PROCEEDINGS OF SCIENCE

Non-perturbative matching of three/four-flavor Wilson coefficients with a position-space procedure

Masaaki Tomii*
Physics Department, University of Connecticut, Storrs, CT 06269, USA
E-mail: masaaki.tomii@uconn.edu

We construct a strategy to non-perturbatively convert Wilson coefficients in the four-flavor theory to those in the three-flavor theory. This non-perturbative matching is expected to reduce one of the biggest systematic uncertainties in RBC/UKQCD's $K \to \pi\pi$ calculation, where the matching was performed perturbatively at scales below the charm threshold. Since our method uses two-point functions in position space, which are a gauge-invariant and are free from contact terms, it prevents irrelevant mixing with gauge noninvariant operators and operators that vanish by the equations of motion. In this report, we present the strategy and our preliminary results for the non-perturbative matching of the Wilson coefficients that multiply the $\Delta S = 1$ four-quark operators associated with $K \to \pi\pi$ decays.

37th International Symposium on Lattice Field Theory - Lattice2019
16-22 June 2019
Wuhan, China

*Speaker.

PoS(LATTICE2019)174

1. Introduction

Numerical simulation of lattice QCD is a good tool to calculate matrix elements of weak decays, while there is a large scale separation between the weak and QCD scales. Flavor-changing processes can be expressed as effective interactions at low energies by integrating out particles heavier than the renormalization scale μ. The property of an effective interaction is expressed by the corresponding weak Hamiltonian, which is composed of effective operators and Wilson coefficients

$$H_W = \sum_i w_i^{S_{n_f}}(\mu) O_i^{S_{n_f}}(\mu). \tag{1.1}$$

Lattice QCD is useful to compute individual matrix elements associated with four-quark or two-quark operators that give the leading contribution to the weak process.

The information of heavy particles that are integrated out is expressed in terms of the Wilson coefficients, which are perturbatively known to one- or two-loop level for many processes. The large scale separation between the weak and QCD scale is accommodated by the renormalization group, which enables us to perform the scale evolution of Wilson coefficients from the weak scale $\mu = m_W$ to lower scales.

While the weak Hamiltonian and its matrix elements are independent of renormalization scheme and scale, we need to choose those of effective operators and Wilson coefficients in actual calculation as in Eq. (1.1). Moreover, we also need to use the same number n_f of active flavors to calculate the Wilson coefficients $w_i^{S_{n_f}}(\mu)$ and the matrix element with effective operators $O_i^{S_{n_f}}(\mu)$. Namely, if we calculate matrix elements without the charm and heavier quarks, we need the corresponding Wilson coefficients in the three-flavor theory to construct the appropriate weak Hamiltonian. The difference between the three- and four-flavor Wilson coefficients is $O(\alpha_s^2)$ if the charm quark cannot be involved in the effective operators. On the other hand, if the charm quark is involved in a weak operator associated with the weak process, there can be larger difference between the three- and four-flavor Wilson coefficients. For example, there are current-current operators with charm in $\Delta S = 1$ four-quark operators associated with $K \to \pi\pi$. In the three-flavor theory, these operators decouple and become a combination of operators composed of the lighter quarks. Therefore the Wilson coefficients of charm-involved operators in the four-flavor theory are absorbed into other Wilson coefficients in the three-flavor theory. Taking into account this absorption is called "matching." Such a matching needs to be considered not only at the charm threshold but also at the bottom threshold. While the matching at the bottom threshold can be done perturbatively with enough precision, the perturbative matching at the charm threshold, 1.3 GeV, may can cause a significant systematic uncertainty [1].

In this work, we construct a nonperturbative strategy to match the Wilson coefficients between the three- and four-flavor theories so that we do not need to employ the perturbative matching procedure. Section 2 designs the theoretical description and our strategy for the nonperturbative matching of the Wilson coefficients. In Section 3 we discuss the $\Delta S = 1$ four-quark operators associated with $K \to \pi\pi$ and indicate the operator bases used in this work. In Section 4 we show the preliminary result of our test calculation on a $16^3 \times 32$ lattice ensemble.

