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# Radiative leptonic decays on the lattice

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Adding a hard photon to the final state of a leptonic pseudoscalar-meson decay lifts the helicity suppression and can provide sensitivity to a larger set of operators in the weak effective Hamiltonian. Furthermore, radiative leptonic $B$ decays at high photon energy are well suited to constrain the first inverse moment of the $B$-meson light-cone distribution amplitude, an important parameter in the theory of nonleptonic $B$ decays. We demonstrate that the calculation of radiative leptonic decays is possible using Euclidean lattice QCD, and present preliminary numerical results for $D_s^+ \to \ell^+ \nu\gamma$ and $K^- \to \ell^- \bar{\nu}\gamma$.

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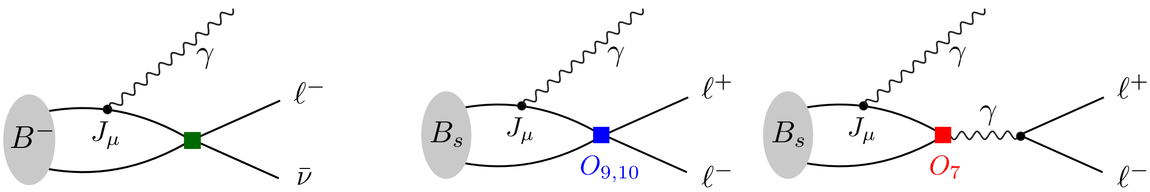


**Figure 1:** Left: A diagram contributing to $B^- \to \ell^- \bar{\nu}\,\gamma$, where the green square corresponds to $W$-boson exchange in the Standard Model and comes with a factor of $V_{ub}$. Right: two diagrams contributing to $B_s \to \ell^+\ell^-\gamma$, via the operators $O_{7,9,10}$ (defined, for example, in Ref. [1]).

## 1. Introduction

Radiative leptonic decays of pseudoscalar mesons probe both the weak interaction and the hadronic structure in useful ways. Adding a sufficiently energetic photon to the final state can actually increase the branching fraction [2], as it removes the helicity suppression. Perhaps the most interesting example is $B^- \to \ell^- \bar{\nu}\,\gamma$, shown in Fig. 1 (left). For large $E_\gamma^{(0)}$, this process is the cleanest probe of the first inverse moment of the $B$-meson light-cone distribution amplitude, $1/\lambda_B = \int_0^\infty \frac{\Phi_{B^+}(\omega)}{\omega}\, d\omega$, an important input in QCD-factorization predictions for nonleptonic $B$ decays that is presently poorly determined [3, 4, 5, 6, 7, 8, 9]. A recent search for this decay by Belle gave an upper limit $\mathscr{B}(B^- \to \ell^- \bar{\nu}\gamma,\, E_\gamma^{(0)} > 1\,\mathrm{GeV}) < 3.0 \times 10^{-6}$, close to the Standard-Model expectation [10]. Lattice QCD results for the $B^- \to \ell^- \bar{\nu}\gamma$ form factors could be used to constrain $\lambda_B$. Also very interesting are the flavor-changing neutral-current decays $B^0 \to \ell^+\ell^-\gamma$ and $B_s \to \ell^+\ell^-\gamma$ (shown in Fig. 1, right). While the purely leptonic decays are sensitive to $C_{10,S,P} - C'_{10,S,P}$ only, the radiative leptonic decays probe all Wilson coefficients in the weak effective Hamiltonian, including $C_9$, in which global fits of experimental results for other $b \to s\ell^+\ell^-$ decays indicate a deviation from the Standard Model that violates lepton flavor universality (LFU) [1]. Since the radiative leptonic decays are not helicity-suppressed, they are well-suited for testing LFU with light leptons [11, 12]. For the charmed-meson radiative leptonic decays $D^+ \to e^+\nu\gamma$ and $D_s^+ \to e^+\nu\gamma$, the BESIII collaboration has reported upper limits on the branching fractions with $E_\gamma^{(0)} > 10\,\mathrm{MeV}$ of $3.0 \times 10^{-5}$ and $1.3 \times 10^{-4}$, respectively [13, 14]. Finally, in contrast to the heavy-meson decays, there are already precise measurements of the differential branching fractions of $K^- \to e^-\bar{\nu}\gamma$, $K^- \to \mu^-\bar{\nu}\gamma$, $\pi^- \to e^-\bar{\nu}\gamma$, and $\pi^- \to \mu^-\bar{\nu}\gamma$, as reviewed in Ref. [15]. These decay modes can therefore be used to test the lattice QCD methods.

