

On the two-loop amplitude for $gg \rightarrow ZZ$ production with full top-mass dependence

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In this talk, we discuss top-quark contributions to ZZ production through gluon fusion at two loops. We use syzygies and modular arithmetic to compute the reductions to master integrals. In order to numerically evaluate the amplitude, we express it in terms of finite integrals using a new method to construct integrable linear combinations of divergent integrands.

14th International Symposium on Radiative Corrections (RADCOR2019)

9-13 September 2019

Palais des Papes, Avignon, France

*Speaker.

1. Introduction

ZZ production in gluon fusion is an important process at the LHC. It is a background to the on-shell production of a Higgs boson with subsequent decay to four leptons through an (off-shell) Z pair. For off-shell Higgs production, interference effects with continuum diboson production can amount to $O(10\%)$ [1]. Even though the gluon-induced process only starts at one-loop and is formally NNLO for the hadronic process $pp \rightarrow ZZ$, its contribution is $O(60\%)$ [2] of the total NNLO correction due to the high gluon luminosity at the LHC. The NLO corrections to $gg \rightarrow ZZ$ were found to be sizable [3], resulting in an $O(5\%)$ increase of the total cross section [4].

The one-loop amplitude for $gg \rightarrow ZZ$ has been known for a long time [5]. The two-loop corrections with massless quarks in the loops were computed in [6, 7]. Top-quark contributions are expected to be relevant, too, in particular at high invariant mass. They may therefore also impact indirect Higgs width determinations as proposed in [8]. Top-loop contributions to the two-loop amplitudes were calculated in [9, 10] using the heavy top-mass approximation and in [11] using Padé approximants. In [12], the authors considered the form factors relevant to the interference with the Higgs production amplitude, incorporating also an expansion around the top-quark pair production threshold at two loops. In this talk, we focus on the calculation of the two-loop amplitude for on-shell ZZ production through gluon fusion with full top mass dependence.

2. Amplitude and form factors

We consider the two-loop QCD corrections to the process

$$g(p_1) + g(p_2) \rightarrow Z(p_3) + Z(p_4), \quad (2.1)$$

where p_1 and p_2 are incoming momenta and p_3 and p_4 are outgoing momenta. We consider a finite top-quark mass and treat all other quarks as massless. We employ conventional dimensional regularization and decompose the amplitude

$$\mathcal{M} = \mathcal{M}_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) \varepsilon^\mu(p_1) \varepsilon^\nu(p_2) \varepsilon^{*\rho}(p_3) \varepsilon^{*\sigma}(p_4) \quad (2.2)$$

into form factors [6] with the unpolarized cross section in mind. Lorentz invariance allows us to restrict ourselves to 138 tensor structures,

$$\begin{aligned} \mathcal{M}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = & a_1 g^{\mu\nu} g^{\rho\sigma} + a_2 g^{\mu\rho} g^{\nu\sigma} + a_3 g^{\mu\sigma} g^{\nu\rho} \\ & + \sum_{i,j=1}^3 (a_{1,ij} g^{\mu\nu} p_i^\rho p_j^\sigma + a_{2,ij} g^{\mu\rho} p_i^\nu p_j^\sigma + a_{3,ij} g^{\mu\sigma} p_i^\nu p_j^\rho \\ & + a_{4,ij} g^{\nu\rho} p_i^\mu p_j^\sigma + a_{5,ij} g^{\nu\sigma} p_i^\mu p_j^\rho + a_{6,ij} g^{\rho\sigma} p_i^\mu p_j^\nu) \\ & + \sum_{i,j,k,l=1}^3 a_{ijkl} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma. \end{aligned} \quad (2.3)$$

We can reduce the number of tensors with the following constraints. We implement transversality of the gluon polarization vectors,

$$\varepsilon(p_1) \cdot p_1 = 0, \quad \varepsilon(p_2) \cdot p_2 = 0, \quad (2.4)$$

