Radiative leptonic decays on the lattice

Christopher Kane\textsuperscript{a}, Christoph Lehner\textsuperscript{b,c}, Stefan Meinel\textsuperscript{a,d}, Amarjit Soni\textsuperscript{c}

\textsuperscript{a}Department of Physics, University of Arizona, Tucson, AZ 85721, USA
\textsuperscript{b}Department of Physics, University of Regensburg, 93040 Regensburg, Germany
\textsuperscript{c}Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA
\textsuperscript{d}RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA

E-mail: smeinel@email.arizona.edu

Adding a hard photon to the final state of a leptonic pseudoscalar-meson decay lifts the helicity suppression and can provide sensitivity to a larger set of operators in the weak effective Hamiltonian. Furthermore, radiative leptonic $B$ decays at high photon energy are well suited to constrain the first inverse moment of the $B$-meson light-cone distribution amplitude, an important parameter in the theory of nonleptonic $B$ decays. We demonstrate that the calculation of radiative leptonic decays is possible using Euclidean lattice QCD, and present preliminary numerical results for $D_s^+ \rightarrow \ell^+ \nu \gamma$ and $K^- \rightarrow \ell^- \bar{\nu} \gamma$.
1. Introduction

Radiative leptonic decays of pseudoscalar mesons probe both the weak interaction and the hadronic structure in useful ways. Adding a sufficiently energetic photon to the final state can actually increase the branching fraction [2], as it removes the helicity suppression. Perhaps the most interesting example is $B^- \rightarrow \ell^- \bar{\nu} \gamma$, shown in Fig. 1 (left). For large $E_\gamma$, this process is the cleanest probe of the first inverse moment of the $B$-meson light-cone distribution amplitude, $1/\lambda_B = \int_0^1 \frac{d \omega}{\omega} C_{B}(\omega)$, an important input in QCD-factorization predictions for nonleptonic $B$ decays that is presently poorly determined [3, 4, 5, 6, 7, 8, 9]. A recent search for this decay by Belle gave an upper limit $\mathcal{B}(B^- \rightarrow \ell^- \bar{\nu} \gamma, E_\gamma > 1$ GeV $) < 3 \times 10^{-6}$, close to the Standard-Model expectation [10]. Lattice QCD results for the $B^- \rightarrow \ell^- \bar{\nu} \gamma$ form factors could be used to constrain $\lambda_B$. Also very interesting are the flavor-changing neutral-current decays $B^0 \rightarrow \ell^+ \ell^- \gamma$ and $B_s \rightarrow \ell^+ \ell^- \gamma$ (shown in Fig. 1, right). While the purely leptonic decays are sensitive to $C_{10,S,P}$ only, the radiative leptonic decays probe all Wilson coefficients in the weak effective Hamiltonian, including $C_9$, in which global fits of experimental results for other $b \rightarrow s \ell^+ \ell^-$ decays indicate a deviation from the Standard Model that violates lepton flavor universality (LFU) [1]. Since the radiative leptonic decays are not helicity-suppressed, they are well-suited for testing LFU with light leptons [11, 12]. For the charmed-meson radiative leptonic decays $D^+ \rightarrow e^+ \nu \gamma$ and $D^+_s \rightarrow e^+ \nu \gamma$, the BESIII collaboration has reported upper limits on the branching fractions with $E_\gamma > 10$ MeV of $3 \times 10^{-5}$ and $1.3 \times 10^{-4}$, respectively [13, 14]. Finally, in contrast to the heavy-meson decays, there are already precise measurements of the differential branching fractions of $K^- \rightarrow e^- \nu \gamma$, $K^- \rightarrow \mu^- \nu \gamma$, $\pi^- \rightarrow e^- \nu \gamma$, and $\pi^- \rightarrow \mu^- \nu \gamma$, as reviewed in Ref. [15]. These decay modes can therefore be used to test the lattice QCD methods.