In the following, we show how radiative leptonic decays can be calculated on a Euclidean lattice, and we present early numerical results. One of us previously reported on this project at the Lattice 2018 conference [16]. At Lattice 2019, radiative leptonic decays were also discussed by G. Martinelli [17].

## 2. Hadronic tensor and form factors

To define the form factors for charged-current radiative leptonic decays of pseudoscalar mesons, we use the notation for $B^- \to \ell^- \bar{\nu}\,\gamma$. The quark electromagnetic and weak currents are given by $J_\mu = \sum_q e_q \bar{q}\gamma_\mu q$ and $J_\nu^{\mathrm{weak}} = \bar{u}\gamma_\nu(1-\gamma_5)b$. The decay amplitude depends on the hadronic tensor, which is defined as

$$T_{\mu\nu} = -i\int d^4x\, e^{ip_\gamma\cdot x}\langle 0|\mathsf{T}\left(J_\mu(x)\, J_\nu^{\rm weak}(0)\right)|B^-(\mathbf{p}_B)\rangle \tag{2.1}$$

in Minkowski space. Throughout this work, we asssume that the photon is real, i.e., $p_\gamma^2 = 0$. The hadronic tensor can be decomposed as [7]

$$T_{\mu\nu} = \varepsilon_{\mu\nu\tau\rho}\, p_\gamma^\tau v^\rho F_V + i[-g_{\mu\nu}(p_\gamma\cdot v) + v_\mu(p_\gamma)_\nu]F_A - i\frac{v_\mu v_\nu}{p_\gamma\cdot v} m_B f_B + (p_\gamma)_\mu\text{-terms}, \tag{2.2}$$

where $p_B = m_B v$ and the $(p_\gamma)_\mu$-terms will disappear when contracting with the photon polarization vector. The form factors $F_V$ and $F_A$ are functions of the photon energy in the $B$-meson rest frame, $E_\gamma^{(0)} = p_\gamma\cdot v = (m_B^2 - q^2)/(2m_B)$. Also appearing in Eq. (2.2) is the $B$-meson decay constant $f_B$.

To prepare for the discussion in the next section, it is useful to write down the spectral representation of $T_{\mu\nu}$ in Minkowski space for the two different time orderings of the currents. By inserting complete sets of energy/momentum eigenstates and performing the time integrals, we find

$$\begin{aligned} T_{\mu\nu}^< &= -i\int_{-\infty(1-i\varepsilon)}^0 dt \quad e^{iE_\gamma t}\int d^3x\, e^{-i\mathbf{p}_\gamma\cdot\mathbf{x}}\langle 0|J_\nu^{\rm weak}(0)\, J_\mu(t,\mathbf{x})|B^-(\mathbf{p}_B)\rangle \\ &= -\sum_n \frac{1}{2E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)}}\frac{\langle 0|J_\nu^{\rm weak}(0)|n(\mathbf{p}_B-\mathbf{p}_\gamma)\rangle\langle n(\mathbf{p}_B-\mathbf{p}_\gamma)|J_\mu(0)|B(\mathbf{p}_B)\rangle}{E_\gamma + E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)} - E_B - i\varepsilon}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} T_{\mu\nu}^> &= -i\int_0^{\infty(1-i\varepsilon)} dt \quad e^{iE_\gamma t}\int d^3x\, e^{-i\mathbf{p}_\gamma\cdot\mathbf{x}}\langle 0|J_\mu(t,\mathbf{x})\, J_\nu^{\rm weak}(0)|B^-(\mathbf{p}_B)\rangle \\ &= \sum_m \frac{1}{2E_{m,\mathbf{p}_\gamma}}\frac{\langle 0|J_\mu(0)|m(\mathbf{p}_\gamma)\rangle\langle m(\mathbf{p}_\gamma)|J_\nu^{\rm weak}(0)|B(\mathbf{p}_B)\rangle}{E_\gamma - E_{m,\mathbf{p}_\gamma} - i\varepsilon} \end{aligned} \tag{2.4}$$

