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Quantum tunneling time in the light-front

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In this paper we study the quantum tunneling time in the light-front using the technique of dwell time (or delay time). We introduce this method of calculation considering the delay time for the Schrödinger equation. Then, as we consider the relativistic Klein-Gordon equation, we face a problem, because the probability density for this case is not positive definite, and the delay time technique becomes inapplicable here. Other techniques have been devised to tackle this issue, but comparison of results becomes difficult because methods used are different. Therefore, there is a discrepancy in the tunneling time assessment for the low-energy non relativistic and relativistic case. By treating the Klein-Gordon equation in the light-front, we show that this discrepancy disappears.

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1. Introduction

Tunneling is a quantum phenomenon in which a particle is able to penetrate and, in some cases, cross a potential barrier. According to the laws of classical physics, if a particle has energy E which is smaller than the height of the potential barrier V, it cannot cross the barrier over. However, according to quantum mechanics, there is a nonzero small probability of that particle to cross a potential barrier even when its height is superior to the particle's energy.

In quantum mechanics, the variable time is a parameter and not an observable. Thus, we have difficulty in measuring it, though we need to know the time it takes for a particle to cross over a potential barrier. Several experiments have been proposed to measure this time, considering how one can measure a particle's behavior without interfering in its state [1–3].

The stationary delay time [4] shows us how we can understand the time that a wave remains inside a potential barrier V simply by studying what happens inside the potential, being given by

$$T_{\mathrm{p}} = \frac{\int_{x_1}^{x_2} \rho(x)dx}{|J(x)|}, \tag{1.1}$$

where T_{p} is the delay time as the ratio between the integral in space of the probability density in the stationary situation, $\rho(x) = \psi(x)\psi^*(x)$, which is nothing more than the probability to find "our" particle/wave in a given region of space and the probability density current for the incident wave $J(x)$ in the potential barrier, given by

$$J(x) = \frac{\hbar}{2im}\left[\psi^*(x)\nabla\psi(x) - \psi(x)\nabla\psi^*(x)\right]. \tag{1.2}$$

Using the method of stationary delay time [4], we calculate the tunneling time for a particle with initial kinetic energy $E < V$, represented by a wave packet incident on a rectangular potential barrier of height V defined in $0 < x < a$, namely: $V(x) = V$ if $0 \leq x \leq a$ and $V(x) = 0$ otherwise.

With this potential, the stationary solutions for the Schrödinger's equation are given by:

$$\begin{aligned} \psi_{\mathrm{I}}(x) &= A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx},\ x < 0; \\ \psi_{\mathrm{II}}(x) &= C\mathrm{e}^{-\rho x} + D\mathrm{e}^{\rho x},\ 0 \leq x \leq a; \\ \psi_{\mathrm{III}}(x) &= F\mathrm{e}^{ikx} + G\mathrm{e}^{-ikx},\ x > a. \end{aligned} \tag{1.3}$$

in which $k = (\sqrt{2mE})/\hbar$ and $\rho = [\sqrt{2m(V-E)}]/\hbar$. The coefficients A, B, C, D, F, and G that appear in Eq. (1.3) above are determined by the continuity conditions for the wave function and its derivatives at the boundaries of the potential. These six boundary conditions allow us to write the six coefficients in terms of two of them. The choice of these two coefficients is arbitrary, and for convenience, we choose A and G because they are exactly coefficients that belong to the incoming wave packet and the outgoing wave packet respectively. According to the probability current Eq. (1.2) for the incident eigenfunction $\psi_{\mathrm{I}}(x)$ and using the definition of dwell time Eq. (1.1) for the eigenfunction $\psi_{\mathrm{II}}(x)$, we get the stationary quantum delay time in the region of the barrier:

$$T_{\mathrm{Schr}}(k,\rho;a) = \frac{mk}{\hbar|d(k,\rho;a)|^2}\left[\frac{(k^2+\rho^2)^2}{k\rho}\sinh(2\rho a) - 2ka(k^2-\rho^2)\right], \tag{1.4}$$

where $d(k,\rho;a) = (k^2-\rho^2)\sinh(\rho a) + 2ik\rho\cosh(\rho a)$ with the incident current $J(x) = \hbar k|A|^2/m$.

