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Epstein-Glaser's Causal Light-Front Field Theory

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Epstein-Glaser's ideas for the formulation of a distributional well-defined perturbative causal field theory are developed in light-front dynamics over the invariant null-plane coordinates introduced by Rohrlich. Explicitly, the causality theorems which warrant the method are adapted to that dynamics, and the causal distribution splitting formulae are re-derived in accordance with it, exhibiting important differences with respect to its instant dynamics version. Application of these splitting formulae to the (anti)commutation relations of the fermion and radiation fields naturally leads to the well known instantaneous terms of their Feynman propagators, while the scalar field's one retains its form from instant dynamics. Additionally, the developed method is applied to Scalar QED (SQED) at second order, taking for the first order distribution the product of the radiation field with only the linear in the coupling constant part of the current. We analyse Moeller scattering, for which the equivalence with instant dynamics is established, and Compton scattering, for which the vertex coming from the second order term in the current is automatically generated in the normalization procedure once the residual gauge invariance which remains from the imposition of the null-plane gauge condition is exploited.

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1. Introduction

The S-operator, as a map between asymptotically free incoming particles and outgoing ones, *must* be constructed with the well defined free fields, as realized by Stückelberg and Rivier [1], who were the first to consider a causality axiom, then simplified by Bogoliubov and Shirkov [2] by the introduction of a switching function regulating the intensity of the interaction. In 1973, Epstein and Glaser [3] constructed a finite perturbative theory in which UV divergences do not appear; they used translation invariance and a causality condition, then the S-operator is constructed inductively, order by order. Currently, this theory is called: Causal Perturbation Theory (CPT) [4]. Moreover, although since Dirac's paper [5] the development of light-front field theories has proven to be useful in many models, the singular character of instantaneous terms has been an intricate problem. In view of the successful CPT applications, this approach could resolve this problem. Nonetheless, the passage from the instant dynamics version of CPT to the light-front dynamics one is not trivial: CPT has to be reconstructed, because the causality condition, which is at the base of the theory, must be reformulated. Show how to do it and the application of the resulting theory to scattering processes on SQED is the purpose of this paper (for a detailed exposition see [6]).

2. Null Plane Causal Perturbation Theory

The $S(g)$ operator connects asymptotically free initial and final states for an interaction whose intensity is regulated by the *switching function* $g \in \mathscr{S}(\mathbb{R}^4)$. We formally express it as the series:

$$S(g) := 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n T_n(x_1; \cdots ; x_n) g(x_1) \cdots g(x_n) \quad . \tag{2.1}$$

Here the product $g(x_1)\cdots g(x_n)$ is symmetric, so do the T_n distributions. Defining the set $X := \{x_j \in \mathbb{M} : j = 1, \cdots, n\}$, we denote: $T_n = T_n(X), g(X) \equiv g(x_1)\cdots g(x_n)$, $dX \equiv d^4x_1 \cdots d^4x_n$, so: $S(g) = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX T_n(X) g(X)$. The inverse operator is: $S(g)^{-1} = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX \tilde{T}_n(X) g(X)$.

For $X, Y \subset \mathbb{M}$ define:[1] $X < Y :\Leftrightarrow \forall x \in X, y \in Y : x^{(+)} < y^{(+)}$ and $X \sim Y :\Leftrightarrow \forall x \in X, y \in Y : (x-y)^2 < 0$. Causality is implemented as follows: (1) For $g_1, g_2 \in \mathscr{S}(\mathbb{R}^4)$ with $\mathrm{supp}(g_1) < \mathrm{supp}(g_2)$: $S(g_1 + g_2) = S(g_2)S(g_1)$; (2) for $g_1, g_2 \in \mathscr{S}(\mathbb{R}^4)$ with $\mathrm{supp}(g_1) \sim \mathrm{supp}(g_2)$, the same decomposition is valid, but $S(g_1)$ and $S(g_2)$ commute. Now, suppose that we know all $T_m, \tilde{T}_m$ for $m = 1, \cdots, n-1$ and we want to find the next order T_n distribution. Define the distributions:

