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# Lattice QCD calculation of the two-photon contributions to $K_L \to \mu^+\mu^-$ and $\pi^0 \to e^+e^-$ decays

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The rare second-order weak decay $K_L \to \mu^+\mu^-$ is precisely measured and sensitive to the structure of the weak interactions at short distances. However, these effects are obscured by a large third-order, long-distance contribution to this decay in which the muon pair is created by two photons. We will discuss the prospects for computing this third-order electroweak process using lattice QCD. As a first step in such a calculation a method will be presented for the lattice calculation of the simpler two-photon decay $\pi^0 \to e^+e^-$.

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The rare decays of $K$ mesons provide important opportunities for the discovery of physics beyond the standard model. For such processes a combination of highly sensitive experiments with increasingly accurate lattice QCD calculations should result in continually more sensitive tests of the standard model. The $K_L$-$K_S$ mass difference, the decay $K^+ \to \pi^+\nu\overline{\nu}$ and the CP-violating parameters $\varepsilon$ and $\varepsilon'$ are well-known examples. Here we will explore a less familiar example: the strangeness changing neutral current process $K_L \to \mu^+\mu^-$.

This decay arises as a one-loop, second-order weak process involving the exchange of two $W^\pm$ bosons or one $W^\pm$ and one $Z^0$ [1]. The second-order decay amplitude can be computed in the standard model with the kaon decay constant $f_K$ as the only required hadronic input. However, this test of the standard model at second order is complicated by the presence of an order $\alpha_{\rm EM}^2 G_F$ decay amplitude of approximately the same size.

Thus, we might hope to use lattice QCD to perform an accurate calculation of this background, two-photon process so that the complete decay could then be computed in the standard model. While such a calculation contains many new and potentially difficult challenges, it is similar in some respects to the calculation of the hadronic light-by-light (HLbL) scattering contribution to the anomalous magnetic moment of the muon, a process that does not include a weak interaction Hamiltonian but does involve three (not two) internal photon lines and which can be computed successfully using lattice QCD [2].

## 1. $K_L \to \mu^+\mu^-$ decay

We focus on the $K_L \to \mu^+\mu^-$ decay because, in contrast to the corresponding $K_S$ decay, its branching fraction is accurately measured: $\mathrm{BR}(K_L \to \mu^+\mu^-) = 6.84 \pm 0.11) \times 10^{-9}$ [3]. The $O(\alpha_{\rm EM}^2 G_F)$ background is described by a complex amplitude whose imaginary part is determined by the optical theorem and the known $K_L \to \gamma\gamma$ decay rate. Using the notation of Ref. [1]:

$$\frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(K_L \to \gamma\gamma)} = 2\beta_\mu \left(\frac{\alpha_{\rm EM}}{\pi}\frac{m_\mu}{M_K}\right)^2 \left(|F_{\rm imag}|^2 + |F_{\rm real}|^2\right). \tag{1.1}$$

The known decay rate and imaginary part determine $|F_{\rm real}| = 1.167 \pm 0.094$. Finally we can write $F_{\rm real} = (F_{\rm real})_{\rm EM} + (F_{\rm real})_{\rm weak}$ where the standard model predicts at one loop: $(F_{\rm real})_{\rm weak} = -1.82 \pm 0.04$. Thus, a lattice calculation of $(F_{\rm real})_{\rm EM}$ with 10% accuracy would determine $(F_{\rm real})_{\rm weak}$ to 6% or 17% depending on whether $F_{\rm real}$ and $(F_{\rm real})_{\rm weak}$ have the same or opposite signs.

While there are encouraging parallels between the two-photon contribution to $K_L \to \mu^+\mu^-$ decay and the HLbL amplitude, there is a very important difference: the entire HLbL calculation can be Wick rotated and computed directly in Euclidean space. The $K_L \to \mu^+\mu^-$ amplitude is complex, with threshold singularities and cannot be easily expressed as a Euclidean space calculation which would be required to directly apply lattice QCD.

A better analogue to the $K_L \to \mu^+\mu^-$ decay calculation is the calculation of the $K_L$-$K_S$ mass difference, $\Delta M_K$. Figures 1 and 2 provide schematic representations for each process. As in the calculation of $\Delta M_K$, the $K_L \to \mu^+\mu^-$ decay amplitude can be written as a Euclidean-space Green's function in which the five weak or electromagnetic operators are integrated over a finite time region $-T/2 \le t \le T/2$ which lies between the initial kaon operator and the operators destroying the final two muons, each carrying the three-momentum required by energy conservation. For both the $\Delta M_K$



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and $K_L \to \mu^+\mu^-$ cases, one can recognize the correct non-covariant, perturbation theory expression for the amplitude of interest among the terms that result from the two or five time integrals.


