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$\pi\pi$ scattering in a renormalized Hamiltonian matrix *

María Gómez-Rocha†
Universidad de Granada and Instituto Carlos I de Física Teórica y Computacional
E-mail: mgomezrocha@ugr.es

Enrique Ruiz Arriola
Universidad de Granada and Instituto Carlos I de Física Teórica y Computacional
E-mail: earriola@ugr.es

A Wilsonian approach to $\pi\pi$ scattering based in the Glazek-Wilson Similarity Renormalization Group (SRG) for Hamiltonians is analyzed in momentum space up to a maximal CM energy of $\sqrt{s} = 1.4$ GeV. To this end, we identify the corresponding relativistic Hamiltonian by means of the 3D reduction of the Bethe-Salpeter equation in the Kadyshevsky scheme, introduce a momentum grid and provide an isospectral definition of the phase-shift based on a spectral shift of a Chebyshev angle. We also propose a new method to integrate the SRG equations based on the Crank-Nicolson algorithm with a single step finite difference so that isospectrality is preserved at any step of the calculations. We discuss issues on the unnatural high momentum tails present in the fitted interactions and reaching far beyond the maximal CM energy of $\sqrt{s} = 1.4$ GeV and how these tails can be integrated out explicitly by using Block-Diagonal generators of the SRG.

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†Speaker.

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1. Introduction

Scattering experiments provide an important source of information regarding hadronic interactions. This is so, for instance, for $\pi\pi$ scattering where accurate determinations of phase-shifts exist [1] but give no clue on how the 3π system possibly explaining ω and A_1 should be handled. Hamiltonian methods involving potentials become very valuable when treating the mentioned hadronic interactions as a few-body problem where one pursues the determination of bound states and resonances of multi-hadron systems in terms of the corresponding potentials. However, existing model $\pi\pi$ potentials that fit the interaction display an annoying long momentum tail (cf. [2, 3]), which involves high energy scales not natural to the physical problem. In this work we address this issue using the similarity renormalization group (SGR) approach to Hamiltonian dynamics [4, 5, 6] and employ a new method for calculating phase shifts that has been investigated recently [7, 3].

2. The Kadyshevsky Hamiltonian: Phase-shifts and the spectral-shift method

The theoretical determination of scattering phase-shifts from potentials requires solving an integral equation, which can be tackled numerically applying a discretization of integrals using a momentum grid. We choose the Kadyshevsky equation [8], which allows for a Hamiltonian formulation and can be easily extended to the more general case of the three-body problem [1]. The reaction matrix in the Kadyshevsky version is related to the scattering phase shifts through:

$$R_l(p',p,\sqrt{s}) = V_l(p',p) + \int_0^\infty dq\,\frac{q^2}{4E_q^2}\frac{V_l(p',q)R_l(q,p,\sqrt{s})}{\sqrt{s}-2E_q}\,, \quad -\tan\delta_l(p) = \frac{\pi}{8}\frac{p}{E_p}R_l(p,p,\sqrt{s})\,, \tag{2.1}$$

where l is the orbital angular momentum, s is the Mandestam variable, and the subscript q (or, analogously, p) in E_q indicates that $E_q = \sqrt{\vec{q}^2+m^2}$.

The Hamiltonian corresponding to the Kadyshevsky equation Eq. (2.1) is given by

$$H\Psi_l(p) \equiv 2E_p\Psi_l(p) + \int_0^\infty dq\,\frac{q^2}{4E_q^2}v_l(p,q)\Psi_l(q)\,, \tag{2.2}$$

which in general needs to be solved numerically. We choose the Gauss-Chebyshev quadrature:

$$p_n = \frac{\Lambda_{\rm num}}{2}\left[1-\cos\phi_n\right]\,, \quad w_n = \frac{\Lambda_{\rm num}}{2}\frac{\pi}{N}\phi_n\,, \quad \phi_n = \frac{\pi}{N}\left(n-\frac{1}{2}\right) \tag{2.3}$$

where $n=1,\ldots,N$, and ϕ_n is the *Chebyshev angle*. On the momentum grid, Eq. (2.2) becomes

$$2E_n\Psi_n + \sum_k w_k\frac{p_k^2}{4E_k^2}V_{n,k}\Psi_k = \sqrt{s}\Psi_n\,. \tag{2.4}$$