(in infinite volume, the sums over $n$ and $m$ include integrals over the continuous spectrum of multi-particle states).

## 3. Extracting the hadronic tensor from a Euclidean three-point function

In this section, we show that $T_{\mu\nu}$ can be extracted from the Euclidean three-point function

$$C_{\mu\nu}(t,t_B) = \int d^3x\int d^3y\, e^{-i\mathbf{p}_\gamma\cdot\mathbf{x}}e^{i\mathbf{p}_B\cdot\mathbf{y}}\left\langle J_\mu(t,\mathbf{x})\; J_\nu^{\rm weak}(0,\mathbf{0})\;\phi_B^\dagger(t_B,\mathbf{y})\right\rangle, \tag{3.1}$$

where $\phi_B \sim \bar{u}\gamma_5 b$ is an interpolating field for the $B$ meson, and $t$, $t_B$ now denote the Euclidean time. We define the integrals

$$I_{\mu\nu}^<(t_B,T) = \int_{-T}^0 dt\; e^{E_\gamma t}\, C_{\mu\nu}(t,t_B), \qquad I_{\mu\nu}^>(t_B,T) = \int_0^T dt\; e^{E_\gamma t}\, C_{\mu\nu}(t,t_B), \tag{3.2}$$

with a finite integration range $T$. Here we take $t_B$ to be large and negative (with $t_B < -T$), such that ground-state saturation is achieved for the $B$ meson. Inserting again complete sets of energy/momentum eigenstates, we find, for the first time ordering,

$$\begin{aligned} I_{\mu\nu}^<(t_B,T) &= \langle B(\mathbf{p}_B)|\phi_B^\dagger(0)|0\rangle\frac{1}{2E_B}e^{E_B t_B} \\ &\quad\times\sum_n \frac{1}{2E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)}}\frac{\langle 0|J_\nu^{\rm weak}(0)|n(\mathbf{p}_B-\mathbf{p}_\gamma)\rangle\langle n(\mathbf{p}_B-\mathbf{p}_\gamma)|J_\mu(0)|B(\mathbf{p}_B)\rangle}{E_\gamma + E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)} - E_B} \\ &\quad\times\left(1 - e^{-(E_\gamma + E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)} - E_B)T}\right). \end{aligned} \tag{3.3}$$



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The sum over states in Eq. (3.3) differs from the sum in Eq. (2.3) by the factor in the last line. However, the exponential $e^{-(E_\gamma+E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)}-E_B)T}$ will vanish for large $T$ if $E_\gamma+E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)} > E_B$. Because the states $|n(\mathbf{p}_B-\mathbf{p}_\gamma)\rangle$ have the same quark-flavor quantum numbers as the $B$ meson, we have $E_{n,(\mathbf{p}_B-\mathbf{p}_\gamma)} \geq E_{B,(\mathbf{p}_B-\mathbf{p}_\gamma)} = \sqrt{m_B^2+(\mathbf{p}_B-\mathbf{p}_\gamma)^2}$. Thus, we need $\sqrt{\mathbf{p}_\gamma^2}+\sqrt{m_B^2+(\mathbf{p}_B-\mathbf{p}_\gamma)^2} > \sqrt{m_B^2+\mathbf{p}_B^2}$. This is in fact always true if $\mathbf{p}_\gamma \neq 0$.