$$A'_n(X) := \sum_{P_2} \tilde{T}_{n_1}(X_1) T_{n-n_1}(X_2 \cup \{x_n\}) \quad , \quad R'_n(X) := \sum_{P_2} T_{n-n_1}(X_2 \cup \{x_n\}) \tilde{T}_{n_1}(X_1) \quad , \tag{2.2}$$

with P_2 a partition of $X \setminus \{x_n\}$ in the disjoint subsets X_1, X_2 such that $X_1 \neq \emptyset$. Also, we define the advanced and retarded distributions of order n by:

$$A_n(X) := \sum_{P_2^{(0)}} \tilde{T}_{n_1}(X_1) T_{n-n_1}(X_2 \cup \{x_n\}) \quad , \quad R_n(X) := \sum_{P_2^{(0)}} T_{n-n_1}(X_2 \cup \{x_n\}) \tilde{T}_{n_1}(X_1) \quad . \tag{2.3}$$

[1] We use Rohrlich's invariant null-plane coordinates [7]: Light-front dynamics is introduced by the definition of the null-plane tetrad basis: $\mathsf{e}_{(\pm)} = \left(\tilde{\mathsf{e}}_{(0)} \pm \tilde{\mathsf{e}}_{(3)}\right)/\sqrt{2}, \mathsf{e}_{(1)} = \tilde{\mathsf{e}}_{(1)}, \mathsf{e}_{(2)} = \tilde{\mathsf{e}}_{(2)}$. Then every vector A has invariant components: $A^{(a)} := \mathsf{A} \cdot \mathsf{e}^{(a)}$, with which it can be expressed as: $\mathsf{A} = A^{(a)} \mathsf{e}_{(a)}$. The scalar product of the vectors A and B is: $\mathsf{A} \cdot \mathsf{B} = A^{(+)}B^{(-)} + A^{(-)}B^{(+)} - A^{(\perp)}B^{(\perp)}$. Finally, we choose $x^{(+)}$ as the invariant NP-time.

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Here, $P_2^{(0)}$ means that now it is allowed that $X_1 = \emptyset$. Then T_n can be found as: $T_n(X) = R_n(X) - R'_n(X)$. Define the **causal distribution**: $D_n(X) := R_n(X) - A_n(X) = R'_n(X) - A'_n(X)$. Also, denote: $\tilde{V}^+(x) := \left\{y \in \mathbb{M} : (y-x)^2 \geq 0; y^{(+)} \geq x^{(+)}\right\}$, $\tilde{V}^-(x) := \left\{y \in \mathbb{M} : (y-x)^2 \geq 0; y^{(+)} \leq x^{(+)}\right\}$, and the sets of n points in $\tilde{V}^{\pm}(x)$: $\Gamma_n^{\pm}(x)$. **Theorem 1:** For $n \geq 3$: $\text{supp}\,(D_n(x_1;\cdots;x_n)) \subseteq \Gamma_n^+(x_n) \cup \Gamma_n^-(x_n)$. $\square$ For $n=2$, the causal support of $D_2(X)$ must be proved explicitly. This theorem tells us that $D_n(X)$ can be split into R_n and A_n with $\text{supp}(R_n) \subseteq \Gamma_n^+(x_n)$ and $\text{supp}(A_n) \subseteq \Gamma_n^-(x_n)$. Also, **Theorem 2:** The constructed distribution $T_n(X)$ satisfies the causality conditions. $\square$ Now, the causal distribution has the form: $D_n(X) = \sum_k d_n^k(X) \;:C_k(\psi^A):$, with $d_n^k(X)$ a numerical distribution determining $\text{supp}(D_n)$ and $:C_k(\psi^A):$ a normal product of the free field operators ψ^A. Define the distribution $d \in \mathscr{S}(\mathbb{R}^{4n-4})$ as: $d(X) := d_n^k(x_1 - x_n;\cdots;x_{n-1}-x_n;0), \text{supp}(d) \subseteq \Gamma_{n-1}^+(0) \cup \Gamma_{n-1}^-(0)$. For the splitting of d as: $d = r - a$, with $\text{supp}(r) \subseteq \Gamma_{n-1}^+(0)$ and $\text{supp}(a) \subseteq \Gamma_{n-1}^-(0)$, we must take into account its behaviour in the neighbourhood of the intersection of the surface $X^{(+)} = 0$ and $\text{supp}(d)$, i.e., the $X^{(-)}$-axis. **Definition 1:** Let $d \in \mathscr{S}'(\mathbb{R}^m)$. If the limit $\lim_{s\to 0^+} s^{\omega_-} s^{3m/4} d\left(sX^{(+)}; sX^{(\perp)}; X^{(-)}\right) = d_-(X)$ exists in $\mathscr{S}'(\mathbb{R}^m)$ and is non-zero, then d_- is the quasi-asymptote of d at the $X^{(-)}$-axis, which then has singular order ω_-. $\square$ For the splitting, consider a continuous non-decreasing monotonous function $\chi(t)$ such that: $\chi(t) = 0$ for $t < 0$, $\chi(t) < 1$ for $0 < t < 1$, $\chi(t) = 1$ for $t \geq 1$. Then, define the retarded and advanced distributions by:

