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Large N_c behaviour of lattice QCD in the heavy dense regime

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Combining strong coupling and hopping expansion one can derive a dimensionally reduced effective theory of lattice QCD. This theory has a reduced sign problem, is amenable to analytic evaluation and was successfully used to study the cold and dense regime of QCD for sufficiently heavy quarks. We show results from the evaluation of the effective theory for arbitrary N_c up to κ^4. The inclusion of gauge corrections is also investigated. We find that the onset transition to finite baryon number density steepens with growing N_c even for $T \neq 0$. This suggests that in the large N_c limit the onset transition is first order up to the deconfinement transition. Beyond the onset, the pressure is shown to scale as $p \sim N_c$ through three orders in the hopping expansion, which is characteristic for a phase termed quarkyonic matter in the literature.

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PoS(LATTICE2019)206

1. Introduction

Large parts of the phase structure of QCD for finite baryon chemical potential are still unknown. This is due to a sign problem which prohibits simulations using importance sampling at finite density. Various approximate methods that extend the Monte-Carlo method to finite densities exist, they are, however, limited to low densities or high temperature $\mu_B/T \lessapprox 3$. In this region, no sign of criticality or a first order phase transition has been found using these methods [1]. Regarding non-perturbative continuum approaches, based on the functional renormalisation group a critical endpoint has recently been reported at $(T_{\mathrm{CEP}}, \mu_{B,\mathrm{CEP}}) = (107, 635)$ MeV [2], however with systematic uncertainties growing large in the density region around the critical endpoint.

It is therefore interesting to study effective lattice theories with a sign problem that is mild enough to also allow simulations in the cold and dense region, or a completely analytic evaluation. One approach to obtain such a theory is to integrate the spatial gauge links via a combined strong coupling and hopping expansion, leading to a 3-dimensional effective theory of lattice QCD. This theory is valid for sufficiently heavy quarks on reasonably fine lattices and can be used to investigate the cold and dense regime around the onset to baryon matter. It can be simulated using reweighting and complex Langevin [3, 4] and can be evaluated analytically using a linked cluster expansion known from statistical mechanics [5]. Here we extend this analytic evaluation to arbitrary N_c up to order κ^4. This enables us to make a connection to another approach to dense and cold QCD, namely that based on large N_c considerations. Specifically, in [6] the authors argue for the existence of a phase termed quarkyonic matter, which has both quark and baryon-like properties and is characterised by pressure scaling as $p_c \sim N_c$.

2. The effective theory for general N_c

The effective theory is based on lattice QCD with standard Wilson fermions. Finite temperature is implemented by a compact euclidean time dimension with $N_\tau = {}^1\!/_{aT}$ slices and (anti-) periodic boundary conditions for (fermions) bosons. Finite chemical potential is introduced by a factor of $\exp((-)a\mu)$ in front of temporal (anti-) quark hops. Integrating out the Grassmann fields and spatial gauge links $U_i \in SU(N_c)$ leads to an effective theory which only depends on temporal links $U_0 \in SU(N_c)$:

$$Z = \int DU D\Psi D\bar{\Psi}\, e^{-S_G[U] - S_f^{(W)}[U,\Psi,\bar{\Psi}]} \tag{2.1}$$

$$=: \int DU_0\, e^{-S_{\mathrm{eff}}[U_0]} = \int DW\, e^{-S_{\mathrm{eff}}[W]} \tag{2.2}$$

$$\Rightarrow S_{\mathrm{eff}}[U_0] = -\log\left(\int DU_i D\Psi D\bar{\Psi}\, e^{-S_G[U] - S_f^{(W)}[U,\Psi,\bar{\Psi}]}\right) \tag{2.3}$$