For the other time ordering, we find

$$I^{>}_{\mu\nu}(t_B,T) = -\langle B(\mathbf{p}_B)|\phi_B^\dagger(0)|0\rangle \frac{1}{2E_B} e^{E_B t_B} \times \sum_m \frac{1}{2E_{m,\mathbf{p}_\gamma}} \frac{\langle 0|J_\mu(0)|m(\mathbf{p}_\gamma)\rangle\langle m(\mathbf{p}_\gamma)|J_\nu^{\rm weak}(0)|B(\mathbf{p}_B)\rangle}{E_\gamma - E_{m,\mathbf{p}_\gamma}} \left(1-e^{(E_\gamma-E_{m,\mathbf{p}_\gamma})T}\right). \quad (3.4)$$

The unwanted exponential $e^{(E_\gamma-E_{m,\mathbf{p}_\gamma})T}$ in the last line goes to zero for large $T$ if $E_{m,\mathbf{p}_\gamma} > E_\gamma$. Because the states $|m(\mathbf{p}_\gamma)\rangle$ are hadronic and have nonzero masses, their energies are larger than the energy of a photon with the same spatial momentum, showing that this condition is also always satisfied. In summary, for $\mathbf{p}_\gamma \neq 0$,

$$T_{\mu\nu} = -\lim_{T\to\infty}\lim_{t_B\to-\infty} \frac{2E_B\, e^{-E_B t_B}}{\langle B(\mathbf{p}_B)|\phi_B^\dagger(0)|0\rangle} I_{\mu\nu}(t_B,T), \quad (3.5)$$

where $I_{\mu\nu}$ is the integral from $-T$ to $T$. The energy $E_B$ and the overlap factor $\langle B(\mathbf{p}_B)|\phi_B^\dagger(0)|0\rangle$ can be obtained from the two-point function $\int d^3x\, e^{-i\mathbf{p}_B\cdot\mathbf{x}}\langle \phi_B(t,\mathbf{x})\,\phi_B^\dagger(0)\rangle$.

Note that similar nonlocal matrix elements appear in processes with two photons, whose lattice calculation has been discussed, for example, in Refs. [18, 19, 20].

## 4. Preliminary numerical results

In this section, we present some early numerical results for the $D_s^+ \to \ell^+\nu\gamma$ and $K^- \to \ell^-\bar{\nu}\gamma$ form factors. These results are from only 25 configurations of the "24I" RBC/UKQCD ensemble