$$r(X) = \lim_{s\to 0} \chi\left(X^{(+)}/s\right) d(X) \quad , \quad a(X) = -\lim_{s\to 0} \chi\left(-X^{(+)}/s\right) d(X) \quad . \tag{2.4}$$

This definition is satisfactory only if these limits do exist. Cauchy's convergence condition implies its existence if $\omega_- < 0$, but it could not exist for $\omega_- \geq 0$. Accordingly, for $\omega_- < 0$ the splitting is trivial: $r(X) = \lim_{s\to 0} \chi\left(X^{(+)}/s\right) d(X) \equiv \Theta\left(X^{(+)}\right) d(X)$. For $\omega_- \geq 0$, we must project the test function onto the space of functions for which the limit exists; this is done with an operator W. Then the retarded distribution is defined as: $(r;\varphi) := \left(\lim_{s\to 0} \chi\left(X^{(+)}/s\right) d(X); (W\varphi)(X)\right) \equiv (d;\Theta W\varphi)$.

In momentum space the splitting formulae are (we show only the case $n=2$), for $\omega_- < 0$:

$$\hat{r}(p) = \frac{i}{2\pi} \text{sgn}\left(p_{(+)}\right) \int_{-\infty}^{+\infty} \frac{\hat{d}\left(tp_{(+)};\boldsymbol{p}\right)}{1 - t + \text{sgn}\left(p_{(+)}\right) i0^+} dt \quad , \tag{2.5}$$

and for $\omega_- \geq 0$ we have the *central solution*:

$$\hat{r}_0(p) = \frac{i}{2\pi} \int \frac{dq}{q + i0^+} \left\{ \hat{d}\left(p_{(+)} - q;\boldsymbol{p}\right) - \sum_{b=0}^{\lfloor \omega_- \rfloor} \frac{\left(p_{(+,\perp)}\right)^b}{b!} D^b_{(+,\perp)} \hat{d}\left(-q;0_{(\perp)};p_{(-)}\right) \right\} . \tag{2.6}$$

Finally, if $(r;a)$ and $(\tilde{r};\tilde{a})$ are two solutions of the splitting problem, $d = r - a = \tilde{r} - \tilde{a}$, then $r - \tilde{r} = a - \tilde{a}$. It is only possible if r and $\tilde{r}$ differ, at most, by *normalization terms*: $r(X) - \tilde{r}(X) = \sum_{a=0}^{\lfloor \omega_-^r \rfloor} C_a\left(X^{(-)}\right) D^a_{(+,\perp)} \delta\left(X^{(+,\perp)}\right)$, with $C_a\left(X^{(-)}\right)$ some distributions which must be fixed by physical conditions besides causality. In momentum space those terms are: $\sum_{a=0}^{\lfloor \omega_-^r \rfloor} \hat{C}_a\left(P_{(-)}\right) P^a_{(+,\perp)}$.