Figure 1: Schematic representation of the calculation of the $K_L$-$K_S$ mass difference. The two weak operators $H_W$ should be integrated over a temporal volume $-T/2 \le t \le T/2$ lying between the initial $K^0$ and final $\overline{K}^0$. The thick solid line represents the hadronic part of the amplitude.

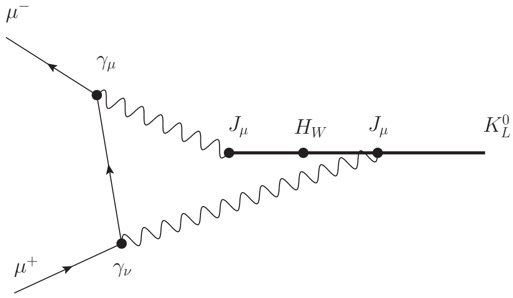


Figure 2: Schematic representation of the calculation of the $K_L \to \mu^+\mu^-$ decay. Here all five operators represented by dark dots should be integrated over a temporal volume $\pm T/2$ lying between the initial $K^0$ and the final two muons. The thick solid line represents the hadronic part of the amplitude for a specific time order.

However, also for both cases, additional terms will appear in these time-integrated Green's functions coming from the limits $\pm T/2$ of integration. These terms would vanish in a standard Minkowski space calculation. In Euclidean space most such terms also vanish but a few, associated with intermediate states less massive than the initial kaon, will contribute unphysical contributions growing exponentially with $T$ that must be removed. While such removal is practical for the case of $\Delta M_K$, it represents a substantial barrier to the lattice calculation of $K_L \to \mu^+\mu^-$. Examining the possible 120 different time orderings of the five internal vertices in Fig. 2 and identifying and evaluating the exponentially growing terms presents a serious challenge.

Since much of the diagram appearing in Fig. 2 involves standard relativistic Feynman perturbation theory, one might hope that for those parts of the diagram, one could avoid the complexities of non-covariant perturbation theory outlined above. In fact, such a simplified treatment is possible and is best illustrated by considering the less complicated decay $\pi^0 \to e^+e^-$.

## 2. $\pi^0 \to e^+e^-$

We now present a method to compute the simpler decay $\pi^0 \to e^+e^-$ using lattice methods which exploits Feynman perturbation theory for the internal electron and photon lines. We begin with the standard Minkowski-space expression for the decay amplitude

$$\mathscr{A}_{\pi^0\to e^+e^-} = \int d^4w L_{\mu\nu}(k_-,k_+,w)\langle 0|T\left\{J_\mu(w/2)J_\nu(-w/2)\right\}|\pi^0(\vec{P}=0)\rangle \qquad (2.1)$$

where the leptonic factor in this amplitude is given by the usual Feynman expression

$$L(k_-,k_+,w)_{\mu\nu} = \int dp_0 \int d^3p\, \overline{u}(\vec{k}_-)\gamma_\mu \frac{\not{p}-\not{k}_+ + m_e}{(p-k_+)^2+m_e^2-i\varepsilon}\gamma_\nu v(\vec{k}_+) \cdot \frac{1}{(p-\frac{P}{2})^2-i\varepsilon}\frac{1}{(p+\frac{P}{2})^2-i\varepsilon}e^{-ip\cdot w}. \qquad (2.2)$$

The amplitude $L(k_-,k_+,w)_{\mu\nu}$ is the standard Feynman expression for the three electron lines and two photon lines in Fig. 2. To be concrete, the Feynman diagram corresponding $L(k_-,k_+,w)_{\mu\nu}$ is repeated with the momentum assignments in Fig. 3. Here $P = (\vec{0}, P_0)$ is the four-momentum of the $\pi^0$, chosen to be at rest. The two factors in Eq. (2.1) are connected by their common position space arguments $\pm w/2$. Here $w$ is the difference between the space-time locations of the two hadronic electromagnetic currents. In obtaining Eq. (2.1) we have integrated over the average position of the two currents and removed the resulting delta function equating the pion's four-momentum $P$ and the sum of the two momenta $k_+$ and $k_-$ carried by the positron and the electron. Thus, $k_\pm$ must be assigned values conserving energy and momentum.