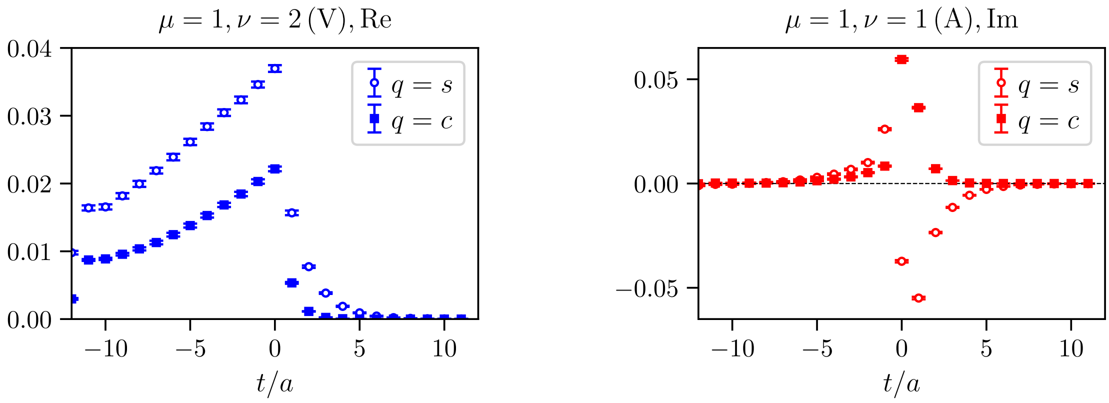


**Figure 2:** The unintegrated, scaled three-point functions $-\frac{2E_{D_s}e^{-E_{D_s}t_{D_s}}}{\langle D_s(\mathbf{p}_{D_s})|\phi_{D_s}^\dagger(0)|0\rangle}C_{\mu\nu}(t,t_{D_s})$ as a function of the electromagnetic-current insertion time $t$, for $t_{D_s}/a=-12$ and $\mathbf{p}_\gamma=(0,0,1)\frac{2\pi}{L}$. The left plot shows a combination of indices sensitive to $F_V$, while the right plot shows a combination sensitive to $F_A$. The contributions from the $s$ and $c$ quark in the electromagnetic current are shown separately, without charge factors.



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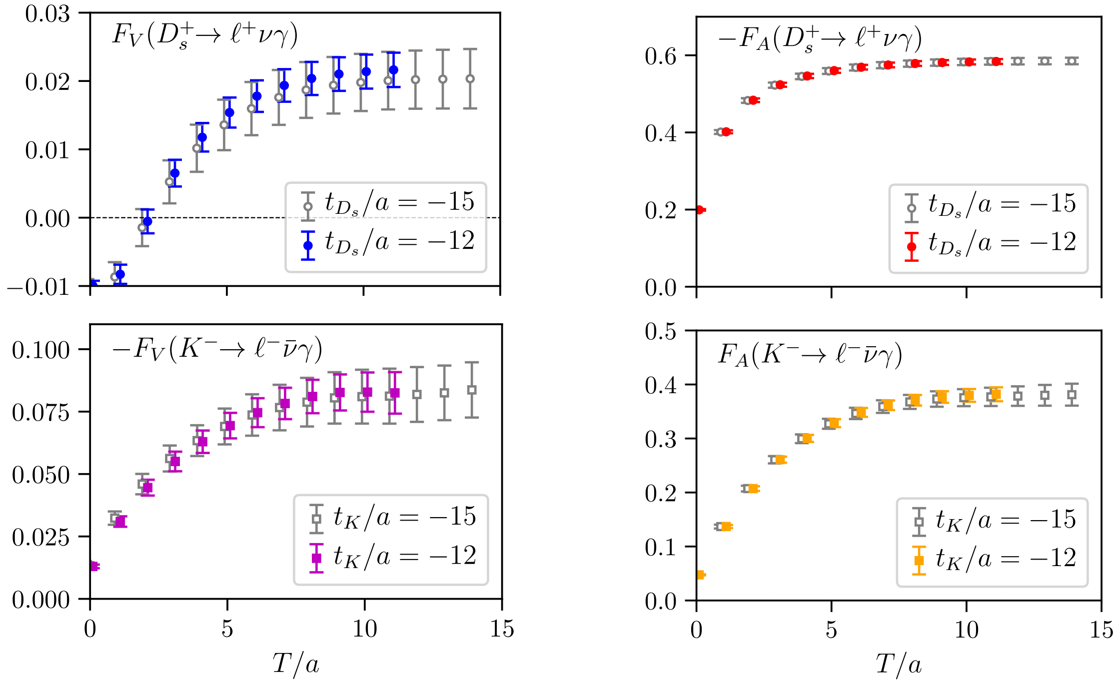


**Figure 3:** The $D_s^+ \to \ell^+\nu\gamma$ and $K^- \to \ell^-\bar{\nu}\gamma$ form factors at $\mathbf{p}_\gamma = (0,0,1)\frac{2\pi}{L}$ as a function of the summation range $T$, for two different meson-field insertion times.