3. Field Propagators

Scalar field: Fourier transform of the commutation function for the scalar field is $\hat{D}(p) = \frac{i}{2\pi} \text{sgn}\left(p_{(-)}\right) \delta(p^2 - m^2)$, which is of singular order $\omega_- = -2 < 0$. We find its retarded part by using Eq. (2.5) and get: $\hat{D}^{ret}(p) = -(2\pi)^{-2}\frac{1}{p^2-m^2+\mathrm{sgn}\left(p_{(-)}\right)i0^+}$. Feynman propagator is:

$$\hat{D}^F(p) := \hat{D}^{ret}(p) - \hat{D}^{(-)}(p) = -(2\pi)^{-2}\frac{1}{p^2-m^2+i0^+} \quad .$$

Fermion field: The anticommutation function of the fermion field in momentum space is: $\hat{S}(p) = \frac{i}{2\pi}(\not{p}+m)\mathrm{sgn}\left(p_{(-)}\right)\delta(p^2-m^2)$, and has singular order $\omega_- = -1 < 0$. Its retarded part is found to be: $\hat{S}^{ret}(p) = -(2\pi)^{-2}\left(\frac{\not{p}+m}{p^2-m^2+\mathrm{sgn}\left(p_{(-)}\right)i0^+} - \frac{\gamma^{(+)}}{2p_{(-)}}\right)$. Feynman propagator is then:

$$\hat{S}^F(p) := \hat{S}^{(-)}(p) - \hat{S}^{ret}(p) = (2\pi)^{-2}\left(\frac{\not{p}+m}{p^2-m^2+i0^+} - \frac{\gamma^{(+)}}{2p_{(-)}}\right) \quad .$$

Radiation field in the NP-gauge: The commutation function for this field is: $\hat{D}_{(a)(b)}(k) = \frac{i}{2\pi}\mathrm{sgn}\left(k_{(-)}\right)\delta(k^2)\left(\eta_{(a)(b)} - \frac{k_{(a)}\eta_{(b)}+\eta_{(a)}k_{(b)}}{k_{(-)}}\right)$. Some terms of this distribution have singular order $\omega_- = -2$, and others have $\omega_- = -1$. In any case, the singular order is negative. The retarded part is: $\hat{D}^{ret}_{(a)(b)}(k) = -\frac{(2\pi)^{-2}}{k^2+\mathrm{sgn}\left(k_{(-)}\right)i0^+}\left\{\eta_{(a)(b)} - \frac{k_{(a)}\eta_{(b)}+\eta_{(a)}k_{(b)}}{k_{(-)}} + \frac{k^2}{k^2_{(-)}}\eta_{(a)}\eta_{(b)}\right\}$. Therefore:

$$\hat{D}^F_{(a)(b)}(k) := \hat{D}^{ret}_{(a)(b)}(k) - \hat{D}^{(-)}_{(a)(b)}(k) = -\frac{(2\pi)^{-2}}{k^2+i0^+}\left\{\eta_{(a)(b)} - \frac{k_{(a)}\eta_{(b)}+\eta_{(a)}k_{(b)}}{k_{(-)}} + \frac{k^2}{k^2_{(-)}}\eta_{(a)}\eta_{(b)}\right\} \quad .$$

4. Scalar QED

We start with the one-point distribution: $T_1(x) = -i \; :j_{(a)}(x): \; A^{(a)}(x)$; with the current: $:j_{(a)}(x): = ie \; :\varphi^\dagger(x)\overleftrightarrow{\partial}_{(a)}\varphi(x):$. Being that way, $T_1(x)$ is a truly first order in e distribution.

Moeller Scattering: Considering the scattering of two scalar particles, the causal distribution for this process is: $D_2^M(x_1;x_2) = -iD_{\mu\nu}(y) \; :j^\mu(x_1)j^\nu(x_2):$ with $y = x_1 - x_2$. The corresponding retarded distribution is: $R_2^M(x_1;x_2) = -iD^{ret}_{\mu\nu}(y) \; :j^\mu(x_1)j^\nu(x_2):$, so that:

$$T_2^M(x_1;x_2) = R_2^M(x_1;x_2) - R'^M_2(x_1;x_2) = -iD^F_{\mu\nu}(y) \; :j^\mu(x_1)j^\nu(x_2): \quad . \tag{4.1}$$