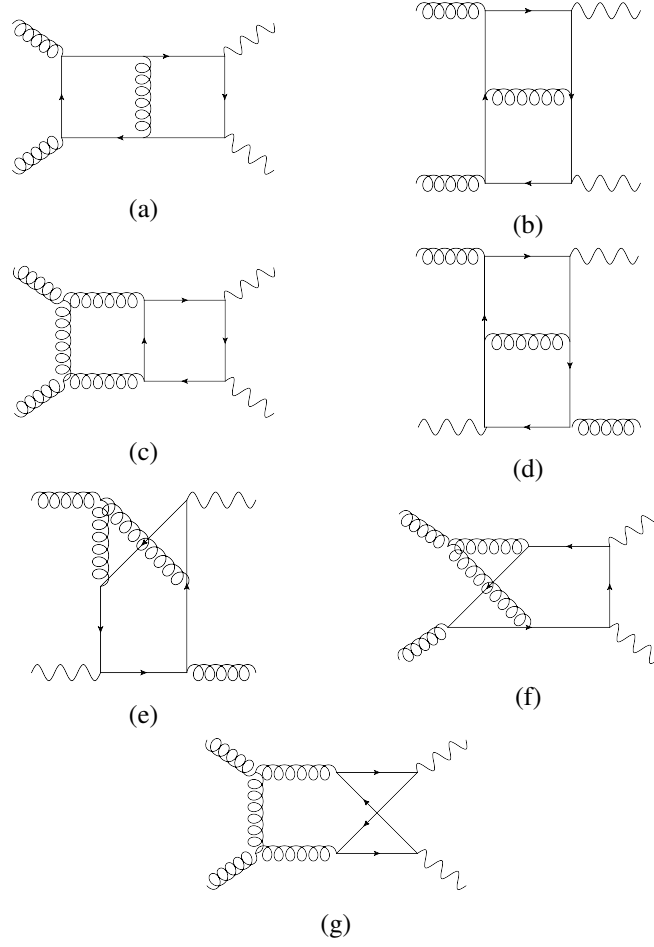


Figure 1: Representative Feynman diagrams with irreducible topologies.

and choose an explicit gauge for the external particles,

$$\varepsilon(p_1) \cdot p_2 = 0, \quad \varepsilon(p_2) \cdot p_1 = 0, \quad \varepsilon(p_3) \cdot p_3 = 0, \quad \varepsilon(p_4) \cdot p_4 = 0. \quad (2.5)$$

The amplitude can then be written as

$$\mathcal{M}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = \sum_{i=1}^{20} A_i(s, t, m_t^2, p_3^2, p_4^2) T_i^{\mu\nu\rho\sigma} \quad (2.6)$$

with the tensor structures

$$\begin{aligned} T_1^{\mu\nu\rho\sigma} &= g^{\mu\nu} g^{\rho\sigma} & T_2^{\mu\nu\rho\sigma} &= g^{\mu\rho} g^{\nu\sigma} & T_3^{\mu\nu\rho\sigma} &= g^{\mu\sigma} g^{\nu\rho} & T_4^{\mu\nu\rho\sigma} &= p_1^\mu p_1^\nu g^{\rho\sigma} \\ T_5^{\mu\nu\rho\sigma} &= p_1^\mu p_2^\nu g^{\rho\sigma} & T_6^{\mu\nu\rho\sigma} &= p_2^\mu p_1^\nu g^{\rho\sigma} & T_7^{\mu\nu\rho\sigma} &= p_2^\mu p_2^\nu g^{\rho\sigma} & T_8^{\mu\nu\rho\sigma} &= p_1^\nu p_3^\rho g^{\mu\sigma} \\ T_9^{\mu\nu\rho\sigma} &= p_2^\nu p_3^\rho g^{\mu\sigma} & T_{10}^{\mu\nu\rho\sigma} &= p_1^\mu p_3^\rho g^{\nu\sigma} & T_{11}^{\mu\nu\rho\sigma} &= p_2^\mu p_3^\rho g^{\nu\sigma} & T_{12}^{\mu\nu\rho\sigma} &= p_1^\nu p_3^\sigma g^{\mu\rho} \\ T_{13}^{\mu\nu\rho\sigma} &= p_2^\nu p_3^\sigma g^{\mu\rho} & T_{14}^{\mu\nu\rho\sigma} &= p_1^\mu p_3^\sigma g^{\nu\rho} & T_{15}^{\mu\nu\rho\sigma} &= p_2^\mu p_3^\sigma g^{\nu\rho} & T_{16}^{\mu\nu\rho\sigma} &= p_3^\rho p_3^\sigma g^{\mu\nu} \\ T_{17}^{\mu\nu\rho\sigma} &= p_1^\mu p_1^\nu p_3^\rho p_3^\sigma & T_{18}^{\mu\nu\rho\sigma} &= p_1^\mu p_2^\nu p_3^\rho p_3^\sigma & T_{19}^{\mu\nu\rho\sigma} &= p_2^\mu p_1^\nu p_3^\rho p_3^\sigma & T_{20}^{\mu\nu\rho\sigma} &= p_2^\mu p_2^\nu p_3^\rho p_3^\sigma. \end{aligned} \quad (2.7)$$