In the following, we show how radiative leptonic decays can be calculated on a Euclidean lattice, and we present early numerical results. One of us previously reported on this project at the Lattice 2018 conference [16]. At Lattice 2019, radiative leptonic decays were also discussed by G. Martinelli [17].

2. Hadronic tensor and form factors

To define the form factors for charged-current radiative leptonic decays of pseudoscalar mesons, we use the notation for $B^- \rightarrow \ell^- \bar{\nu} \gamma$. The quark electromagnetic and weak currents are given by $J_\mu = \sum_q e_q \bar{q} \gamma_\mu q$ and $J_\mu^{\text{weak}} = \bar{u} \gamma_\mu (1 - \gamma_5) b$. The decay amplitude depends on the hadronic tensor,
which is defined as

$$T_{\mu\nu} = -i \int d^4x \, e^{ip_{\gamma} x} \langle 0 | T (J_{\mu}(x) J_{\nu}^{\text{weak}}(0)) | B^{-}(p_B) \rangle$$  \hspace{1cm} (2.1)$$

in Minkowski space. Throughout this work, we assume that the photon is real, i.e., $p_{\gamma}^2 = 0$. The hadronic tensor can be decomposed as [7]

$$T_{\mu\nu} = \epsilon_{\mu\nu\tau\rho} p_B^\tau v^\rho F_{\tau\rho} + i[-g_{\mu\nu}(p_B \cdot v) + v_{\mu}(p_B)] F_A - i \frac{V_{\mu\nu}}{p_B \cdot v} m_B f_B + (p_B)_{\mu\tau} \epsilon_{\tau\rho\nu} v^\rho,$$  \hspace{1cm} (2.2)$$

where $p_B = m_B v$ and the $(p_B)_{\mu\tau}$-terms will disappear when contracting with the photon polarization vector. The form factors $F_{\tau\rho}$ and $F_A$ are functions of the photon energy in the $B$-meson rest frame, $E_{\gamma}^{(0)} = p_B \cdot v = (m_B^2 - q^2)/(2m_B)$. Also appearing in Eq. (2.2) is the $B$-meson decay constant $f_B$.

To prepare for the discussion in the next section, it is useful to write down the spectral representation of $T_{\mu\nu}$ in Minkowski space for the two different time orderings of the currents. By inserting complete sets of energy/momentum eigenstates and performing the time integrals, we find

$$T_{\mu\nu}^- = -i \int_{-\infty}^0 dt \, e^{iE_{\gamma} t} \int d^3x \, e^{-ip_{\gamma} x} \langle 0 | J_{\mu}(t, x) J_{\nu}^{\text{weak}}(0) | B^{-}(p_B) \rangle$$

\hspace{1cm} (2.3)$$

and

$$T_{\mu\nu}^+ = -i \int_0^{\infty} dt \, e^{iE_{\gamma} t} \int d^3x \, e^{-ip_{\gamma} x} \langle 0 | J_{\mu}(t, x) J_{\nu}^{\text{weak}}(0) | B^{-}(p_B) \rangle$$

\hspace{1cm} (2.4)$$

(in infinite volume, the sums over $n$ and $m$ include integrals over the continuous spectrum of multi-particle states).

3. Extracting the hadronic tensor from a Euclidean three-point function

In this section, we show that $T_{\mu\nu}$ can be extracted from the Euclidean three-point function

$$C_{\mu\nu}(t, t_B) = \int d^3x \int d^3y \, e^{-ip_{\gamma} x} e^{ip_{\gamma} y} \langle J_{\mu}(t, x) J_{\nu}^{\text{weak}}(0, 0) \phi^{\gamma \tau}(t_B, y) \rangle,$$  \hspace{1cm} (3.1)$$

where $\phi_B \sim \bar{u} \gamma_\tau b$ is an interpolating field for the $B$ meson, and $t, t_B$ now denote the Euclidean time. We define the integrals