PoS(LATTICE2019)174

2. Theoretical foundation

In this work we consider the mixing of charm-involved operators into other operators composed of the lighter quarks and its effect on the Wilson coefficients. We define O_{n_f} as a vector of operators that is used in the weak Hamiltonian in the n_f-flavor theory. At long distances ($\gg 1/m_c$), the charm quark decouples and operators with the charm quark behave the same as a combination of operators composed of the lighter quarks. Thus the decoupling of the charm quark in weak operators can be expressed as

$$O_{4;\alpha}^{S_4}(\mu) \to \sum_i M_{\alpha i}^{S_{4/3}}(\mu,\mu')O_{3;i}^{S_3}(\mu'), \tag{2.1}$$

with a matrix $M_{\alpha i}^{S_{4/3}}(\mu,\mu')$. Here S_{n_f} denotes a renormalization scheme S in the n_f-flavor theory and μ and μ' are the renormalization scale in the four- and three-flavor theories, respectively. This decoupling means a correlation function of $O_{4;\alpha}^{S_4}$ and any operator $\overline{O}$ at long distances can be expressed in terms of correlation functions of $O_{3;i}$'s and $\overline{O}$ after proper renormalizations:

$$\left\langle O_{4;\alpha}^{S_4}(\mu;x)\overline{O}^{X_4}(\bar{\mu};y)^\dagger\right\rangle_4 \xrightarrow{|x-y|\gg 1/m_c} \sum_i M_{\alpha i}^{S_{4/3}}(\mu,\mu')\left\langle O_{3;i}^{S_3}(\mu';x)\overline{O}^{X_3}(\bar{\mu};y)^\dagger\right\rangle_3. \tag{2.2}$$

Here, $\langle\ldots\rangle_{n_f}$ stands for the vacuum expectation value in the n_f-flavor theory. This relation provides a part of the condition to determine the matrix $M_{\alpha i}^{S_{4/3}}(\mu,\mu')$. By calculating these correlation functions with a sufficient number of $\overline{O}$'s we can determine $M_{\alpha i}^{S_{4/3}}(\mu,\mu')$.

The matching matrix $M_{\alpha i}^{S_{4/3}}(\mu,\mu')$ also gives the relation between Wilson coefficients $w_{n_f}^{S_{n_f}}$ in the three- and four-flavor theories. Inserting the decoupling relation (2.1) to the equivalence of the weak hamiltonian

$$H_W = \sum_i w_{3;\alpha}^{S_3}(\mu')O_{3;i}^{S_3}(\mu') = \sum_\alpha w_{4;\alpha}^{S_4}(\mu)O_{4;\alpha}^{S_4}(\mu), \tag{2.3}$$

we obtain

$$w_{3;i}^{S_3}(\mu') = \sum_\alpha w_{4;\alpha}^{S_4}(\mu)M_{\alpha i}^{S_{4/3}}(\mu,\mu'). \tag{2.4}$$

Thus the three-flavor Wilson coefficients can be obtained from the four-flavor Wilson coefficients and the matching matrix, $M_{\alpha i}^{S_{4/3}}(\mu,\mu')$, which can be nonperturbatively determined by the conditions (2.2) imposed for various operators $\overline{O}$.

In this work we choose $\overline{O} = O_{3;i}$ and neglect the effect of sea charm quarks, i.e. we calculate all correlators on $2+1$-flavor lattice ensembles. Then Eq. (2.2) becomes

$$\sum_\beta Z_{O_4;\alpha\beta}^{S_4/\mathrm{lat}}(\mu;1/a)G_{\beta i}^{4\text{-}3}(x-y) = \sum_{j,k} M_{\alpha j}^{S_{4/3}}(\mu,\mu')Z_{O_3;jk}^{S/\mathrm{lat}}(\mu';1/a)G_{ki}^{3\text{-}3}(x-y) \tag{2.5}$$