Gauge invariance necessitates that the dependence of the effective action on temporal links is in terms of traces of powers of Wilson lines W,

$$W(\mathbf{n}) = \prod_{\tau=1}^{N_\tau} U_0(\tau, \mathbf{n}). \tag{2.4}$$

PoS(LATTICE2019)206

At this point the effective theory is uniquely determined by eq. (2.3). However, the integration of the spatial links lead to interactions to all distances, so in practice truncations have to be introduced. Here, we use a character (resummed strong coupling) expansion for the gauge part and a hopping expansion for the fermion part of the action. Truncating these expansions at a certain order enables an analytic evaluation of the path integral to that order. We start in the strong coupling limit, setting the lattice gauge coupling $\beta = \frac{2N_c}{g^2} = 0$, and discuss the systematics of the hopping expansion. This is done in an alternative approach in comparison to earlier publications. It is based on integrating the spatial gauge fields before integrating Grassmann valued fields, well known from the approach to staggered fermions in the strong coupling limit [7, 8]. This makes the systematic inclusion of low N_τ-contributions, which were neglected in [5] and are relevant to get the correct N_c scaling, easier.

2.1 Systematics of the hopping expansion

At $\beta = 0$, the link integration factorizes and splitting temporal and spatial hops results in

$$-S_{\text{eff}}[U_0] = \log\left[\int D\Psi D\bar{\Psi}\, e^{\bar{\Psi}(-1+T[U_0])\Psi} \prod_{n\in\Lambda}\prod_{i=1}^{3}\int \mathrm{d}U_i(n)\, e^{\kappa\,\mathrm{tr}\left(J_i(n)U_i(n)+U_i^\dagger(n)K_i(n)\right)}\right]. \tag{2.5}$$

Here, Λ refers to the $N_\tau \times N_s$ lattice, κ is the hopping expansion parameter and is related to the bare quark mass via $\kappa = 1/(2am_q+8)$, T is the temporal hopping matrix

$$T[U_0](n,m) = \kappa e^{\mu}(1-\gamma_0)U_0(n)\delta_{n_0,n_0+1}\delta_{\mathbf{n},\mathbf{m}} + \kappa e^{-\mu}(1+\gamma_0)U_0(n)^\dagger\delta_{n_0,n_0+1}\delta_{\mathbf{n},\mathbf{m}}, \tag{2.6}$$

and the spatial hops are encoded in J and K in the following way:

$$J_i(n)_{ab} = \bar{\Psi}(n)^f_{\alpha,b}(1-\gamma_i)_{\alpha\beta}\Psi(n+\mathbf{e}_i)^f_{\beta,a} \tag{2.7}$$

$$K_i(n)_{ab} = \bar{\Psi}(n+\mathbf{e}_i)^f_{\alpha,b}(1+\gamma_i)_{\alpha\beta}\Psi(n+\mathbf{e}_i)^f_{\beta,a}. \tag{2.8}$$

In $\bar{\Psi}^f_{\alpha,b}$, f refers to flavour, α to Dirac and b to $SU(N_c)$ indices. All flavours are assumed to be degenerate. The single site integral in eq. (2.5) was given for $U(N_c)$ in [9] in terms of irreducible characters χ_r of the general linear group. Although we are interested in $SU(N_c)$ here, the only difference to $U(N_c)$ is due to spatial baryon hoppings, which have a prefactor κ^{kN_c}, meaning they are suppressed for large N_c. Since we are ultimately interested in large N_c considerations, we neglect them here, which means we can use the $U(N_c)$ results for the spatial integration. As a result one obtains a multinomial M of the Ψs with which the effective action can be represented as

$$\begin{aligned}-S_{\text{eff}}[U_0] = \log\Bigg[&\int D\Psi D\bar{\Psi}\, e^{\bar{\Psi}(-1+T[U_0])\Psi}\\ &\times\prod_{n\in\Lambda}\prod_{i=1}^{3}\left(1+M(\Psi(n),\Psi(n+\mathbf{e}_i),\bar{\Psi}(n),\bar{\Psi}(n+\mathbf{e}_i))\right)\Bigg].\end{aligned} \tag{2.9}$$