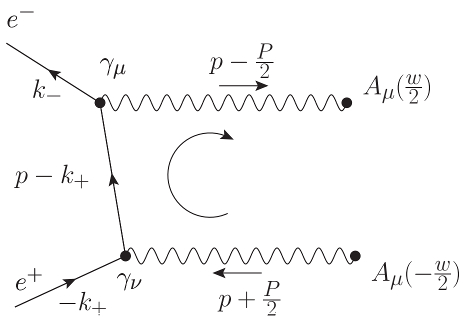


Figure 3: That portion of the $\pi^0 \to e^+e^-$ decay amplitude represented by the standard Feynman perturbation theory expression given in Eq. (2.2). The momentum assignments in this figure correspond to those used in that equation.

Next we observe that the Minkowski-space expression given in Eq. (2.1) for $\pi^0 \to e^+e^-$ decay can be directly evaluated using lattice QCD if we simply Wick-rotate the $w_0$ contour in that equation. Instead of integrating along the real axis from $-\infty$ to $+\infty$, we instead choose $w_0$ to follow the contour $w_0 = \widetilde{w}_0 e^{-i\phi}$ where the real parameter $\widetilde{w}_0$ continues to vary between $-\infty$ and $+\infty$ but the angle $\phi$ increases from 0 to $\pi/2$. To carry out this deformation of the $w_0$ integration contour we must analytically continue both the Feynman amplitude $L_{\mu\nu}(k_+,k_-,w)$ and the hadronic Green's function.

The dependence of the latter on the complex variable $w_0$ is easily determined by substituting a sum over intermediate states between the two currents:

$$\langle 0|T\left\{J_\mu(\frac{w}{2})J_\nu(-\frac{w}{2})\right\}|\pi^0\rangle = \sum_n \Big\{\theta(w_0)\langle 0|J_\mu(\frac{\vec{w}}{2},0)|n\rangle\langle n|J_\nu(-\frac{\vec{w}}{2},0)|\pi^0\rangle e^{-iw_0(E_n-M_\pi/2)}$$
$$+\theta(-w_0)\langle 0|J_\nu(-\frac{\vec{w}}{2},0)|n\rangle\langle n|J_\mu(\frac{\vec{w}}{2},0)|\pi^0\rangle e^{+iw_0(E_n-M_\pi/2)}\Big\}. \qquad (2.3)$$

The analytic dependence of the exponential function on $w_0$ permits this change of contour and the resulting behavior at large $w_0$ changes from oscillatory to exponentially damped with the least-rapid decrease coming from the two-pion intermediate state with the behavior $\exp(-3M_\pi|w_0|/2)$. With this change of contour the hadronic Minkowski-space matrix element given in Eq. (2.3) is now evaluated at imaginary time making it a conventional Euclidean space amplitude that can be directly evaluated using lattice QCD.

The analytic continuation of the factor $L(k_-,k_+,w)_{\mu\nu}$ requires more discussion since it is expressed as in integral over the variable $p_0$ in Eq. (2.2) and a naive change of phase for $w_0$ will cause that integral to diverge. However, as we vary $w_0$ to explore the asymptotic behavior of this integral in the complex plane we are free to use Cauchy's theorem to vary the $p_0$ integration contour in Eq. (2.2) to obtain maintain the convergence of the $p_0$ integration and to obtain the smallest upper bound on the growth the $L(k_-, k_+, w)_{\mu\nu}$ for large $|w_0|$. The convergence of the $p_0$ integral can be maintained if, for large $|p_0|$, the phase of $p_0$ is adjusted to cancel that of $w_0$. For large $p_0$ we can choose the $p_0$ contour so that $p_0 = \widetilde{p}_0 e^{+i\phi}$ where $\phi$ is the phase introduced into $w_0$ above.

As indicated in Fig. 4 we cannot make a simple Wick rotation of the $p_0$ contour for small values of $p_0$ because of the poles at $p_0 = \pm M_\pi/2 \mp |\vec{p}|(1 - i\varepsilon)$ in the first and third quadrants. The resulting non-zero real part of $p_0$ implies that for large Wick-rotated $w_0$, the integral $L(k_-, k_+, w)_{\mu\nu}$ will grow as $e^{|w_0|M_\pi/2}$. However, this growth is canceled by the $e^{-3|w_0|M_\pi/2}$ asymptotic behavior of the hadronic factor so that the $w_0$ integral along the imaginary axis is convergent. This method has been successfully used to compute $\mathscr{A}_{\pi^0\to e^+e^-}$ and the results reported in the companion talk [4].