where

$$G_{\alpha i}^{4\text{-}3}(x-y) = \left\langle O_{4;\alpha}^{\mathrm{lat}}(1/a;x)O_{3;i}^{\mathrm{lat}}(1/a;y)^\dagger\right\rangle_{2+1},\quad G_{ij}^{3\text{-}3}(x-y) = \left\langle O_{3;i}^{\mathrm{lat}}(1/a;x)O_{3;j}^{\mathrm{lat}}(1/a;y)^\dagger\right\rangle_{2+1}, \tag{2.6}$$

and $Z_{O_4;\alpha\beta}^{S_4/\mathrm{lat}}(\mu;1/a)$ and $Z_{O_3;ij}^{S_3/\mathrm{lat}}(\mu;1/a)$ are the renormalization matrices of the four- and three-flavor operators. While we neglect the effect of sea charm quarks, we still need to distinguish these two renormalization matrix since $Z^{S_4/\mathrm{lat}}_{O_4;\alpha\beta}(\mu;1/a)$ mixes operators with the charm quark, whereas $Z^{S_4/\mathrm{lat}}_{O_4;\alpha\beta}(\mu;1/a)$ does not. The renormalization matrices on the right of the correlator matrices $G^{4\text{-}3}_{\beta i}(x-y)$ and $G^{3\text{-}3}_{ij}(x-y)$ are omitted since they are equivalent with each other after neglecting the effect of sea charm quarks. If $G^{3\text{-}3}_{ij}(\mu;x-y)$ is invertible it means we have a sufficient number of conditions to determine $M^S_{\alpha i}(\mu,\mu')$:

$$M^S_{\alpha i}(\mu,\mu') = \sum_{j,k,\beta} Z^{S/\mathrm{lat}}_{O_4;\alpha\beta}(\mu;1/a) G^{4\text{-}3}_{\beta j}(x-y) \left[G^{3\text{-}3}(x-y)^{-1}\right]_{jk} \left[Z^{S/\mathrm{lat}}_{O_3}(\mu';1/a)^{-1}\right]_{ki}. \tag{2.7}$$

3. $\Delta S = 1$ four-quark operators

In this work, we apply the matching procedure explained in the previous section to the $\Delta S = 1$ four-quark operators, which are associated with $K \to \pi\pi$ matrix elements. In this report, we focus on the matching matrix between the lattice operators, which can be used to determine the three-flavor Wilson coefficients in any scheme

$$M^{\mathrm{lat}}_{\alpha i}(1/a,1/a;x) = \sum_j G^{4\text{-}3}_{\alpha j}(x) \left[G^{3\text{-}3}(x)^{-1}\right]_{ji}. \tag{3.1}$$

In general the calculation of two-point functions of four-quark operators requires all-to-all propagators as there are diagrams that contain a quark loop at the sink point. These diagrams can induce a quadratic divergence $\sim a^{-2}$ as a result of mixing with the lower-dimensional operators $\bar{s}d$ and $\bar{s}\gamma_5 d$. We remove such divergence by replacing the four-quark operators as

$$O^{\mathrm{lat}}_{4;\alpha/3;i} \to \mathscr{O}^{\mathrm{lat}}_{4;\alpha/3;i} = O^{\mathrm{lat}}_{4;\alpha/3;i} - C^S_{4;\alpha/3;i}\bar{s}d - C^P_{4;\alpha/3;i}\bar{s}\gamma_5 d. \tag{3.2}$$

Here, the power divergent coefficients $C^S_{4;\alpha/3;i}$ and $C^P_{4;\alpha/3;i}$ are determined by the condition

$$\left\langle \bar{s}d(x)\mathscr{O}^{\mathrm{lat}}_{4;\alpha/3;i}(y)^\dagger \right\rangle\Big|_{|x-y|=x_0} = 0, \quad \left\langle \bar{s}\gamma_5 d(x)\mathscr{O}^{\mathrm{lat}}_{4;\alpha/3;i}(y)^\dagger \right\rangle\Big|_{|x-y|=x_0} = 0, \tag{3.3}$$

with a specific distance x_0, which we choose 2.5 GeV^{-1}. In what follows the correlator matrices $G^{4\text{-}3}_{\alpha i}$ and $G^{3\text{-}3}_{ij}$ stand for the correlator matrices after the subtraction of the power divergence.