The stationary delay time given in Eq. (1.4) indicates the time interval in which the quantum system remains within the region $0 < x < a$, without any mention to the arrival or departure time nor the times between the moments in which the particle arrives and leaves the barrier region [2].

As we consider the relativistic version of the stationary delay time Eq. (1.1), we encounter a problem as we try to apply it to the Klein-Gordon equation. There is an inherent fundamental problem in the continuity equation, because for the Klein-Gordon equation, the probability density ρ can be positive ($\rho > 0$), or negative ($\rho < 0$). This precludes the possibility of having a relativistic version for the stationary delay time. In the following, we shall consider how this problem can be circumvented by defining the Klein-Gordon equation in the light-front coordinates.

2. Klein-Gordon equation in the light-front

We introduce the light-front coordinates by defining $x^{\pm} = \left(x^0 \pm x^3\right)/\sqrt{2}$ and $\mathbf{x}^{\perp} = x^1\hat{\imath} + x^2\hat{\jmath}$, with the canonically conjugate momenta components following the same pattern, i.e., $k^{\pm} = (k^0 \pm k^3)/\sqrt{2}$ and $\mathbf{k}^{\perp} = k^1\hat{\imath} + k^2\hat{\jmath}$, in which $(\hat{\imath}, \hat{\jmath})$ are unit vectors in the $x-$ and $y-$direction respectively. Scalar product between two vectors is therefore $x \cdot y = x_\mu y^\mu = x^+y^- + x^-y^+ - \mathbf{x}^{\perp}\mathbf{y}^{\perp}$. Of particular importance is the scalar momentum squared, $k_\mu k^\mu = 2k^+k^- - \mathbf{k}_{\perp}^2 = m_0^2c^2$, where m_0 is the rest mass of a particle, from which we have this peculiar energy-momentum relation, $k^- = (\mathbf{k}_{\perp}^2 + m_0^2c^2)/(2k^+)$.

The energy of a free particle in the Minskowski space-time is given by $k^0 = E = \pm c\sqrt{\vec{k} + m_0^2}$, a quadratic dependence between k^0 and $\vec{k}$ in contrast to the linear dependence between the k^- and $1/k^+$ in the light-front. Moreover, the energy-momentum relation in the light-front shows us that the signs of k^- and k^+ are inexorably correlated, i.e., if k^- is positive, k^+ is also positive and if k^- is negative, k^+ is also negative. The minus sign for the energy is interpreted to mean positive energy for antiparticles propagating into the future. Since the signs are correlated, this also means that in the light-front milieu, particles and antiparticles cannot appear simultaneously.

The D'Alambertian operator $\Box \equiv \partial_\mu\partial^\mu$ in the light-front coordinates is $\Box = 2\partial^+\partial^- - \partial_{\perp}^2$, so that the Klein-Gordon equation with interacting potential V is

$$\left[2\partial^+\partial^- - \partial_{\perp}^2 + \frac{m^2c^2}{\hbar^2} + V\right]\Psi_{\mathrm{LF}}(x) = 0, \tag{2.1}$$

in which $\Psi_{\mathrm{LF}}(x) \equiv \Psi_{\mathrm{LF}}(x^+, x^-, \mathbf{x}^{\perp}) = e^{i(k^-x^+ + k^+x^- - \mathbf{k}^{\perp}\mathbf{x}^{\perp})}$. From Eq. (2.1) we get the continuity equation for the Klein-Gordon equation in the light-front:

$$\partial^+\rho_+ + \partial^{\perp}\mathbf{J}_{\perp} = 0, \tag{2.2}$$

in which the probability density and current in the light-front are respectively

$$\rho_+ = \frac{\hbar}{im}\left(\Psi_{\mathrm{LF}}^*\partial_+\Psi_{\mathrm{LF}} - \Psi_{\mathrm{LF}}\partial_+\Psi_{\mathrm{LF}}^*\right), \tag{2.3}$$

$$\mathbf{J}_{\perp} = \frac{i\hbar}{2m}\left(\Psi_{\mathrm{LF}}\partial_{\perp}\Psi_{\mathrm{LF}}^* - \Psi_{\mathrm{LF}}^*\partial_{\perp}\Psi_{\mathrm{LF}}\right). \tag{2.4}$$