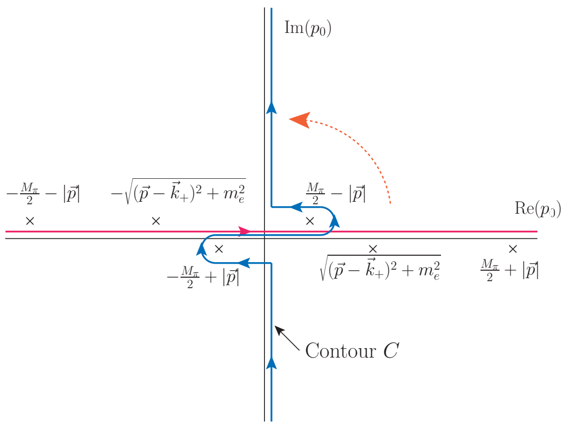


Figure 4: The initial (red, horizontal) and final (blue, vertical) $p_0$ contour employed when using Cauchy's theorem to deform the $p_0$ contour appearing Eq. (2.2). Because of the on-shell pion energy $M_\pi$ the $p_0$ singularities $\pm m_\pi/2 \mp |\vec{p}| \pm i\varepsilon$ can appear in the first and third quadrants in the complex $p_0$ plane and prevent a conventional Wick rotation where the $p_0$ contour simply follows the imaginary axis.

Equations (2.1) and (2.2) describe the $\pi^0 \to e^+e^-$ decay easily without the difficulties associated with two-photon states whose energy lies below the $\pi^0$ mass. Instead these equations give the correct, complex Minkowski space amplitude amplitude with an imaginary part determined by the optical theorem and the $\pi^0 \to \gamma\gamma$ decay amplitude. Because of Cauchy's theorem, our changes to the $w_0$ and $p_0$ contours do not alter the final result for $\mathscr{A}_{\pi^0\to e^+e^-}$ and allow us to evaluate this complex Minkowski-space amplitude using lattice QCD.

## 3. Two-photon contribution to $K_L \to \mu^+\mu^-$

If we attempt to apply this same approach to the more interesting decay $K_L \to \mu^+\mu^-$, we find that the simple rotation of contours that allowed the direct calculation of $\pi^0 \to e^+e^-$ encounters difficulties. In Fig. 5 we show the three different time orders that occur as one integrates over the time coordinates of the two E&M currents and the weak Hamiltonian.

Case (A) in Fig. 5 offers no special difficulties. Here the change in strangeness caused by the action of $H_W$ occurs after the two currents have acted and the states that can be produced by the first current acting on the $K_L$ must be more massive that the kaon. Specifically $E_Y \geq M_K + M_\pi$ where the intermediate state $Y$ entering between the two currents is labeled in diagram (A) of Fig. 5. Following the same approach used for the $\pi^0 \to e^+e^-$ above, the Wick rotation of the

