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Ward-Takahashi Identity in QED_4 at the One-loop Level on the Light-Front

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The Ward-Takahashi identity in quantum electrodynamics (QED_4), first considered by J. C. Ward and Y. Takahashi, correlates the wave function renormalization for the electron to its vertex renormalization function. It guarantees the cancellation of ultraviolet (UV) divergences to all orders of perturbation theory. Since the QED in the light-front gauge is a constrained theory that brings a more demanding UV renormalization program due to the inevitable non-local terms, we check the Ward-Takahashi identity to the one-loop level, using the Mandelstam-Leibbrandt prescription to handle the characteristic light-front poles that appear in the Feynman integrals. Our calculations for the vertex correction and inverse electron propagators show that the strict light-front part contributions do indeed satisfy the Ward-Takahashi identity to one-loop order.

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1. Introduction

Ward-Takahashi identities [1, 2] are those that arise from the symmetries of a given theory, among which gauge symmetry is an example. These entail a definite relationship between correlation functions or amplitudes in quantum field theories. Such identities should hold even amid and after the process of regularization and renormalization that the quantum theory calls for. Since the renormalization program for the light-front fixed gauge theories are beset by singularities, we take the model quantum electrodynamics to demonstrate how these identities are preserved and recovered in the milieu of specific light-cone calculations.

The study of different forms for relativistic dynamics was pioneered by P.A.M.Dirac [3], who considered three possibilities for such description. One of them is known as the front form, in which the time evolution occurs along the plane $x^+ = (x^0 + x^3)/\sqrt{2}$, traditionally understood as the light-front "time", while $x^- = (x^0 - x^3)/\sqrt{2}$ defines the longitudinal space dimension, with the transverse components remaining as usual in the Minkowski space-time, that is, $\mathbf{x}^\perp = (x^1, x^2) \equiv (x, y)$. Their conjugate momenta are similarly defined as $k^+ = (k^0 + k^3)/\sqrt{2}$ (the longitudinal component of momentum), $k^- = (k^0 - k^3)/\sqrt{2}$ (the light-front "energy") and $\mathbf{k}^\perp = (k^1, k^2) \equiv (k_x, k_y)$ (the transverse components of momentum).

In this paper we are going to focus our attention on the quantum electrodynamics calibrated in the light-front gauge and see how the different specific light-front contributions to the Ward-Takahashi identity behave at the one-loop level of quantum corrections. We show that the identity is preserved ensuring the renormalizability of the theory.

2. The Ward-Takahashi Identities

Ward-Takahashi identities are the quantum versions of the classical Noether's theorem for conservation laws associated with symmetries exhibited (i.e. obeyed) by the physical system. They comprise a family of several possible types of relations between various correlation functions or amplitudes. In the specific case of quantum electrodynamics (QED), we have for example, identities that involve vertices with N external photons and no external electrons (only virtual electrons) such that $(k_j)_{\mu_j} V_N^{\mu_1\mu_2\cdots\mu_N}(k_1, k_2, \cdots, k_N) = 0$, $\forall j$ where k_j is the momentum of the j^{th} photon in the vertex. The simplest case of this class is for $N = 2$ photons and the correlation function is known as the vacuum polarization tensor, for which we have $k_\mu \Pi_2^{\mu\nu}(k) = 0$. Next, there is a family of identities in which we have two external electrons, with momentum p and p' respectively, and N photons. These identities become more complicated due to the relations between different correlation functions, namely, $(k_j)_{\mu_j} V_N^{\mu_1\mu_2\cdots\mu_N}(p, p', k_1, k_2, \cdots, k_N) = eS_{N-1}^{\mu_1\mu_2\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_N}(p', p - k_j) - eS_{N-1}^{\mu_1\mu_2\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_N}(p' + k_j, p)$, $\forall j$ where momentum conservation requires $p - p' = k_1 + k_2 + \cdots + k_N$. The simplest example for this class is the identity that relates the vertex and the inverse electron propagators, which we are going to treat in detail in the following sections. We see that, in principle, we could consider other general cases, involving amplitudes containing A electrons and N photons, with increasing complexity, but of course, for the renormalization program in QED, the latter one is of paramount importance.