In the four-flavor theory, there are 12 $\Delta S = 1$ four-quark operators [2, 3], which give a contribution to $K \to \pi\pi$ decays from the Standard Model and are obtained from the full theory by integrating out the weak bosons, the top and bottom quarks. Since there are 3 identities [4] among these operators, the number of independent operators is 7 in the three-flavor theory and 9 in the four-flavor theory. The independent operators involve 4 operators with the charm and anti-charm quarks, which are all in the (8,1) representation of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry. These charm operators play an important role in the matching of the Wilson coefficients and mix with a combination of $(8,1)$ operators composed of the lighter quarks. Therefore we focus on the $(8,1)$ operators and take the operator bases as

$$\begin{aligned}
O_{3;1} = O_{4;1} &= \frac{1}{\sqrt{10}}\left((\bar{s}_a d_a)_L(\bar{u}_b u_b)_L + (\bar{s}_a d_b)_L(\bar{u}_b u_a)_L + 2(\bar{s}_a d_a)_L(\bar{d}_b d_b)_L + 2(\bar{s}_a d_a)_L(\bar{s}_b s_b)_L\right), \\
O_{3;2} = O_{4;2} &= \frac{1}{\sqrt{2}}\left((\bar{s}_a d_a)_L(\bar{u}_b u_b)_L - (\bar{s}_a d_b)_L(\bar{u}_b u_a)_L\right), \\
O_{3;3} = O_{4;3} &= \frac{1}{\sqrt{3}}\left((\bar{s}_a d_a)_L(\bar{u}_b u_b)_R + (\bar{s}_a d_a)_L(\bar{d}_b d_b)_R + (\bar{s}_a d_a)_L(\bar{s}_b s_b)_R\right), \\
O_{3;4} = O_{4;4} &= \frac{1}{\sqrt{3}}\left((\bar{s}_a d_b)_L(\bar{u}_b u_a)_R + (\bar{s}_a d_b)_L(\bar{d}_b d_a)_R + (\bar{s}_a d_b)_L(\bar{s}_b s_a)_R\right), \\
O_{4;5} &= (\bar{s}_a d_a)_L(\bar{c}_b c_b)_L, \quad O_{4;6} = (\bar{s}_a d_b)_L(\bar{c}_b c_a)_L, \\
O_{4;7} &= (\bar{s}_a d_a)_L(\bar{c}_b c_b)_R, \quad O_{4;8} = (\bar{s}_a d_b)_L(\bar{c}_b c_a)_R,
\end{aligned} \tag{3.4}$$

where the summation over color indices a, b is understood and the spin contraction is taken as

$$(\bar{s}d)_L(\bar{q}q)_{R/L} = \sum_\mu \bar{s}\gamma_\mu(1-\gamma_5)d \cdot \bar{q}\gamma_\mu(1\pm\gamma_5)q. \tag{3.5}$$

By taking these bases, the matching matrix is an 8×4 matrix but it is trivial that the upper 4×4 sub matrix is the unit matrix before renormalization. Thus only the lower 4×4 sub matrix is nontrivial and represents the remnant effects of the charm quark at long distances. The relation between the three- and four-flavor Wilson coefficients (2.4) therefore becomes

$$w^{\mathrm{lat}}_{3;i}(1/a) = w^{\mathrm{lat}}_{4;i}(1/a) + \sum_{\alpha=5}^{8} w^{\mathrm{lat}}_{4;\alpha}(1/a) M^{\mathrm{lat}}_{\alpha i}(1/a, 1/a). \tag{3.6}$$

While this unrenormalized formula may not be useful for actual matching of renormalized Wilson coefficients, it may be all that is needed to interpret our lattice results.