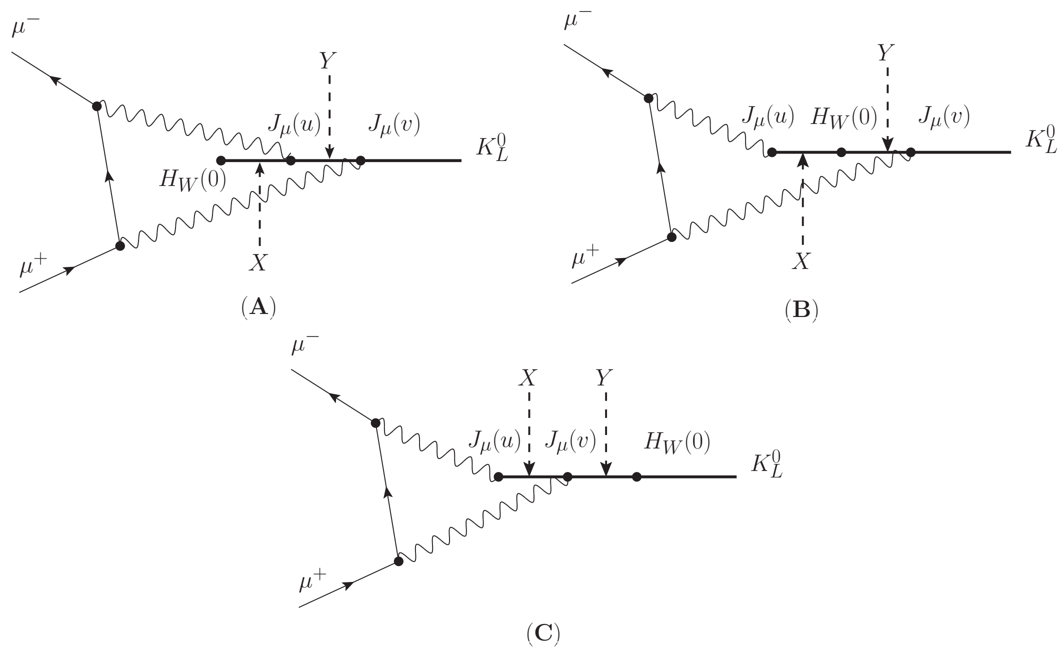


Figure 5: Diagrams showing the three different time orders for the two hadronic E&M currents and the weak Hamiltonian which appear in the $K_L^0 \to \mu^+\mu^-$ decay amplitude. In each case $X$ and $Y$ label the possible hadronic intermediate states.

$w_0$ contour gives a convergent Euclidean-time integral that can be evaluated using lattice QCD. In analogy with the $\pi^0 \to\ e^+e^-$ case, the Feynman loop integral will introduce a factor which grows exponentially with increasing $w_0$, behaving as $e^{|w_0|M_K/2}$. However, the product of the two E&M currents contributes a compensating factor $e^{-|w_0|(E_Y-M_K/2)}$, making the Euclidean-time $w_0$ integral convergent as in the $\pi^0 \to\ e^+e^-$ case.

Case (B) is less favorable with the desired Wick rotation prevented by the apparent exponential growth of the integrand for large $|w_0|$ arising from states $X$ whose energy is less than the kaon mass. For such a state the exponential growth of the one-loop Feynman integral identified above, $e^{|w_0|M_K/2}$, will not be overcome by the time dependence of the hadronic factor which will behave as $e^{-|w_0|(E_X-M_K/2)}$. However, we might regulate this $e^{|w_0|(M_K-E_X)}$ growth by introducing a finite upper limit for the $w_0$ integral. We can then perform the integral over $w_0$ and explicitly remove the contribution of those states $X$ with $E_X < M_K$ to the term arising from upper limit of the $w_0$ integral.

The most important states with $E_X < M_K$ are two-pion states, possibly with a non-zero spatial momentum which is quantized by the finite spatial volume in which a lattice calculation would be performed. It seems likely that this treatment will give a correct contribution to the real part of the hadronic amplitude in the limit that the spatial volume approaches infinity, as is the case when a similar procedure is applied when computing $\Delta M_K$. However, for this approach to be useful we need to have theoretical control over the finite volume effects caused by replacing a principal part integral by a few discrete energy denominators of the form $1/[E_X-(M_K+p_0)/2]$.

While such a theory has been worked out for the case of $\Delta M_K$ [5, 6], a similar treatment for the present case is not known and may not exist. For case (B) integrals which are present in infinite volume (integrals over the two-pion center-of-mass energy and 3-momentum of the two-pion center-of-mass) are being represented by a few finite-volume states $X$ when $E_X \sim M_K$. Determining the finite volume correction needed for the imaginary part of this amplitude appears especially challenging. Case (C) is even more problematic. Now the Wick-rotated $w_0$ integral will not converge if either $E_X$ or $E_Y$ or both are less than $M_K$. A more powerful approach may be needed.

Perhaps the next process to examine more carefully is the decay $K_L \to \gamma\gamma$. For this decay the spatial momentum of each final-state photon must have a magnitude equal to $M_K/2$. This insures that those states $X$ causing difficulty in cases (B) and (C) must have energy larger than that of the photon into which they decay, resulting in convergent Euclidean time integrals. The only difficulty arises from orderings analogous to case (C) with $E_Y \le M_K$ which appears to offer no challenges beyond those already overcome in the calculation of $\Delta M_K$.

In this presentation we have discussed some of the difficulties that must be overcome if the two-photon contribution to the decay process $K_L \to \mu^+\mu^-$ is to be computed using lattice QCD. Such a calculation would allow the interesting, strangeness changing neutral current decay amplitude of $O(G_F^2)$ to be determined from the existing accurate measurement of the $K_L \to \mu^+\mu^-$ decay rate. While much remains to be understood before such a calculation becomes practical, the method proposed to use lattice QCD to calculate the simpler decay $\pi^0 \to e^+e^-$ appears to be a promising first step. The Wick rotation method introduced here allows the complete complex decay amplitude to be determined from a straight-forward lattice QCD calculation with finite volume errors that are exponentially suppressed in the lattice size. This method has the further advantage that it exploits the covariant Feynman amplitude for that portion of the matrix element which involves only photon and electron propagators. This offers a substantial simplification over the use of non-covariant perturbation that is customary in similar applications of lattice QCD.

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