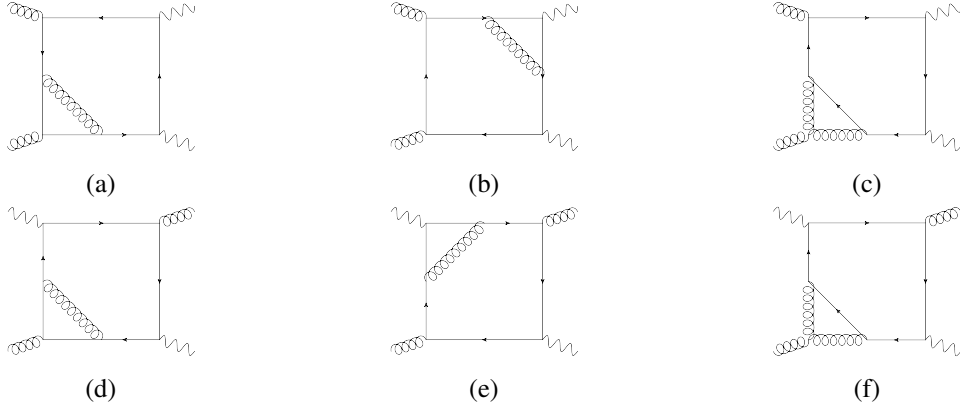


Figure 2: Representative Feynman diagrams with reducible topologies.

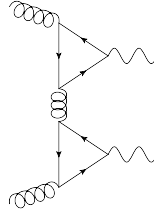


Figure 3: Representative diagram with triangles generating axial anomaly contributions.

The form factors $A_i(s, t, m_t^2, p_3^2, p_4^2)$ can be related to the amplitude using projection operators $P_i^{\mu\nu\rho\sigma}$ which can themselves be decomposed in terms of the $T_i^{\mu\nu\rho\sigma}$'s as

$$P_i^{\mu\nu\rho\sigma} = \sum_{j=1}^{20} B_{ij}(s, t, m_t^2, p_3^2, p_4^2) (T_j^{\mu\nu\rho\sigma})^\dagger \quad (2.8)$$

The exact forms of B_{ij} 's are provided at the [VVamp project website](#). While the above formulae hold in general, in this talk we only consider the top-quark contributions to on-shell Z pair production, i.e. $p_3^2 = p_4^2 = m_Z^2$. In addition, we use $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and, of course, $p_1^2 = p_2^2 = 0$. For further simplification, we set the ratio of the Z -boson mass squared and the top-quark mass squared to the numerical value $m_Z^2/m_t^2 = 5/18$.

We employ `Qgraf` [13] to generate the Feynman diagrams relevant for the top-quark contributions. In total, we find 174 diagrams out of which 49 vanish due to their colour structure. The remaining diagrams with a single fermion loop can be described using four complete sets of propagators (integral families); there are 13 top-level topologies (trivalent graphs), out of which 7 are irreducible (figure 1) and 6 are reducible (figure 2).

In addition to the diagrams involving a single fermion loop, we also include certain anomalous diagrams involving two closed fermion loops as depicted in 3. The sum of the vector-vector contribution of these diagrams can be shown to vanish for both massless and massive quarks from Furry's theorem, while the axial-axial piece vanishes only after summing over a degenerate weak $SU(2)$ doublet. For the third generation, the cancellation of the axial-axial part is incomplete and a finite remainder is generated due to the mass splitting for the top-bottom doublet. Therefore, in

addition to the top quark contribution, we also need to include the bottom quark contribution to these diagrams for consistency.

After generating the unreduced amplitude using FORM [14], we find 29749 distinct integrals before applying any symmetries, with integrals having up to 5 irreducible scalar products in the numerator. Using Reduze 2 [15, 16, 17] we perform a numerical reduction by substituting numbers for kinematic variables and find, after applying all symmetries and sector relations, 286 master integrals, including crossed topologies.