$$I_{\mu\nu}^<(t_B, T) = \int_{-T}^0 dt \, e^{E_{\gamma} t} C_{\mu\nu}(t, t_B), \hspace{1cm} I_{\mu\nu}^>(t_B, T) = \int_0^T dt \, e^{E_{\gamma} t} C_{\mu\nu}(t, t_B),$$  \hspace{1cm} (3.2)$$

with a finite integration range $T$. Here we take $t_B$ to be large and negative (with $t_B < -T$), such that ground-state saturation is achieved for the $B$ meson. Inserting again complete sets of energy/momentum eigenstates, we find, for the first time ordering,

$$I_{\mu\nu}^<(t_B, T) = \langle B(p_B) | \phi^{\gamma \tau}_B(0) | 0 \rangle \frac{1}{2E_B} e^{E_{\gamma} t_B}$$

\hspace{1cm} (3.3)$$

\times \sum_n \frac{1}{2E_n(p_B - p_{\gamma})} \frac{\langle 0 | J_{\nu}^{\text{weak}}(0) | n(p_B - p_{\gamma}) \rangle \langle n(p_B - p_{\gamma}) | J_{\mu}(0) | B(p_B) \rangle}{E_{\gamma} + E_n(p_B - p_{\gamma}) - E_B} \times \left( 1 - e^{-(E_{\gamma} + E_n(p_B - p_{\gamma}) - E_B)T} \right).
The sum over states in Eq. (3.3) differs from the sum in Eq. (2.3) by the factor in the last line. However, the exponential \( e^{-(E_T + E_{m,p_T}) - E_B T} \) will vanish for large \( T \) if \( E_T + E_{m,p_T} > E_B \). Because the states \(|n(p_B - p_\gamma)|\) have the same quark-flavor quantum numbers as the \( B \) meson, we have \( E_{m,p_T} \geq E_B(p_B - p_\gamma) = \sqrt{m_B^2 + (p_B - p_\gamma)^2} \). Thus, we need \( \sqrt{p_T^2 + m_B^2 + (p_B - p_\gamma)^2} > \sqrt{m_B^2 + p_B^2} \). This is in fact always true if \( p_\gamma \neq 0 \).

For the other time ordering, we find
\[
I_{\mu\nu}(t_B, T) = -\langle B(p_B)|\phi_B^\dagger(0)|0 \rangle \frac{1}{2E_B} e^{E_B t_B} \times \sum_m \frac{1}{2E_{m,p_T}} \langle 0|J_\mu(0)|m(p_T)\rangle \langle m(p_T)|J^{\text{weak}}_\nu(0)|B(p_B)\rangle \frac{1}{E_T - E_{m,p_T}} \left(1 - e^{-(E_T - E_{m,p_T})T}\right). \tag{3.4}
\]

The unwanted exponential \( e^{(E_T - E_{m,p_T})T} \) in the last line goes to zero for large \( T \) if \( E_{m,p_T} > E_T \). Because the states \(|m(p_T)\rangle\) are hadronic and have nonzero masses, their energies are larger than the energy of a photon with the same spatial momentum, showing that this condition is also always satisfied.

In summary, for \( p_\gamma \neq 0 \),
\[
T_{\mu\nu} = -\lim_{T \to -\infty} \lim_{t_B \to -\infty} \frac{2E_B e^{-E_B t_B}}{\langle B(p_B)|\phi_B^\dagger(0)|0 \rangle} I_{\mu\nu}(t_B, T), \tag{3.5}
\]
where \( I_{\mu\nu} \) is the integral from \(-T\) to \( T \). The energy \( E_B \) and the overlap factor \( \langle B(p_B)|\phi_B^\dagger(0)|0 \rangle \) can be obtained from the two-point function \( \int d^3x e^{-ip_0 x} \langle \phi_B(t, x) \phi_B^\dagger(0) \rangle \).