From Eq. (2.3) it can be shown that $\rho_+ = 2\hbar k^-/m$ which is strictly positive as long as we keep $k^+ > 0$ in the energy-momentum relation. Since we are going to deal with one-dimensional potential barrier, in the K-G equation (2.1) we set $x^- = 0$ and $\mathbf{x}^\perp = x\hat{\imath}$. This choice for a specific direction in our coordinate system is the most natural one, since we are restricting ourselves to the study of one-dimensional tunneling in the usual x-direction, for which a point in space-time is (x^0, x). This, in the light-front corresponds to $(x^+, \mathbf{x}^\perp)$, in which x^+ represents the 'time' evolution parameter [5] and $\mathbf{x}^\perp = x$ the one-dimensional space variable, with the other space coordinate $x^- \equiv 0$. Since the four-momentum vector in the light front is $(k^-, k^+, \mathbf{k}^\perp)$, in one space dimensional case we have correspondingly $(k^-, k^\perp)$ and using the operator correspondence for $k^\perp = -i\hbar\partial^\perp/\partial x$ we have:

$$\left(-\frac{\hbar^2\partial^{\perp 2}}{2k^+} + V_{\rm LF}\right)\Psi_{\rm LF} = k^-\Psi_{\rm LF}, \tag{2.5}$$

in which $V_{\rm LF} \equiv \dfrac{m^2c^2}{2k^+\hbar^2} + \dfrac{V}{2k^+}$. Eq. (2.5) has a similar structure as (and therefore can be thought of as being equivalent to) the non relativistic Schrödinger equation, in which instead of the usual m in the denominator, we find k^+ and the potential gets and additional term.

3. Tunneling time in the light-front

The stationary delay time in the light-front version, with the help of the Klein-Gordon equation (2.5) is given by

$$T_{\rm LF} = \frac{\int_{x_1}^{x_2} \rho_{\rm LF}(x)dx}{|J_{\rm LF}|}, \tag{3.1}$$

in which $\rho_{\rm LF}(x) = \rho_+$ is the light-front probability density and $|J_{\rm LF}| = \mathbf{J}^\perp = \hbar\mathbf{k}^\perp/m$ is the current in the light-front satisfying by continuity equation (2.2).

The integral in Eq. (3.5) can be evaluated using the following trick: Derive the K-G equation (2.5) with respect to k^- and multiply on the left by $\Psi^*_{\rm LF}$ to get

$$-\frac{\hbar^2}{2k^+}\Psi^*_{\rm LF}\frac{\partial^3\Psi_{\rm LF}}{\partial k^-\partial\mathbf{x}^{\perp 2}} + [V_{\rm LF} - k^-]\Psi^*_{\rm LF}\frac{\partial\Psi_{\rm LF}}{\partial k^-} = \Psi^*_{\rm LF}\Psi_{\rm LF}. \tag{3.2}$$

Next, we take the complex conjugate of Eq. (2.5), multiply to the right by $\partial\Psi_{\rm LF}/\partial k^-$ and subtracting the obtsined resulf from the Eq. (3.2) we get

$$\Psi^*_{\rm LF}\Psi_{\rm LF} = \frac{\hbar^2}{2k^+}\left[\frac{\partial^2\Psi^*_{\rm LF}}{\partial\mathbf{x}^{\perp 2}}\frac{\partial\Psi_{\rm LF}}{\partial k^-} - \Psi^*_{\rm LF}\frac{\partial^3\Psi_{\rm LF}}{\partial\mathbf{x}^{\perp 2}\partial k^-}\right]. \tag{3.3}$$