3. Integration by parts reduction and syzygies

After performing numerator algebra, we can express the amplitude in terms of scalar integrals. A general two-loop scalar Feynman integral with N edges can be represented by

$$I(a_1, \dots, a_N) = \int d^d k_1 d^d k_2 \prod_{i=1}^N \frac{1}{(q_i^2 - m_i^2)^{a_i}} \quad (3.1)$$

where k_1, k_2 are the loop momenta, q_i are the momenta of the edges, m_i are the masses of the edges, a_i are (integer) exponents of the propagators, and $d = 4 - 2\epsilon$. In dimensional regularization, an integral of a total derivative vanishes, which leads to linear relations between integrals with different propagator exponents [18]

$$0 = \int d^d k_1 d^d k_2 \frac{\partial}{\partial k_j^\mu} v^\mu \prod_{i=1}^N \frac{1}{(q_i^2 - m_i^2)^{a_i}}, \quad (3.2)$$

where v_μ is some linear combination of external and loop momenta. Using these linear relations, we can eliminate most of the integrals appearing in the amplitude with everything expressed in terms of a basis set of integrals, usually referred to as master integrals. An algorithm developed by S. Laporta [19] can be used to systematically reduce the linear system, and there are many public codes available for this purpose [15, 20, 21, 22, 23].

However, conventional simple choices for the vectors v_μ lead to the introduction of auxiliary integrals with many additional ‘‘dots’’, i.e. higher powers a_i of the propagators. As a consequence, the resulting linear systems are relatively large and computationally expensive to reduce. A method was proposed in [24] to avoid these higher powers of propagators by constructing suitable generating vectors v_μ from syzygies. These syzygies may be computed either using a Gröbner basis, or up to a specific degree, using linear algebra [25]. Subsequent work [26, 27] refined these constructions in the momentum space representation but also in Baikov’s representation [28].

Indeed, by a change of variables, the two-loop integral (3.1) can be written as

$$I(a_1, \dots, a_N) = \mathcal{N} \int dz_1 \dots dz_N \frac{1}{\prod_{i=1}^N z_i^{a_i}} P^{\frac{d-L-E-1}{2}}, \quad (3.3)$$

where P is the Baikov polynomial and \mathcal{N} is a normalization factor, $L = 2$ is the number of loops and $E = 3$ is the number of linearly independent external momenta. In Baikov’s representation, integration-by-parts identities are given by

$$0 = \int dz_1 \dots dz_N \sum_{i=1}^N \left(\frac{\partial f_i}{\partial z_i} + \frac{d-L-E-1}{2P} f_i \frac{\partial P}{\partial z_i} - a_i \frac{f_i}{z_i} \right) \frac{1}{\prod_{i=1}^N z_i^{a_i}} P^{\frac{d-L-E-1}{2}}. \quad (3.4)$$

Terms in the above equation that appear with $1/P$ can lead to dimensionally shifted integrals. In order to avoid these integrals, one can impose the constraint

$$\left(\sum_{i=1}^N f_i \frac{\partial P}{\partial z_i} \right) + gP = 0. \quad (3.5)$$

This is known as a syzygy constraint in algebraic geometry. Explicit solutions to this equation were pointed out in [29]. The resulting f_i are simple polynomials of degree 1 in the z_i and the kinematic invariants. To enforce the absence of doubled propagators, one requires that for all i with $a_i \geq 1$ the f_i is proportional to z_i to cancel the $1/z_i$ in the relation,

$$f_i = b_i z_i \quad \forall i \text{ with } a_i \geq 1. \quad (3.6)$$

While it is straight-forward to fulfil both constraints (3.5) and (3.6) separately, a simultaneous solution requires a non-trivial calculation. Formally, the task at hand is the determination of the intersection of two syzygy modules [30]. In practice, computer algebra packages implement algorithms to solve this task. For performance reasons we decided to develop a custom syzygy solver based on linear algebra and finite field arithmetic [31, 32], which converts the intersection problem up to a specific degree of the syzygies to the row reduction of a matrix. In this way, we were able to obtain the required syzygies for our calculation.

In a subsequent step, we employed the syzygies to generate linear relations between Feynman integrals with an in-house linear solver based on finite field arithmetic and rational reconstruction. In this way, we successfully reduced all of the Feynman integrals in our calculation to master integrals.