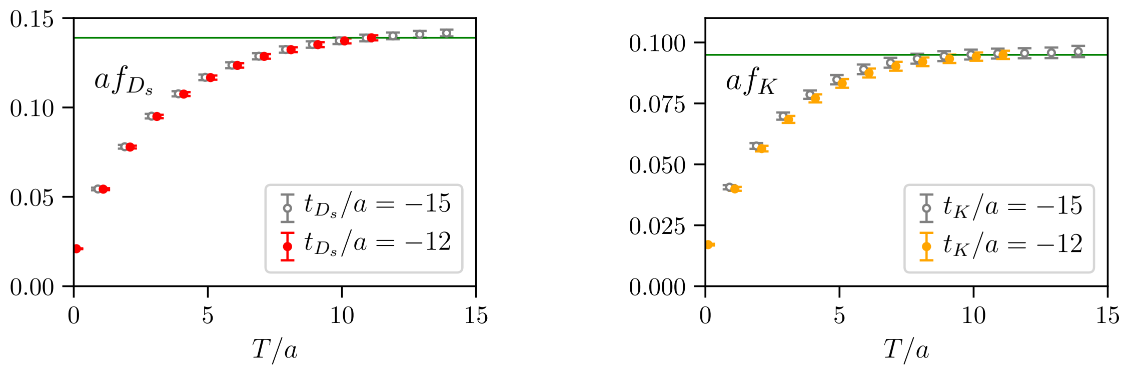


**Figure 4:** The $D_s$ and $K$ decay constants extracted from $T_{\mu\nu}$ at $\mathbf{p}_\gamma = (0,0,1)\frac{2\pi}{L}$, as a function of the summation range $T$, for two different meson-field insertion times. For the $D_s$, the horizontal line shows the physical value from Ref. [21]. For the $K$, the horizontal line shows the value computed on the same ensemble with the standard method in Ref. [22].

[22] with 2+1 flavors of domain-wall fermions and the Iwasaki gauge action, with $a^{-1} = 1.785(5)$ GeV and $m_\pi = 340(1)$ MeV. For the light and strange valence quarks, we use the same domain-wall action as in Ref. [22]. The valence charm quark is implemented with a Möbius domain-wall action with stout-smeared gauge links ($N = 3$, $\rho = 0.1$), $L_5/a = 12$, $aM_5 = 1.0$, $am_f = 0.6$ [23], which approximately corresponds to the physical charm-quark mass. We use local currents with "mostly nonperturbative" renormalization. Gaussian smearing is performed for the lighter quark in the meson interpolating field. We start with a $\mathbb{Z}_2$ random-wall source at the time slice of the weak current (denoted as time "0" here) and perform sequential inversions through the meson in-

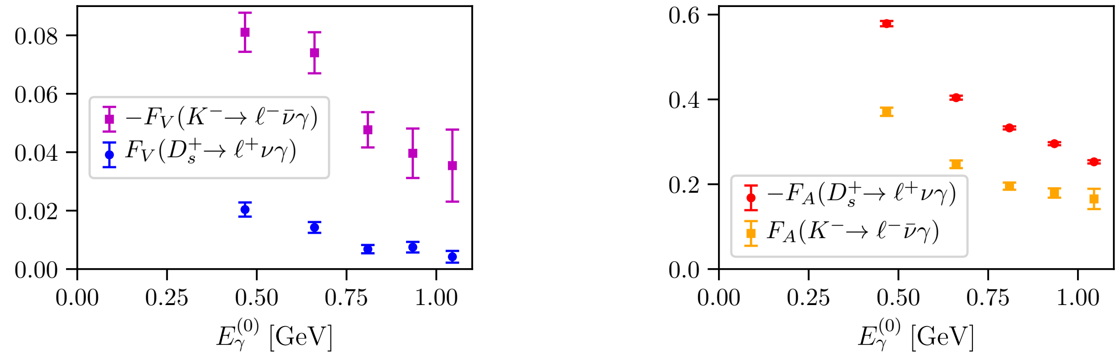


**Figure 5:** The $D_s^+ \to \ell^+\nu\gamma$ and $K^- \to \ell^-\bar{\nu}\gamma$ form factors as a function of the photon energy. The results shown here were obtained with $T/a = 8$ and $t_{K/D_s}/a = -12$. Only the statistical uncertainties are given.

terpolating field; disconnected diagrams are presently neglected. All-mode averaging [24] with 16 sloppy and 1 exact samples per configuration is employed; the 16 sloppy samples correspond to 16 different starting time slices. Our initial calculations used $\mathbf{p}_{K/D_s} = 0$ and $\mathbf{p}_\gamma^2 \in \{1,2,3,4,5\}\left(\frac{2\pi}{L}\right)^2$.