3. Tree-level Ward-Takahashi Identity for QED

Let us consider the electron propagator $S_0(p) = \frac{i}{\not p - m}$ and the three-legged vertex QED process $V_1^\mu(p',p,k) = \frac{i}{\not p' - m}(-ie\gamma^\mu)\frac{i}{\not p - m}$ at the tree-level, with p, p', k being respectively the incoming electron and outgoing electron and photon, with momentum conserved, $p' = p - k$. The contraction of the photon's momentum k_μ with the vertex $V_1^\mu(p',p,k)$ yields $k_\mu V_1^\mu(p',p,k) = \frac{-ie}{\not p - m} + \frac{ie}{\not p' - m} \equiv -eS_0(p,p) + eS_0(p',p')$. We can re-express this result to resemble the general formula given earlier by writing explicitly $k_\mu V_1^\mu(p',p,k) = eS_0(p',p-k) - eS_0(p'+k,p)$.

An alternative form that often is used for the Ward-Takahashi identity can be worked out as follows: Take the vertex $V_1^\mu(p',p,k)$ and express it as $V_1^\mu(p',p,k) \equiv S_0(p')\Gamma_1^\mu(p',p,k)S_0(p)$, in which $\Gamma_1^\mu(p',p,k)$ to the zeroth-order is simply $-ie\gamma^\mu$. Multiplying from the left by $S_0^{-1}(p')$ and from the right by $S_0^{-1}(p)$, we get the right-hand-side in terms of inverse electron propagators. Then the Ward-Takahashi identity becomes

$$k_\mu \Gamma_1^\mu(p',p,k) = eS_0^{-1}(p) - eS_0^{-1}(p'). \tag{3.1}$$

4. One-loop level Ward-Takahashi Identity for QED

Now, we want to go to the next level of perturbation, i.e, one-loop order correction to the vertex and propagators. Then, we have

$$\begin{aligned} \Gamma_1^\mu(p',p,k) &= -i\left\{e\gamma^\mu + \Lambda_{1loop}^\mu(p',p,k)\right\}, \\ S_0^{-1}(p) &= -i\left\{\not p - m - \Sigma_{1loop}(p)\right\}. \end{aligned} \tag{4.1}$$

Substituting Eq. (4.1) into Eq. (3.1) we obtain the relevant one-loop relation:

$$k_\mu \Lambda_{1loop}^\mu(p',p,k) = e\left\{\Sigma_{1loop}(p') - \Sigma_{1loop}(p)\right\}, \tag{4.2}$$

in which

$$\Lambda_{1loop}^\mu(p,k) = (-ie)^3 \int \frac{d^Dq}{(2\pi)^D}\gamma^\alpha\left[iS_o(p-k-q)\right]\gamma^\mu\left[iS_0(p-q)\right]\gamma^\beta\left[iD_{\alpha\beta}(q)\right], \tag{4.3}$$

$$\Sigma_{1loop}(p) = (-ie)^2 \int \frac{d^Dq}{(2\pi)^D}\gamma^\alpha\left[iS_0(p-q)\right]\gamma^\beta\left[iD_{\alpha\beta}(q)\right], \tag{4.4}$$

where $D_{\alpha\beta}(q)$ is the gauge dependent photon propagator. This identity which relates the vertex correction with the electron self-energies can be represented grapically as (see Figure 2):

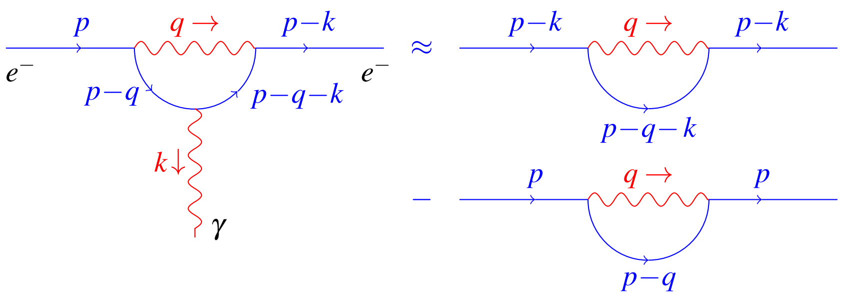

Figure 2 - Feynman diagrams for one-loop Ward-Takahashi identity.