Phase shifts can be determined from the spectrum of the Hamiltonian, as it has been explained by DeWitt [9] and Fukuda and Newton [10], who linked the shift produced in the spectrum after

[1] The standard relativistic approach to describe $\pi\pi$ scattering is the Bethe-Salpeter equation (BSE). In practice, due to the complications that the four-dimensional nature of the integrals presents, one employs judicious 3D reductions of the BSE, which are closer in spirit to the nonrelativistic Lippmann-Schwinger equation.

introducing the interaction with the scattering phase shifts appearing in the S-matrix. The relation is based on the use of an equidistant energy and momentum grids, respectively. For our Gauss-Chebyshev grid, which is equidistant in the Chebyshev angle, the corresponding relation becomes [7, 3]

$$\delta_n = -\pi \frac{\Phi_n - \phi_n}{\Delta\phi_n} \equiv -\pi \frac{\Delta\Phi_n}{\Delta\phi_n} \,, \tag{2.5}$$

where, $\Delta\phi_n = \frac{\pi}{N}$, and the "distorted" angles Φ_n are calculated inverting in Eqs. (2.3) and replacing p_n by P_n, which is extracted from the Hamiltonian eigenvalue $\sqrt{s} = 2E_n = \sqrt{m^2 + P_n^2}$.

For our discussion we take the $\pi\pi$ potentials introduced in Ref. [2]. In Figure 1 we show our ϕ-shift results compared with the model fit [2, 3] and with the experimental data [1] [2]. As it can be seen, the prediction is very accurate. Even for a grid with a relative small number of points, the results are much better and less sensitive to discretization effects than the results obtained from standard Lippann-Schwinger (LS) equation (cf. Figure 2 and [3] for details). Note also the improvement with respect to the *momentum* or *energy-shift* method described in [11, 3].

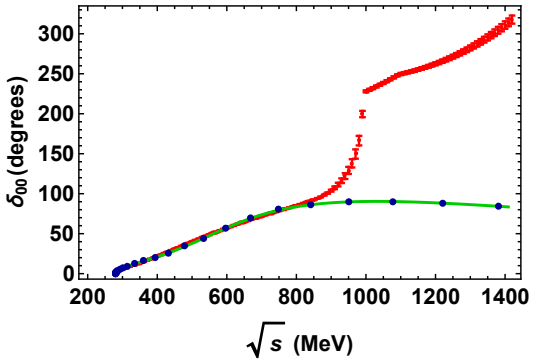

Figure 1: $\pi\pi$ scattering phase shifts in the $S0$ calculated using the ϕ-shift method using a grid of N =50 points (blue dots) compared with the model fit, (green, smooth line) and experimental data taken from [1] (red dots with error bars). See all channels in [3].