Accordingly: $S_2^M = -\frac{i}{2}(2\pi)^{-2}\int d^4k d^4x_1 d^4x_2 e^{-iky}\left(\hat{D}^F_{\mu\nu}(k) + \hat{C}\left(k_{(-)}\right)\right) \; :j^\mu(x_1)j^\nu(x_2):$. If we choose $\hat{C}\left(k_{(-)}\right) = (2\pi)^{-2}\eta_\mu\eta_\nu/k^2_{(-)}$, this normalization term cancels the instantaneous part in Feynman propagator (locality condition). Also, choosing the initial and final states $b^\dagger(\boldsymbol{q}_1)b^\dagger(\boldsymbol{p}_1)\Omega$ and $b^\dagger(\boldsymbol{q}_2)b^\dagger(\boldsymbol{p}_2)\Omega$, respectively, the terms in $\hat{D}^F_{\mu\nu}(k)$ linear in the momentum do not contribute because of the on-mass-shell condition. We conclude that the final result is the same as taking $-(2\pi)^{-2}\frac{g_{\mu\nu}}{k^2+i0^+}$ as the photon propagator, proving the equivalence with instant dynamics.

Compton Scattering: Now consider the scattering of a scalar particle with a photon. Performing the splitting of the causal distribution of second order and writing the allowed normalization terms, we find the two-points distribution as being:

$$T_2^C(x_1;x_2) = ie^2 \ :A^{(a)}(x_1)A^{(b)}(x_2): \ \{-\left(:\varphi(x_1)\varphi^\dagger(x_2): + :\varphi^\dagger(x_1)\varphi(x_2):\right)\partial_{(a)}\partial_{(b)}D^F(y)$$
$$+\left(:\partial_{(a)}\varphi(x_1)\varphi^\dagger(x_2): + :\partial_{(a)}\varphi^\dagger(x_1)\varphi(x_2):\right)\partial_{(b)}D^F(y) - \left(:\varphi(x_1)\partial_{(b)}\varphi^\dagger(x_2):\right.$$
$$\left. + :\varphi^\dagger(x_1)\partial_{(b)}\varphi(x_2):\right)\partial_{(a)}D^F(y) + \left(:\partial_{(a)}\varphi(x_1)\partial_{(b)}\varphi^\dagger(x_2): + :\partial_{(a)}\varphi^\dagger(x_1)\partial_{(b)}\varphi(x_2):\right)D^F(y)\}$$
$$+ :A^{(\alpha)}(x_1)A^{(\beta)}(x_2): \ \eta_{(\alpha)(\beta)}\left\{C\left(y^{(-)}\right) :\varphi(x_1)\varphi^\dagger(x_2): +C'\left(y^{(-)}\right) :\varphi^\dagger(x_1)\varphi(x_2):\right\}\delta\left(y^{(+,\perp)}\right) .$$

Imposing charge conjugation invariance, i.e. $\left(\mathsf{U}_\varphi \otimes \mathsf{U}_A\right) T_n(x_1;\cdots;x_n)\left(\mathsf{U}_\varphi \otimes \mathsf{U}_A\right)^\dagger = T_n(x_1;\cdots;x_n)$, we obtain: $C\left(y^{(-)}\right) = C'\left(y^{(-)}\right)$. Also, imposing the invariance of S_2^C under the residual gauge freedom remaining from the choice of the NP gauge $A^{(+)} = 0$, we arrive at: $C\left(y^{(-)}\right) = \delta\left(y^{(-)}\right)$. The contribution of this normalization terms is, in the adiabatic limit $g \to 1$:

$$\frac{1}{2!}\int d^4x_1 d^4x_2 T_2^C(x_1;x_2) = \cdots - ie^2\int d^4x_1 \ :A^{(\perp)}(x_1)A^{(\perp)}(x_1): \ :\varphi^\dagger(x_1)\varphi(x_1): \quad . \qquad (4.2)$$

We see that the e^2 term which arises from the minimal coupling prescription in the conventional theory, appears as a normalization term required for invariance under residual gauge transformations in the second order perturbation term in the causal approach.

5. Conclusions

We have reformulated CPT in a way compatible with LF dynamics. Particularly, we have shown that when the causality axiom is referred to the $x^{(+)}$ coordinate, causality theorems allow the retarded and advanced distributions to be non-null on the entire $x^{(-)}$-axis, so that the normalization terms are defined on it. In the applications to SQED, these normalization distributions can always be chosen in such a way that locality is preserved, cancelling the instantaneous terms of the propagators which arise in the splitting procedure of the commutation distributions of the fields; those normalization distributions are identified with the non-local terms in the Lagrangian density in the usual approach.

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