After expanding the product, the integration can be done using Wick's theorem. The corresponding propagator $(1-T[U_0])^{-1}$ has been obtained in [4] and is diagonal with respect to its spatial arguments but not with respect to its temporal arguments. Therefore, to organize the expansion we

PoS(LATTICE2019)206

write the lattice as a product of time slices of spatial lattices $\Lambda = \Lambda_\tau \times \Lambda_s$ and rewrite the product in eq. (2.9) to

$$\begin{aligned}
&\prod_{n\in\Lambda}\prod_{i=1}^{3}\big(1+M(\Psi(n),\Psi(n+\mathbf{e}_i),\bar{\Psi}(n),\bar{\Psi}(n+\mathbf{e}_i))\big) \\
&\quad = \prod_{\mathbf{n}\in\Lambda_s}\prod_{i=1}^{3}\prod_{\tau\in\Lambda_\tau}\big(1+M(\Psi(\tau,\mathbf{n}),\Psi(\tau,\mathbf{n}+\mathbf{e}_i),\bar{\Psi}(\tau,\mathbf{n}),\bar{\Psi}(\tau,\mathbf{n}+\mathbf{e}_i))\big)
\end{aligned} \tag{2.10}$$

$$\quad =: \prod_{\mathbf{n}\in\Lambda_s}\prod_{i=1}^{3}\Big(1+\ \mathbf{n}\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet(\mathbf{n}+\mathbf{e}_i)\ \Big), \tag{2.11}$$

The expansion of the product can then be represented as a sum of connected and disconnected subgraphs of the spatial lattice, with the graph in eq. (2.11) as an elementary building block of more complicated graphs. Denoting by Φ the evaluation of a graph according to

$$\Phi\Big(\ \mathbf{n}\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet(\mathbf{n}+\mathbf{e}_i)\ \Big) = \frac{\int D\Psi D\bar{\Psi}\, e^{\bar{\Psi}(-1+T[U_0])\Psi}\ \mathbf{n}\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet(\mathbf{n}+\mathbf{e}_i)}{\int D\Psi D\bar{\Psi}\, e^{\bar{\Psi}(-1+T[U_0])\Psi}}, \tag{2.12}$$

the evaluation factorizes over disconnected graphs. Then, the moment-cumulant formalism [10, 11] can be used to expand the logarithm in eq. (2.9), resulting in an expansion in connected clusters of graphs

$$-S_{\text{eff}}[U_0] = \sum_{\mathbf{n}\in\Lambda_s}\log(z_0(W(\mathbf{n}))) + \sum_{n=1}^{\infty}\sum_{g_1,\dots,g_n\in\mathscr{G}_c}\frac{1}{n!}[g_1,\dots,g_n]\Phi(g_1)\cdots\Phi(g_n). \tag{2.13}$$

In this equation, $[\dots]$ takes integer values that are non-vanishing exactly when all the graphs in its argument form a cluster of connected graphs. While eq. (2.13) is exact, in practice one truncates the infinite sum by including only contributions up to a certain power in κ.

2.2 Evaluation of the effective theory

Evaluating the effective theory eq. (2.13) along the lines of [5], the free energy to $\mathscr{O}(\kappa^2)$ reads

$$-f = \log(z_0(h_1)) - 6N_f\frac{\kappa^2 N_\tau}{N_c}\left(\frac{z_{11}(h_1)}{z_0(h1)}\right), \tag{2.14}$$

with the $SU(N_c)$ integrals

$$z_0 = \int_{SU(N_c)} \mathrm{d}W \det(1+h_1 W)^{2N_f}, \tag{2.15}$$

$$z_{11} = \int_{SU(N_c)} \mathrm{d}W \det(1+h_1 W)^{2N_f}\,\mathrm{tr}\left(\frac{h_1 W}{1+h_1 W}\right), \tag{2.16}$$

where $h_1 = (2\kappa e^{a\mu})^{N_\tau} = e^{\frac{\mu-m}{T}}$ and $am = -\log(2\kappa)$ is the leading order expression of the constituent quark mass of a baryon in lattice units. We have also neglected all contributions containing factors of $\bar{h}_1 = (2\kappa e^{-a\mu})^{N_\tau}$, which is justified because we first want to consider low temperatures. The integrals can be solved in the Polyakov gauge [12], for details and the $\mathscr{O}(\kappa^4)$ contribution to the free energy we refer to [13].