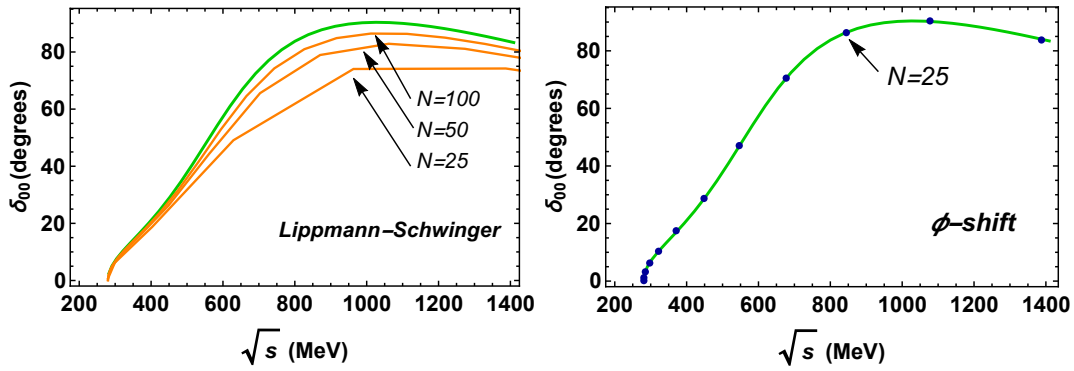

Figure 2: Phase shifts for the $S0$ channel calculated using the LS equation versus the ϕ-shift method. Left panel: results obtained using LS equation for three grids of different number of points. Right panel: ϕ-shift method for a grid of N =25 points. The green, smooth line is the model fit in both graphics. See a deeper analysis with all channels in [3].

[2] Phenomenologically, the steep raise of the $\pi\pi$ phase-shift in the $S0$ channel is due to the onset of the subthreshold $\bar{K}K$ channel not included here for simplicity.

3. Similarity renormalization group and $\pi\pi$ scattering

The phenomenological potentials present very long tails (up to 30 GeV, see Fig. 3), an annoying fact that appears unnatural considering the experimental region reaches only ~ 1.5 GeV. To handle

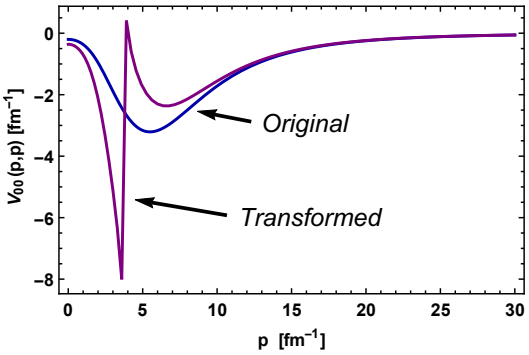

Figure 3: Diagonal matrix elements of the block-diagonal evolved potential for the S0 channel compared to the original one. It can be done analogously for all channels [11, 12]

this we invoke the SRG [4, 5, 6] which has been used mostly in nuclear physics to tame the nuclear core, and resting on the possibility to apply a (scale dependent) unitary transformation on the initial Hamiltonian, evolving it in a continuous way into a more convenient basis *preserving* the eigenvalues, and hence the phase-shifts. The SRG scale-dependence is dictated by the equation

$$\frac{dH_t}{dt} = [[G_t, H_t], H_t] \,, \tag{3.1}$$

where t is the renormalization-group parameter and G_t is the generator which determines the basis into which the *running* Hamiltonian, H_t, is transformed. One customary choice is to use a diagonal-like Wilson generator, $G_t = T$, which transforms the Hamiltonian matrix into a narrow band-diagonal one using in practice the Crank-Nicolson method (see e.g. [11]). Here we consider instead a block-diagonal generator $G = PHP + QHQ$, by means of the orthogonal projectors, $P = \theta(\Lambda - p)$ and $Q = \theta(p - \Lambda)$, which define two subspaces separated by a cut-off Λ and which transforms the matrix into a block-diagonal one. Figure 4 illustrates the evolution of the Hamiltonian matrix corresponding to the $S0$ channel potential of Section 2 up to different values of $\lambda = 1/\sqrt{t}$. The block-diagonal generator decouples two subspaces, below and beyond the cutoff $\Lambda \sim 1400$ MeV, as it is visible in the lower-right panel of Figure 4. The matrix elements relevant to the experimental region are now contained in the small block, and the eigenvalues of such a submatrix are the ones of the complete matrix with values below the cutoff *only*, thus eliminating higher energies explicitly.

4. Conclusions and outlook

We have used the $S0$ channel in $\pi\pi$ scattering to illustrate how the SRG can be employed to treat the problem of long tails potential in hadronic physics. Using a block-diagonal generator, we have constructed an effective Hamiltonian that allows to decouple two energy regions and select the matrix block relevant to the experimental region. We have presented a method [7, 3] for calculating phase shifts based on the spectrum modification that appears after introducing the interaction, which is obtained given a momentum grid.