Figure 2 shows examples of the $D_s^+ \to \ell^+\nu\gamma$ three-point functions. Multiplying by $e^{E_\gamma t}$ and summing over $t$ gives $T_{\mu\nu}$ for sufficiently large summation range $T$. The form factors $F_V$ and $F_A$ extracted from $T_{\mu\nu}$ (at the lowest photon momentum) are shown as a function of $T$ in Fig. 3. The results plateau at approximately $T/a = 8$. We also extracted the meson decay constants from the $v^\mu v^\nu$ term in $T_{\mu\nu}$. As can be seen in Fig. 4, the results agree with the known values, which is a valuable test of our calculation. Finally, Fig. 5 shows the form factors $F_V$ and $F_A$ as a function of the photon energy. Note that, with our current choice of momenta, all of the photon energies are above the physical region for $K^- \to \ell^-\bar{\nu}\gamma$. The results for $F_A$ are dominated by the point-like contribution equal to $-e_\ell f_{K/D_s}/E_\gamma^{(0)}$.

## 5. Conclusions and Outlook

We have shown that the form factors describing radiative leptonic decays can be calculated on the lattice; even though they involve a nonlocal matrix element, the use of imaginary time poses no difficulty in this case. The early results shown here for $D_s^+ \to \ell^+\nu\gamma$ and $K^- \to \ell^-\bar{\nu}\gamma$ cover photon energies from approximately 0.5 to 1 GeV. For $K^- \to \ell^-\bar{\nu}\gamma$ we need to reach lower photon energies to compare with experiment; this can be achieved by using moving frames (i.e., nonzero $\mathbf{p}_K$) and/or a larger volume. To study the $B_{(s)}$ radiative leptonic decays with the domain-wall action for the heavy quark, we will need to extrapolate in the mass. We are also considering calculations directly at the physical $b$-quark mass using the "relativistic heavy-quark action" [25], but, because this action is only on-shell improved, additional steps are likely needed to remove unphysical behavior occurring when the electromagnetic and weak currents get close to each other.

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[1] J. Aebischer, W. Altmannshofer, D. Guadagnoli, M. Reboud, P. Stangl, and D. M. Straub, "*B*-decay discrepancies after Moriond 2019," arXiv:1903.10434.

[2] D. Atwood, G. Eilam, and A. Soni, "Pure leptonic radiative decays $B^{\pm}, D_s \to \ell \nu \gamma$ and the annihilation graph," Mod. Phys. Lett. **A11** (1996) 1061, arXiv:hep-ph/9411367.

[3] G. P. Korchemsky, D. Pirjol, and T.-M. Yan, "Radiative leptonic decays of B mesons in QCD," Phys. Rev. **D61** (2000) 114510, arXiv:hep-ph/9911427.

[4] S. Descotes-Genon and C. T. Sachrajda, "Factorization, the light cone distribution amplitude of the B meson and the radiative decay $B \to \gamma \ell \nu_\ell$," Nucl. Phys. **B650** (2003) 356, arXiv:hep-ph/0209216.

[5] E. Lunghi, D. Pirjol, and D. Wyler, "Factorization in leptonic radiative $B \to \gamma e \nu$ decays," Nucl. Phys. **B649** (2003) 349, arXiv:hep-ph/0210091.

[6] V. M. Braun and A. Khodjamirian, "Soft contribution to $B \to \gamma \ell \nu_\ell$ and the *B*-meson distribution amplitude," Phys. Lett. **B718** (2013) 1014, arXiv:1210.4453.

[7] M. Beneke, V. M. Braun, Y. Ji, and Y.-B. Wei, "Radiative leptonic decay $B \to \gamma \ell \nu_\ell$ with subleading power corrections," JHEP **07** (2018) 154, arXiv:1804.04962.