PoS(LATTICE2019)206

3. Large N_c behaviour of the effective theory

Having obtained the free energy in the previous section, we can investigate thermodynamic observables in the $\mu_B = 3\mu$ and T-plane for general and large N_c. To illustrate the general strategy, consider for $N_f = 1$ the κ^2 correction to the pressure, which reads

$$a^4 p_1 = -6\kappa^2 \frac{(\frac{1}{2}N_c(N_c+1)h_1^{N_c} + N_c h_1^{2N_c})^2}{N_c(1+h_1^{N_c}(1+N_c)+h_1^{2N_c})^2}. \tag{3.1}$$

For $h_1 < 1$, the $h_1^{N_c}$ factors are strongly suppressed for $N_c \to \infty$ and a Taylor expansion around $h_1^{N_c} = 0$ results in

$$a^4 p_1 = -\frac{3}{2}\kappa^2 N_c(N_c+1)^2 h_1^{2N_c} + \mathscr{O}(h_1^{3N_c}) \sim -\frac{3}{2}\kappa^2 N_c^3 h_1^{2N_c}. \tag{3.2}$$

Similarly, for $h_1 > 1$, expanding about $1/h_1^{N_c} = 0$ gives

$$a^4 p_1 = -6\kappa^2 N_c + \mathscr{O}(1/h_1^{N_c}) \sim -6\kappa^2 N_c. \tag{3.3}$$

Using this strategy one obtains for $N_f = 2$ degenerate flavours for the pressure and baryon density:

$$p \sim \begin{cases} \frac{1}{6a^4 N_\tau} N_c^3 h_1^{N_c} - \kappa^2 \frac{1}{48a^4} N_c^7 h_1^{2N_c} + \kappa^4 \frac{3N_\tau \kappa^4}{800a^4} N_c^8 h_1^{2N_c} + \mathscr{O}(\kappa^6), & \text{if } h_1 < 1, \\ \frac{4\log(h_1)}{N_\tau a^4} N_c - \kappa^2 \frac{12}{a^4} N_c + \kappa^4 \frac{198}{a^4} N_c + \mathscr{O}(\kappa^6), & \text{if } h_1 > 1, \end{cases} \tag{3.4}$$

$$n_B \sim \begin{cases} \frac{1}{6a^3} N_c^3 h_1^{N_c} - \kappa^2 \frac{N_\tau}{24a^3} N_c^7 h_1^{2N_c} + \kappa^4 \frac{(9N_\tau+1)N_\tau}{1200a^3} N_c^8 h_1^{2N_c} + \mathscr{O}(\kappa^6), & \text{if } h_1 < 1, \\ \frac{4}{a^3} - \kappa^2 \frac{N_\tau}{a^3} \frac{N_c^4}{h_1^{N_c}} - \kappa^4 \frac{(59N_\tau - 19)N_\tau}{20a^3} \frac{N_c^5}{h_1^{N_c}} + \mathscr{O}(\kappa^6), & \text{if } h_1 > 1. \end{cases} \tag{3.5}$$

One can make several observations based on these formulas. For $h_1 < 1$, i. e. before the baryon onset, the observables are exponentially suppressed for $N_c \to \infty$, for all N_τ.