The above equation is the integrand of Eq. (3.5) and integration between $0 \le \mathbf{x}^\perp \le a$ yields

$$\begin{aligned}\int_0^a \Psi^*_{\rm LF}\Psi_{\rm LF}d\mathbf{x}^\perp &= \frac{\hbar^2}{2k^+}\left[\frac{\partial\Psi^*_{\rm LF}}{\partial\mathbf{x}^\perp}\frac{\partial\Psi_{\rm LF}}{\partial k^-} - \Psi^*_{\rm LF}\frac{\partial^2\Psi_{\rm LF}}{\partial\mathbf{x}^\perp\partial k^-}\right]_{\mathbf{x}^\perp=a} \\ &+ \frac{\hbar^2}{2k^+}\left[\frac{\partial\Psi^*_{\rm LF}}{\partial\mathbf{x}^\perp}\frac{\partial\Psi_{\rm LF}}{\partial k^-} - \Psi^*_{\rm LF}\frac{\partial^2\Psi_{\rm LF}}{\partial\mathbf{x}^\perp\partial k^-}\right]_{\mathbf{x}^\perp=0}.\end{aligned} \tag{3.4}$$

Using this last result together with $|J_{\rm LF}| = \mathbf{J}^\perp = \hbar\mathbf{k}^\perp/m$ we finally get for the stationary delay time in the light-front as

$$T_{\rm LF} = \frac{m}{2\hbar k^+\mathbf{k}^\perp}\left[2|\mathscr{T}|^2\left(2ak^+\mathbf{k}^\perp - \frac{d\phi_{\rm T}}{dk^-}\right) + 4k^+|\mathscr{R}|^2\sin(\phi_{\rm R})\right], \tag{3.5}$$

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in which the $|\mathscr{T}|^2 = \mathscr{T}^*_{\rm LF}\mathscr{T}_{\rm LF}$ and $|\mathscr{R}|^2 = \mathscr{R}^*_{\rm LF}\mathscr{R}_{\rm LF}$ are respectively the probability of transmission and reflexion, with

$$\mathscr{T}_{\rm LF} = \frac{2\mathbf{k}^{\perp}\rho^{\perp}e^{i\phi_{\rm T}}}{\sqrt{(\mathbf{k}^{\perp 2}+\rho^{\perp 2})^2\sinh^2(\rho^{\perp}a)+4(\mathbf{k}^{\perp}\rho^{\perp})^2}}, \tag{3.6}$$

$$\mathscr{R}_{\rm LF} = \frac{(\mathbf{k}^{\perp 2}+\rho^{\perp 2})\sinh(\rho^{\perp}a)e^{i\phi_{\rm R}}}{\sqrt{(\mathbf{k}^{\perp 2}+\rho^{\perp 2})^2\sinh^2(\rho^{\perp}a)+4(\mathbf{k}^{\perp}\rho^{\perp})^2}}. \tag{3.7}$$

The $\phi_{\rm T}$ and $\phi_{\rm R}$ are respectively the phases of transmission and reflexion

$$\phi_{\rm T} = \arctan\left[\frac{(\mathbf{k}^{\perp 2}-\rho^{\perp 2})}{2\mathbf{k}^{\perp}\rho^{\perp}}\tanh(\rho^{\perp}a)\right] - \mathbf{k}^{\perp}a, \tag{3.8}$$

$$\phi_{\rm R} = \arctan\left[\frac{(\mathbf{k}^{\perp 2}-\rho^{\perp 2})}{2\mathbf{k}^{\perp}\rho^{\perp}}\tanh(\rho^{\perp}a)\right] + \frac{\pi}{2}, \tag{3.9}$$

in which $\mathbf{k}^{\perp} = \sqrt{E^2-m^2c^4}/\hbar c$, $\rho^{\perp} = \sqrt{m^2c^4-(E-V)^2}/\hbar c$ and $E = K+mc^2$; K being the kinetic energy and m the rest mass.

In Figure 1 we summarize the results for the Schrödinger and K-G in the light-front in a graphic plot and show the similarity and agreement of the results.