[8] Y.-M. Wang and Y.-L. Shen, "Subleading-power corrections to the radiative leptonic $B \to \gamma \ell \nu$ decay in QCD," JHEP **05** (2018) 184, arXiv:1803.06667.

[9] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, "QCD factorization for $B \to \pi\pi$ decays: Strong phases and CP violation in the heavy quark limit," Phys. Rev. Lett. **83** (1999) 1914, arXiv:hep-ph/9905312.

[10] **Belle** Collaboration, M. Gelb *et al.*, "Search for the rare decay of $B^+ \to \ell^+ \nu_\ell \gamma$ with improved hadronic tagging," Phys. Rev. **D98** (2018) 112016, arXiv:1810.12976.

[11] D. Guadagnoli, M. Reboud, and R. Zwicky, "$B_s^0 \to \ell^+ \ell^- \gamma$ as a test of lepton flavor universality," JHEP **11** (2017) 184, arXiv:1708.02649.

[12] F. Dettori, D. Guadagnoli, and M. Reboud, "$B_s^0 \to \mu^+ \mu^- \gamma$ from $B_s^0 \to \mu^+ \mu^-$," Phys. Lett. **B768** (2017) 163, arXiv:1610.00629.

[13] **BESIII** Collaboration, M. Ablikim *et al.*, "Search for the radiative leptonic decay $D^+ \to \gamma e^+ \nu_e$," Phys. Rev. **D95** (2017) 071102, arXiv:1702.05837.

[14] **BESIII** Collaboration, M. Ablikim *et al.*, "Search for the decay $D_s^+ \to \gamma e^+ \nu_e$," Phys. Rev. **D99** (2019) 072002, arXiv:1902.03351.

[15] **Particle Data Group** Collaboration, M. Tanabashi *et al.*, "Review of Particle Physics," Phys. Rev. **D98** (2018) 030001.

[16] A. Soni, "Flavor anomalies & the lattice," 2018. Presentation at Lattice 2018, East Lansing, MI, USA.

[17] **RM123** Collaboration, G. Martinelli, "Electromagnetic Corrections to Decay Amplitudes: Real Emissions in Leptonic Decays," 2019. Presentation at Lattice 2019, Wuhan, China.

[18] X.-d. Ji and C.-w. Jung, "Studying hadronic structure of the photon in lattice QCD," Phys. Rev. Lett. **86** (2001) 208, arXiv:hep-lat/0101014.

[19] J. J. Dudek and R. G. Edwards, "Two Photon Decays of Charmonia from Lattice QCD," Phys. Rev. Lett. **97** (2006) 172001, arXiv:hep-ph/0607140.

[20] X. Feng, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, J.-i. Noaki, and E. Shintani, "Two-photon decay of the neutral pion in lattice QCD," Phys. Rev. Lett. **109** (2012) 182001, arXiv:1206.1375.

[21] **Flavour Lattice Averaging Group** Collaboration, S. Aoki *et al.*, "FLAG Review 2019," arXiv:1902.08191.

[22] **RBC/UKQCD** Collaboration, Y. Aoki *et al.*, "Continuum Limit Physics from 2+1 Flavor Domain Wall QCD," Phys. Rev. **D83** (2011) 074508, arXiv:1011.0892.

[23] **RBC/UKQCD** Collaboration, P. A. Boyle, L. Del Debbio, N. Garron, A. Jüttner, A. Soni, J. T. Tsang, and O. Witzel, "SU(3)-breaking ratios for $D_{(s)}$ and $B_{(s)}$ mesons," arXiv:1812.08791.

[24] E. Shintani, R. Arthur, T. Blum, T. Izubuchi, C. Jung, and C. Lehner, "Covariant approximation averaging," Phys. Rev. **D91** (2015) 114511, arXiv:1402.0244.

[25] N. H. Christ, M. Li, and H.-W. Lin, "Relativistic Heavy Quark Effective Action," Phys. Rev. **D76** (2007) 074505, arXiv:hep-lat/0608006.