4. Exploratory calculation

In this section we show our preliminary result for the matching matrix $M^{\mathrm{lat}}_{\alpha i}$ before renormalization (3.1), whose nontrivial part is the 4×4 sub matrix of $\alpha = 5$–8 as explained above. Our exploratory calculation is carried out on a $16^3\times 32$ domain-wall ensemble at $a^{-1} = 1.78$ GeV. We employ pure random noise sources to calculate all-to-all quark propagators. Correlation functions measured on the lattice have values only at discrete lattice sites violating $O(4)$ symmetry. In order to reduce the discretization error and obtain a result which is a continuous function of the distance $|x|$ only, we apply the spherical average [5].

Figure 1 shows our preliminary results for M^{lat}_{6i} calculated with $am_c = 0.60$ (crosses), 0.36 (circles) and 0.24 (squares). These matrix elements at $\alpha = 6$ are expected to be the largest contribution to the three-flavor Wilson coefficients (2.4) since they are multiplied by the $O(1)$ Wilson coefficient $w^{\mathrm{lat}}_{4;6}$ that corresponds to the current-current operator without gluon exchange $O_{4;6}$, while other elements are multiplied with the other $O(\alpha_s)$ Wilson coefficients. While the results for other matching matrix elements, whose magnitude and $|x|$-dependence are similar to these plots, are omitted because of the restriction of pages, they will be shown in a forthcoming full paper.