4.1 Light-front Case

Eqs. (4.3) and (4.4) can be calculated in different gauge choices. In particular, Eqs. (4.3) and (4.4) can be split into the covariant part and the light-front gauge part:

$$\Lambda^{\mu}_{1loop}(p,k) = \Lambda^{\mu[\mathrm{Cov}]}_{1loop}(p,k) + \Lambda^{\mu[\mathrm{LF}]}_{1loop}(p,k), \tag{4.5}$$

$$\Sigma_{1loop}(p) = \Sigma^{\mathrm{Cov}}_{1loop}(p) + \Sigma^{\mathrm{LF}}_{1loop}(p). \tag{4.6}$$

The gauge propagator $D_{\alpha\beta}(q)$ in the light-front gauge has been extensively discussed in Ref. [4]. Results for the covariant part are standard and are expressed as

$$k_\mu \Lambda^{\mu[\mathrm{Cov}]}_{1loop}(p,k) = \not{k} F_1^{(1loop)}(k^2) + \frac{i}{2m} k_\mu k_\nu \sigma^{\mu\nu} F_2^{(1loop)}(k^2),\; k = p - p', \tag{4.7}$$

in which $F_1(k^2)$ and $F_2(k^2)$ are the vertex functions, given in terms of quite complicated diverging momentum integrals.

For the specific light-front contributions in the light-front gauge, we write for the left-hand-side and the right-hand-side of Eq. (4.2) respectively as

$$k_\mu \Lambda^{\mu[\mathrm{LF}]}_{1loop}(p, p-p') \equiv L_{\mathrm{a}}(p,p') + L_{\mathrm{b}}(p,p'), \tag{4.8}$$

$$e\left\{\Sigma^{[\mathrm{LF}]}_{1loop}(p') - \Sigma^{[\mathrm{LF}]}_{1loop}(p)\right\} \equiv \mathscr{S}_{\mathrm{a}}(p,p') + \mathscr{S}_{\mathrm{b}}(p,p'), \tag{4.9}$$

in which

$$\begin{aligned} L_{\mathrm{a}}(p,p') = & -e^3 \not{k}\Big\{2\left[Q_{[0]}(p) + Q_{[0]}(p')\right] - 2[p^+ Q_{[+]}(p) + p'^{+}_{[+]}(p')] \\ & -\not{k}\gamma^+\gamma_\alpha Q^{\alpha}_{[+]}(p) - \gamma_\alpha\gamma^+\not{k} Q^{\alpha}_{[+]}(p')\Big\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} L_{\mathrm{b}}(p,p') = & \; 2e^3\Big\{2\not{k} Q_{[++]}(p,p') + \not{k}\gamma^+\gamma_\alpha Q'^{\alpha}_{[++]}(p,p') + \gamma_\alpha\gamma^+\not{k} Q^{\alpha}_{[++]}(p,p') \\ & + k^+\gamma_\alpha\gamma^+\gamma_\beta Q^{\alpha\beta}_{[++]}(p,p')\Big\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathscr{S}_{\mathrm{a}}(p,p') = & -2e^3\Big\{(\not{p}' - m) Q_{[0]}(p') - (\not{p} - m) Q_{[0]}(p) \\ & -(p'^+\gamma_\alpha + p'_\alpha\gamma^+) Q^{\alpha}_{[+]}(p') + (p^+\gamma_\alpha + p_\alpha\gamma^+) Q^{\alpha}_{[+]}(p)\Big\}, \end{aligned} \tag{4.12}$$

$$\mathscr{S}_{\mathrm{b}}(p,p') = -2e^3\left\{p'^+ T_{[++]}(p') - p^+ T_{[++]}(p)\right\}, \tag{4.13}$$

with the notation $a^+ = a \cdot n = a_\mu n^\mu$ for a convenient choice for the gauge fixing vector n.