For the baryon density one then has a first order jump to the lattice saturation density $a^3 n_B^{\text{sat}} = 2N_f$ starting at $h_1 > 1$. The saturation density is a discretization artefact determined by the leading order contribution in the hopping expansion, i. e. the static determinant. Higher orders in κ do not contribute. The boundary $h_1 = 1$ does not depend on T, therefore this transition should always be first order, until one reaches another discontinuity. Note however, that we neglected $\bar{h}_1$, so for higher temperatures this might be questionable. The inclusion of $\bar{h}_1$ was discussed in [13], leaving the qualitative observations unchanged.

For the pressure we observe that it scales linearly in N_c after the onset, a property that characterises quarkyonic matter [6]. Just like for the baryon density, the leading order is determined by lattice saturation, which is a discretization artefact. However, for the pressure the linear N_c scaling is also observed in all computed higher order terms, which do not contribute to saturation. Therefore, this suggests to hold to all orders in the hopping expansion, and then would be a genuine effect beyond saturation for all current quark masses.

3.1 Gauge corrections and 't Hooft scaling

So far, all quantities were analysed in the strong coupling limit $\beta = 0$. The continuum studies we are interested in employ the 't Hooft limit [14], which is defined by taking $N_c \to \infty$ with $\lambda_H :=$

PoS(LATTICE2019)206

$g^2 N_c$ kept fixed. Therefore, in order to take the same limit, we have to include corrections due to the gauge part of the action, which are of course anyway important in order to make connections to continuum results. Many aspects related to the gauge sector of the effective theory for general N_c were discussed in [15]. The strong coupling expansion around $\beta = 0$ is done by a character expansion. Here we only include leading gauge corrections from the character expansion to the fermion determinant. This works the same as in [4], the only difference being that the expansion coefficient of the fundamental character $u(\beta)$ has to be replaced by its proper generalisation, which can be found, for example, in [16]. To $\mathcal{O}(\kappa^2)$ the free energy then reads

$$-f = \ln(z_0(h_1)) + \frac{\kappa^2 N_\tau}{N_c}\left[1 + 2\frac{u - u^{N_\tau}}{1-u}\right](-6N_f)\frac{z_{11}(h_1)}{z_0(h_1)}^2, \tag{3.6}$$

with a modified h_1 due to gauge corrections

$$h_1 = (2\kappa)^{N_\tau} e^{a\mu N_\tau} \exp\left[6N_\tau \kappa^2 \frac{u - u^{N_\tau}}{1-u}\right]. \tag{3.7}$$

In taking the ’t Hooft limit by keeping $\lambda_H = \frac{2N_c^2}{\beta}$ fixed as $N_c \to \infty$, one can use that for $\lambda_H > 1$ [17]

$$u(\beta) = \frac{1}{\lambda_H} \tag{3.8}$$

in this limit. Therefore, these corrections shift the point on the chemical potential axis where $h_1 = 1$ and besides that only modify the asymptotic behaviour of the observables by a constant $\sim N_c^0$. In [13], we also discussed the inclusion of the leading order contribution of the pure gauge sector to the effective theory and similarly observed no qualitative changes in the large N_c behaviour. One should, however, mention one important caveat. Based on an analysis of QCD in $1+1$ dimensions, the interchange of the $N_c \to$ and the strong coupling expansion was declared to be “highly suspicious” in [17], which we did so far. Furthermore, the fact that the density at the onset transition immediately jumps to lattice saturation indeed suggests that to get results for continuum physics one should take the continuum limit before taking the large N_c limit. This has been investigated further in [13], where the linear N_c scaling of the pressure was observed before saturation and this behaviour was at least observed to be stable when the lattice is made finer.

4. Conclusion

We have investigated the large N_c behaviour of an effective theory of lattice QCD for heavy quarks, which is based on a combined strong coupling and hopping expansion. We have found that in the baryon condensed phase the pressure scales linearly in N_c and the onset transition becomes first-order. These results are in agreement with the proposed large N_c phase diagram in [6].

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PoS(LATTICE2019)206

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PoS(LATTICE2019)206