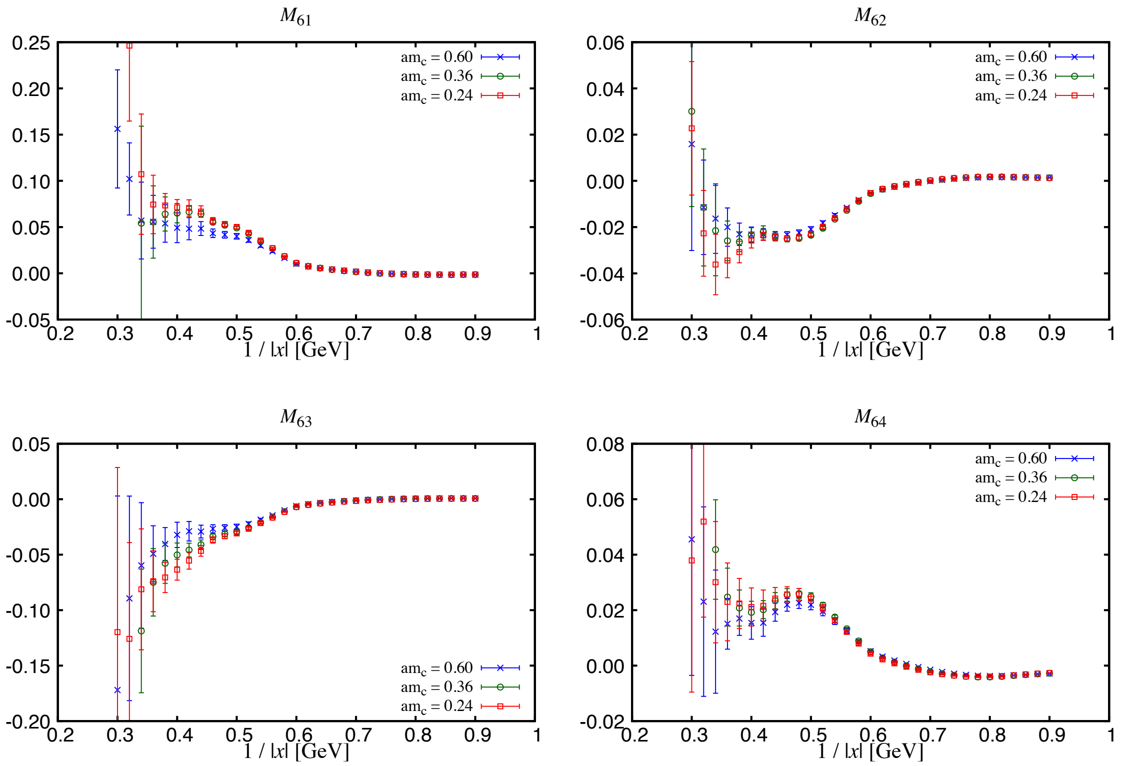

Figure 1: Preliminary results for $M^{\rm lat}_{6i}(1/a, 1/a; x)$ calculated with $am_c = 0.60$ (crosses), $am_c = 0.36$ (circles) and $am_c = 0.24$ (squares).

A naive dimensional analysis tells us that the results in the perturbative region should be proportional to $|x|^6$ because $G^{3\text{-}3}_{ij}(x)$'s involve diagrams with 4 quark propagators connecting the source and sink points and thus they are proportional to $|x|^{-12}$ in the perturbative region, whereas $G^{4\text{-}3}_{\alpha i}(x)$'s for $\alpha = 5$–8 are proportional to $|x|^{-6}$ as they do not involve diagrams in which there are more than 2 quark propagators connecting the source and sink points. This is why the results are close to 0 at short distances. Since the matching matrix must be obtained from long distances $1/|x| \ll m_c$, where the result in principle should be independent of the distance $|x|$, we plan to extract the matching matrix from the region where we see a plateau.

Since $w^{\rm lat}_{4;6}(1/a) M^{\rm lat}_{6i}(1/a, 1/a)$ is perturbatively $O(1)$ and would dominate the second term on the RHS of (3.6) as explained above, we roughly expect the difference between $w^{\rm lat}_{3;1}(1/a)$ and $w^{\rm lat}_{4;1}(1/a)$ is about $w^{\rm lat}_{4;6}(1/a) M^{\rm lat}_{6i}(1/a, 1/a) \approx 0.05(2)$, which is about 25% of $w^{\rm lat}_{4;1}(1/a)$ if we assume $w^{\rm lat}_{4;1}(1/a) \approx O(\alpha_s) \approx 0.2$. Since 25% with a 40% error means the result has $\pm$10% uncertainty. Namely, this matching may make about 20% change in the Wilson coefficients with 10% uncertainty, which is compatible or a little better than the perturbative matching procedure [3]. This is a very rough estimation but may provide us with a prospect for successful matching of the Wilson coefficients with this nonperturbative procedure as we increase the statistics and our sampling strategy.

5. Summary

We make an attempt to match the Wilson coefficients of the $\Delta S = 1$ four-quark operators in the four- and three-flavor theories by studying the decoupling of the charm and anti-charm quarks in those four-quark operators. We obtain a certain signal-to-noise ratio that would be superior to the systematic uncertainty of the perturbative matching procedure. We are about to start our main calculation on the $32^3 \times 64$ ensembles.

References

[1] Z. Bai *et al.* [RBC and UKQCD], Phys. Rev. Lett. **115**, no.21, 212001 (2015) doi:10.1103/PhysRevLett.115.212001 [arXiv:1505.07863 [hep-lat]].

[2] A. J. Buras, M. Jamin and M. E. Lautenbacher, Nucl. Phys. B **408**, 209-285 (1993) doi:10.1016/0550-3213(93)90535-W [arXiv:hep-ph/9303284 [hep-ph]].

[3] G. Buchalla, A. J. Buras and M. E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996) doi:10.1103/RevModPhys.68.1125 [hep-ph/9512380].

[4] T. Blum *et al.* [RBC], Phys. Rev. D **68**, 114506 (2003) doi:10.1103/PhysRevD.68.114506 [arXiv:hep-lat/0110075 [hep-lat]].

[5] M. Tomii and N. H. Christ, Phys. Rev. D **99**, no.1, 014515 (2019) doi:10.1103/PhysRevD.99.014515 [arXiv:1811.11238 [hep-lat]].