4. Finite master integrals using numerators

After obtaining the reductions to master integrals, the next step is to evaluate them. A common approach is to evaluate multi-scale master integrals analytically using differential equations [33, 34, 35, 36, 37, 38, 39]. In this way, the master integrals for the massless two-loop corrections to diboson production were evaluated in [40, 41, 42, 43]. It is rather challenging to analytically evaluate the master integrals considered here, however, owing to the presence of the internal mass, as a consequence of which functions beyond multiple polylogarithms are expected. Instead, we evaluate the master integrals numerically in Feynman parametric representation using Sector Decomposition [44, 45]. The conventional basis of master integrals is, however, not adequate for this purpose since these integrals are often divergent and suffer from stability and performance issues in the quadrature. Instead, we choose a different basis composed of integrals which are finite for $d \rightarrow 4$ [46, 47, 48]. Finite integrals are known to improve numerical integration [49, 50]. Often, they can be chosen such that fewer terms in the ϵ expansion of the more complicated topologies are required.

It is straight-forward to find a basis of finite integrals for any given topology if one allows for dimension shifts and higher powers of existing propagators (dots) e.g. by using the finite integral finder implemented in `Reduze 2`. Examples for finite integrals of this type are shown in figures 4c and 4d. In order to change to this basis of integrals one needs to calculate additional reduction identities involving several dots in addition to the usual relations required for the amplitude

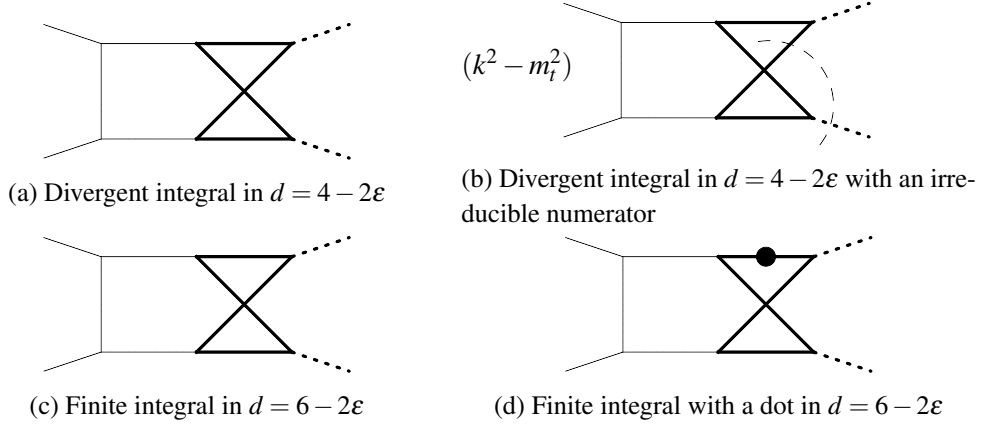


Figure 4: Examples of divergent and finite integrals in the limit $\epsilon \rightarrow 0$ for a non-planar topology. Thick and dashed lines represent massive propagators and legs. Topology (b) contains an irreducible numerator, where k is the difference of the momenta of the edges marked by the thin dash lines.

Integral	Relative error	Timing(s)
Divergent integral in figure 4a	~ 1	123
Divergent integral in figure 4b	~ 0.5	272
Finite integral with dimension shift, in figure 4c	$\sim 8 \cdot 10^{-4}$	81
Finite integral with dimension shift and dot, in figure 4d	$\sim 2 \cdot 10^{-3}$	135
Finite linear combination in eq. 4.1	$\sim 2 \cdot 10^{-3}$	125

Table 1: Numerical performance of different non-planar integrals for a physical phase-space point. Timings are for the evaluation of the leading term in ϵ with `pySecDec` [51] using the `VEGAS` algorithm [52].

reduction. Here, we explore a novel idea of combining divergent integrals to produce finite linear combinations for $\epsilon \rightarrow 0$. The key idea is, that the non-integrable divergences of the different terms cancel at the level of the combined integrand, such that the linear combination can be considered as a single generalized integral for the purpose of numerical integration. Using our new algorithm we systematically construct such linear combinations, allowing also for integrals with numerators (inverse propagators) and integrals from subtopologies as constituent integrals in $d = 4 - 2\epsilon$ dimensions. Choosing conventional integrals for the constituent integrals has the advantage that the basis change can be performed using identities very similar to those needed for the regular reduction of the amplitude. The numerical performance for such integrals is observed to be on par with those of the finite integrals obtained using dimension shifts and dots, as shown in table 1.