Once the effective Hamiltonian has been constructed and the different energy regions have been decoupled, only the small matrix is needed to study the physical problem, and a corresponding

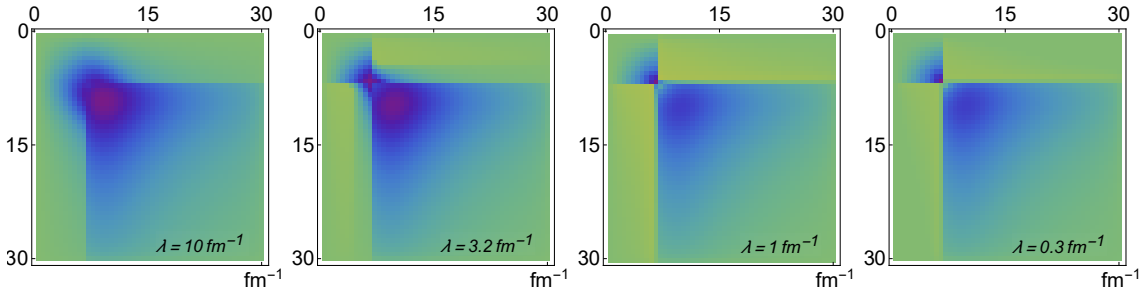

Figure 4: Block-diagonal evolved Hamiltonian in the $S0$ channel corresponding to different values of λ. The calculation is made using a grid of N =50 points.

potential in the reduced region can be identified, thus explicitly ignoring any ultraviolet effects. From the evolved matrix and using Eq. (2.4) in inverse way, we can identify the corresponding effective potential, whose diagonal is plotted in Figure 3. Since the reduced subspace of the effective potential (i.e. from 0 to about 5 fm^{-1}) yields the same phase shifts as the whole original Hamiltonian, we can simply consider the reduced space, which contains much less points, to reproduce experiments and to treat further physical problems that requires such a potential. An example is the three-body problem, e.g., the ω and A_1 resonances decaying to three pions, that includes the two-body potential as a component in its formulation. Moreover, the fact that the ϕ-shift method reproduces quite accurately the experimental results for rather coarse momentum grids, can advantageously be used as an alternative tool that reduces the computational cost in a situation where the binding in the three-body system is dominated by an infrared cut-off corresponding to the finite size of the three-body system.

References

[1] R. Garcia-Martin et al., Phys. Rev. D83 (2011) 074004, 1102.2183.

[2] L. Mathelitsch and H. Garcilazo, Phys. Rev. C32 (1985) 1635.

[3] M. Gómez-Rocha and E. Ruiz Arriola, (2019), arXiv:1911.08990.

[4] S.D. Glazek and K.G. Wilson, Phys. Rev. D48 (1993) 5863.

[5] S.D. Glazek and K.G. Wilson, Phys. Rev. D49 (1994) 4214.

[6] K.G. Wilson et al., Phys. Rev. D49 (1994) 6720, hep-th/9401153.

[7] M. Gómez-Rocha and E. Ruiz Arriola, Phys. Lett. B800 (2020) 135107, 1910.10560.

[8] V.G. Kadyshevsky, Nucl. Phys. B6 (1968) 125.

[9] B.S. DeWitt, Phys. Rev. 103 (1956) 1565.

[10] N. Fukuda and R.G. Newton, Phys. Rev. 103 (1956) 1558.

[11] M. Gómez-Rocha and E. Ruiz Arriola, 15th International Conference on Meson-Nucleon Physics and the Structure of the Nucleon (MENU 2019) Pittsburgh, Pennsylvania, USA, June 2-7, 2019, 2019, 1909.01715.

[12] M. Gómez-Rocha and E. Ruiz Arriola, In preparation.