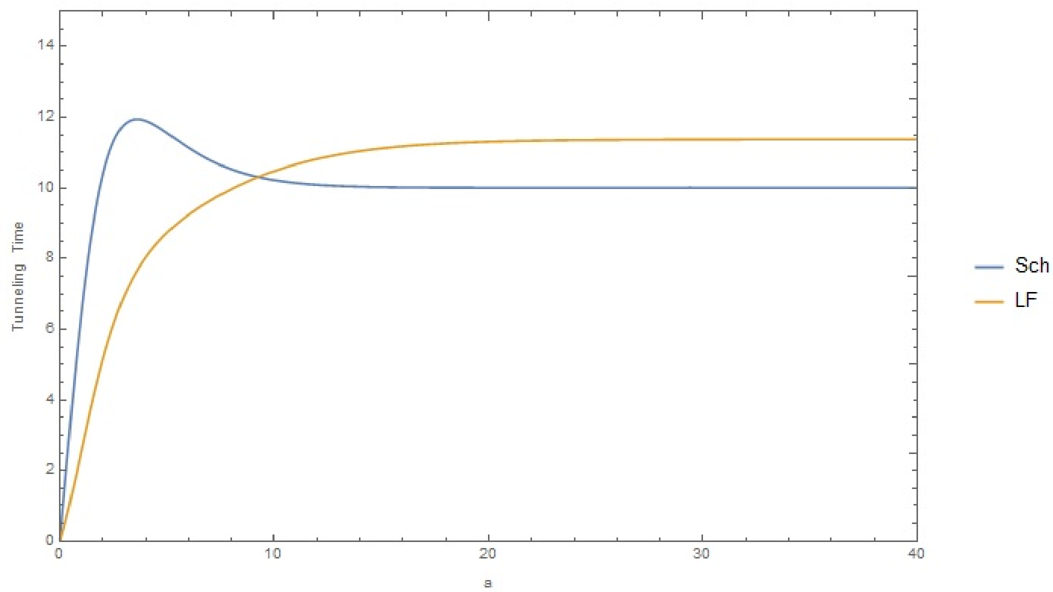

Figure 1: Delay time versus barrier width a for the Schrödinger (bumped curve) and KG in the light-front (smooth curve) equations at $K = 0.01, V = 0.05$, with $m = \hbar = c = 1$ and $0 \leq a \leq 40$.

In Figure 1 we see the plot of two curves, one for the delay time given by Eq.(1.4) and the other given by Eq.(3.5). We observe that these two curves are similar in their behavior with close converging points (for $a \geq 10$ these converging points are $T_{\rm Schr} \approx 10$ and $T_{\rm LF} \approx 11.4$.) We also notice that there is a hump in the $T_{\rm Schr}$ curve, which is caused by numerical fluctuations when $a \to 0$ in the equation for tunneling time in the Schrödinger case.

For opaque barriers characterized by $\rho a >> 1$ (or $a \to \infty$) it follows that the tunneling time becomes independent of the width of the barrier,

$$T_{\text{Schr}} \approx \frac{2mk}{\hbar\rho(k^2+\rho^2)}. \tag{3.10}$$

This result is known as the *Hartman* [6] *effect*. The same effect is observed in the curve that corresponds to the tunneling time in the light front,

$$T_{\text{LF}} \approx \frac{2m\sin(\phi_{\text{R}}|_{a\to\infty})}{\hbar k^{\perp}}, \tag{3.11}$$

in which

$$\phi_{\text{R}}|_{a\to\infty} = \arctan\left(\frac{k^{\perp 2}-\rho^{\perp 2}}{2k^{\perp}\rho^{\perp}}\right). \tag{3.12}$$

To obtain this last equation for the opaque barrier limit, we used the relation $|\mathscr{T}_{\text{LF}}|^2 + |\mathscr{R}_{\text{LF}}|^2 = 1$ and that we do not have emission or absorption of particles, that is, the potential is real.

For opaque barriers in rectangular potentials the tunneling time in the light-front is independent of the size of the potential. Therefore, the Hartman effect is also present in the light-front case. This is the reason why the two curves in the graph of Figure 1 do not tend to zero, but to a fixed value for opaque barriers, a result known as the Hartman effect.

4. Conclusion

The fundamental reason for the existence of so intriguing aspects in the study of tunneling times is that time is not an observable in Quantum Mechanics [1]. Because of this, there are several different definitions for tunneling time, which are, in their majority, not equivalent with each other. A few among of the many revision articles about these tunneling times are given in [1–3]. And as an example of a very interesting characteristic in the process of tunneling is the time independence of tunneling with the extension of the region classically forbidden when the barrier is opaque. This effect was discovered by Hartmnan [6] and today this effect bears his name.

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