The relevant integrals to be calculated are listed below:

$$Q_{[0]}(p) = \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon) D(p)}, \tag{4.14}$$

$$Q_{[+]}(p) = \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon) D(p)} \frac{1}{q^+}, \tag{4.15}$$

$$Q^{\alpha}_{[+]}(p) = \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon) D(p)} \frac{q^\alpha}{q^+}, \tag{4.16}$$

$$Q_{[++]}(p,p') = \int \frac{d^dq}{(2\pi)^d} \frac{1}{D(p') D(p)} \frac{(p'^+ - q^+)(p^+ - q^+)}{(q^+)^2}, \tag{4.17}$$

$$Q'^{\alpha}_{[++]}(p,p') = \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(p'^+ - q^+)(q^\alpha)}{(q^+)^2}, \tag{4.18}$$

$$Q^{\alpha}_{[++]}(p,p') = \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{(p^+ - q^+)(q^\alpha)}{(q^+)^2}, \tag{4.19}$$

$$Q^{\alpha\beta}_{[++]}(p,p') = \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p')D(p)}\frac{q^\alpha q^\beta}{(q^+)^2}, \tag{4.20}$$

$$T_{[++]}(p) = \int \frac{d^dq}{(2\pi)^d}\frac{1}{D(p)}\frac{1}{(q^+)^2}, \tag{4.21}$$

in which we have introduced the shorthand definition $D(p) \equiv (p-q)^2 - m^2 + i\varepsilon$. Integral calculations are carried out using the Mandelstam-Leibbrandt [5, 6] prescription for the $(k^+)^{-\alpha}, \alpha = 1,2$ poles. Results for all the relevant integrals above are tabulated in [7]. Together with the following important Dirac algebra relations in the light-front, $(\gamma^+)^2 = (\gamma^-)^2 = 0$ and $\gamma^\mu\gamma^+\gamma^- + \gamma^-\gamma^+\gamma^\mu = 2\gamma^\mu + 2g^{\mu+}\gamma^- + 2g^{\mu-}\gamma^+$, we have

$$k_\mu \Lambda^{\mu[\mathrm{LF}]}_{1loop}(p, p-p')\Big|_{\mathrm{div}} = -2e^3\left\{\not{k} - k^+\gamma^- - k^-\gamma^+\right\} I_{\mathrm{div}}(d), \tag{4.22}$$

$$e\left\{\Sigma^{[\mathrm{LF}]}_{1loop}(p')\Big|_{\mathrm{div}} - \Sigma^{[\mathrm{LF}]}_{1loop}(p)\Big|_{\mathrm{div}}\right\} = -2e^3\left\{\not{k} - k^+\gamma^- - k^-\gamma^+\right\} I_{\mathrm{div}}(d), \tag{4.23}$$

where $I_{\mathrm{div}} = Q_{[0]}(p) = \frac{2i\pi}{(4-d)} + f_{[0]}(p) + \mathscr{O}(2-d/2)$. Note here that the non-vanishing finite part in $d=4$ is denoted by $f_{[0]}(p)$. So, Ward-Takahashi Identity is verified to one-loop order.

5. Conclusion

Using the Mandelstam-Leibbrandt prescription to handle the light-front poles, we were able to verify the Ward-Takahashi identity in QED to the one-loop order. Question is whether this can be carried out for higher orders, the issue being that then multiplicative factors such as $\frac{1}{q^+(p^+-q^+)\cdots}$ appear and for which some kind of partial fractioning will be needed. Is partial fractioning OK in the Mandelstam-Leibbrandt prescription scheme? That is something that needs further checking.

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