As an example, using the constituent integrals shown in figure 5, we can construct the following finite linear combination:

$$I_{fin} = s(m_z^2 - s - t) I_1 + s I_2 + s I_3 - s I_4 - s I_5 - (m_z^2 - s - t) I_6 - (m_z^2 - t) I_7. \quad (4.1)$$

We find that integrals constructed using such linear combinations of numerator and subtopology integrals are able to cure the (IR) divergences of our more complicated topologies. For simpler

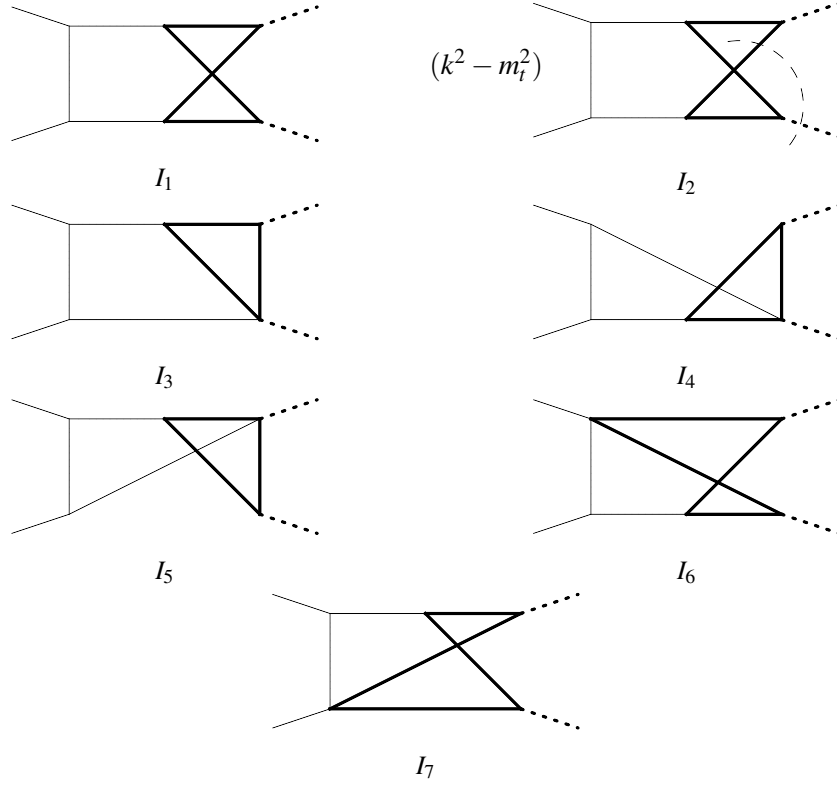


Figure 5: Integrals appearing in eq. (4.1). I_1 is the corner integral of the topology under consideration. I_2 is a second integral in the topology, but with a numerator $(k^2 - m_t^2)$ where k is equal to the difference of the momenta of the edges marked by the thin dashed lines. Integrals I_3, I_4, I_5, I_6, I_7 belong to subtopologies. All integrals are defined in $d = 4 - 2\epsilon$ dimensions.

topologies, we also employ finite integrals with dimension shifts and dots, in particular to cure UV poles. In this way, we express our amplitude in terms of finite master integrals suitable for numerical evaluation.

5. Conclusions

Diboson production is an important process for precision measurements at the LHC. In this talk, we discussed top-quark contributions to onshell ZZ production at two loops with exact dependence on the top-quark mass. For the reduction of the loop integrals we employed syzygies, which we constructed using linear algebra and modular arithmetic. We developed a new algorithm to construct a basis of finite integrals through linear combinations of divergent integrands. The resulting representation of the two-loop amplitude allows for a stable numerical evaluation using `pySecDec` and paves the way for the calculation of precise cross sections and distributions.

Acknowledgments

We would like to thank Gudrun Heinrich, Stephan Jahn and Stephen Jones for their help with `pySecDec` and collaborations on related work, as well as the Max Planck Institute Munich for its

hospitality and support. This work was supported by the National Science Foundation under Grant No. 1719863.

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