Bounding method for total contribution to the muon anomaly, using the weighting function w . Clockwise from the top we have the 48^3 , 64^3 , and 96^3 ensembles.

column includes the finite-volume corrections of Ref. [3], while the fourth column also includes the infinite-volume taste corrections in Table 1. The fifth column adjusts the values shown in the fourth column to a common pion mass of 135 MeV using NLO ChPT, to account for the small mistunings of the pion mass. Continuum extrapolated values of each column are shown in the last row. The left plot of Fig. 2 shows the continuum limits taken to get the results shown in Table 2.

After taking all of the corrections discussed into account, we obtain for our final result

$$a_\mu^{\text{HVP}} = (659 \pm 20 \pm 5 \pm 5 \pm 4) \times 10^{-10} = 659(22) \times 10^{-10} \quad (4.1)$$

where the errors quoted are, respectively, statistical, continuum extrapolation, scale setting, and higher orders in ChPT, and the final result shows them added in quadrature.

To explore a more precise comparison with other results, we adopt the window method of Ref. [5]: $a_\mu^W = 2 \sum_{t=0}^{T/2} C(t) w(t) (\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta))$, with $\Theta(t, t', \Delta) = \frac{1}{2} (1 + \tanh((t - t')/\Delta))$, where $t_1 - t_0$ is the size of the window and Δ is a suitably chosen width that smears out the window at either edge. We choose windows to avoid both lattice artifacts at short distance and large statistical errors at long distance. Results for several windows and both weighting functions are tabulated in Ref. [3].

In Fig. 2 (right figure), we show an example continuum limit combined with the window method with $t_0 = 0.4$ fm, $t_1 = 1$ fm, $\Delta = 0.15$.¹ Squares (crosses) correspond to uncorrected data

¹We note that since this talk was given, we have updated this figure to better compare with the domain wall fermion

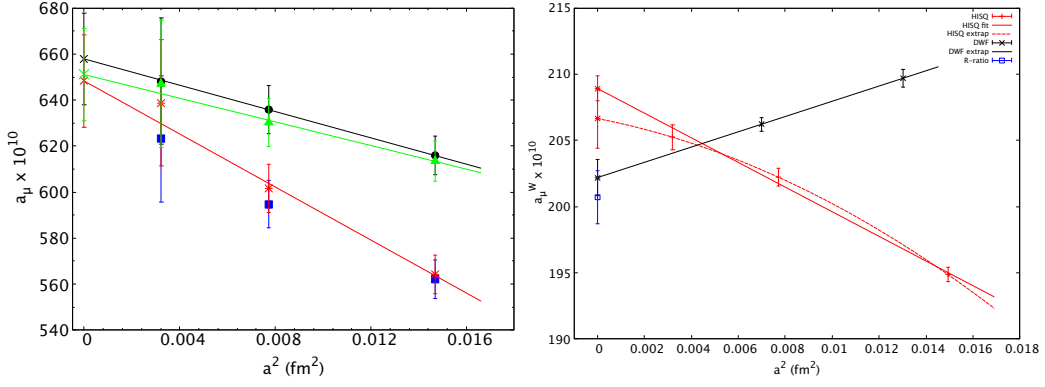


Figure 2: Left: Continuum limit after correcting the data according to columns 3, 4, and 5 (bursts, circles, and triangles, respectively) of Table 2 with the uncorrected data (squares) shown for comparison. Right: Continuum limit combined with the window method as described in the text.

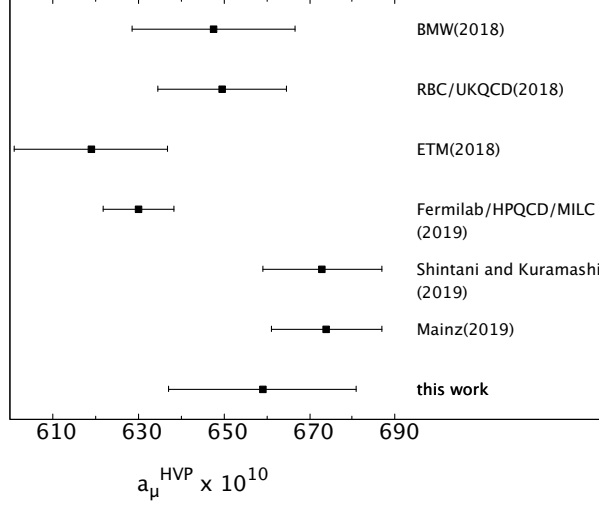


Figure 3: Contributions to the $g - 2$ from the connected light quark vacuum polarization from recent publications [9] (BMW), [5] (RBC/UKQCD), [10] (ETM), [11] (Fermilab/HPQCD/MILC), [12] (Shintani and Kuramashi), [13] (Mainz).

points with weighting function $w(\hat{w})$; filled circles are taste-breaking corrected to NLO of w data points. Solid curves show linear fits in a^2 ; all three agree very well in the continuum limit. Dashed curves denote a fully constrained parametrization (no degrees of freedom) using both a^2 and a^4 terms. Additionally we include the recent RBC/UKQCD computation using domain wall fermions, as the results should agree in the continuum limit up to small systematics.

We also show the corresponding dispersive e^+e^- value, using the R-ratio compilation of Ref. [14]. The largest difference is about 7×10^{-10} , or roughly 1 % of the total HVP contribution to a_μ . Given the uncertainties it is difficult to conclude there is a significant discrepancy,

calculation, and this will appear in the latest version of Ref. [3].

though the spread seems uncomfortably large. A third, smaller, lattice spacing ensemble is being generated by the RBC/UKQCD collaborations [15], which could firmly establish whether or not a discrepancy exists. The window method is a useful approach to cross-check different calculations using the most precise data available for each.

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