Note that similar nonlocal matrix elements appear in processes with two photons, whose lattice calculation has been discussed, for example, in Refs. [18, 19, 20].

4. Preliminary numerical results

In this section, we present some early numerical results for the \( D_s^+ \to \ell^- \nu \gamma \) and \( K^- \to \ell^- \nu \gamma \) form factors. These results are from only 25 configurations of the “24I” RBC/UKQCD ensemble.
Radiative leptonic decays on the lattice

Stefan Meinel

Figure 3: The $D_s^+ \to \ell^+ \nu \gamma$ and $K^- \to \ell^- \bar{\nu} \gamma$ form factors at $p_T = (0, 0, 1) \frac{2\pi}{T}$ as a function of the summation range $T$, for two different meson-field insertion times.

Figure 4: The $D_s$ and $K$ decay constants extracted from $T_{\mu\nu}$ at $p_T = (0, 0, 1) \frac{2\pi}{T}$, as a function of the summation range $T$, for two different meson-field insertion times. For the $D_s$, the horizontal line shows the physical value from Ref. [21]. For the $K$, the horizontal line shows the value computed on the same ensemble with the standard method in Ref. [22].

[22] with $2+1$ flavors of domain-wall fermions and the Iwasaki gauge action, with $a^{-1} = 1.785(5)$ GeV and $m_s = 340(1)$ MeV. For the light and strange valence quarks, we use the same domain-wall action as in Ref. [22]. The valence charm quark is implemented with a Möbius domain-wall action with stout-smeared gauge links ($N = 3$, $\rho = 0.1$), $L_5/a = 12$, $aM_5 = 1.0$, $am_f = 0.6$ [23], which approximately corresponds to the physical charm-quark mass. We use local currents with “mostly nonperturbative” renormalization. Gaussian smearing is performed for the lighter quark in the meson interpolating field. We start with a $Z_2$ random-wall source at the time slice of the weak current (denoted as time “0” here) and perform sequential inversions through the meson in-
terpolating field; disconnected diagrams are presently neglected. All-mode averaging [24] with 16 sloppy and 1 exact samples per configuration is employed; the 16 sloppy samples correspond to 16 different starting time slices. Our initial calculations used $p_{K/D_s} = 0$ and $p_T^2 \in \{1, 2, 3, 4, 5\} \left(\frac{2\pi}{T}\right)^2$.

Figure 2 shows examples of the $D_s^+ \to \ell^+ \nu \gamma$ form factors as a function of the photon energy. The results shown here were obtained with $T/a = 8$ and $t_{K/D_s}/a = -12$. Only the statistical uncertainties are given.

### 5. Conclusions and Outlook

We have shown that the form factors describing radiative leptonic decays can be calculated on the lattice; even though they involve a nonlocal matrix element, the use of imaginary time poses no difficulty in this case. The early results shown here for $D_s^+ \to \ell^+ \nu \gamma$ and $K^- \to \ell^- \bar{\nu} \gamma$ cover photon energies from approximately 0.5 to 1 GeV. For $K^- \to \ell^- \bar{\nu} \gamma$ we need to extrapolate in the mass. We are also considering calculations directly at the physical $b$-quark mass using the “relativistic heavy-quark action” [25], but, because this action is only on-shell improved, additional steps are likely needed to remove unphysical behavior occurring when the electromagnetic and weak currents get close to each other.

**Acknowledgments:** We thank the RBC and UKQCD Collaborations for providing the gauge-field configurations. C.K. and S.M. are supported by the US DOE, Office of Science, Office of HEP under award number DE-SC0009913. S.M. is also supported by the RIKEN BNL Research Center. A.S. and C.L. are supported in part by US DOE Contract No. DESC0012704(BNL). During a part of this work, C.L. was also supported by a DOE Office of Science Early Career Award. This work used resources at TACC that are part of XSEDE, supported by NSF grant number ACI-1548562.