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The four-loop slope of the Dirac form factor

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We have evaluated with 1100 digits of precision the 4-loop contribution to the slope of the Dirac form factor in QED. The value is

$$m^2 F_1^{(4)'}(0) = 0.886545673946443145836821730610315359390424032660064745\ldots \left(\frac{\alpha}{\pi}\right)^4 .$$

We have also obtained a semi-analytical fit to the numerical value. The expression contains harmonic polylogarithms of argument $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$, one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits. We show the correction to the energy levels of the hydrogen atom due to the slope.

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0.886545673946443145836821730610315359390424032660064745368055909320840316465628927454836
48632417733686935127587472183079968759239748884668261476117530119175848314467747526729803
26917402719214651539325519844793100495019624531372119372946716080063429980958425369584945
06068383665985141387321894210012394882759515382378653722038834964485600756898576168775641
02719779603910290276615122356406105399227905150277608224592369504332757036133509352517647
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74780212674247904052222473302950218310742901990299162768291602289058991164264634498789876
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03345902599182245563613125124119880615415537621337112284846277684867421928289686568115480
30353727600787303621093059264752959892234017835732828971749623991833527848841324243696992
64221364032006844000612423529815833966332566753158241741448217616597381276692161976675095
05074064930956136195898802456451163545675716230944173884811565020098334847940590188785421
7006673782208530535419531883786100755181163385192201189714219158725102827198986...
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Table 1: First 1100 digits of A_4.

1. The slope of the Dirac form factor

In QED the vertex can be written

$$\bar{u}(p_1)\left(\gamma_\mu F_1(t)+\frac{\sigma_{\mu\nu}}{2m}q_\nu F_2(t)\right)u(p_2)\ , \tag{1.1}$$

where m is the electron mass, $F_1(t)$ and $F_2(t)$ are the Dirac and Pauli form factors. At $t=0$, the charge conservation implies that

$$F_1(0)=1\ , \tag{1.2}$$

whereas the value of the Pauli form factor is the g-2

$$F_2(0)=\frac{g-2}{2}\ . \tag{1.3}$$

The quantity $\frac{d}{dt}F_1(t)\Big|_{t=0}=F_1'(0)$ is the *slope* of the Dirac form factor.

1.1 Theoretical expression

We expand perturbatively the slope in powers of $\left(\frac{\alpha}{\pi}\right)$

$$m^2F_1'(0)=A_1\left(\frac{\alpha}{\pi}\right)+A_2\left(\frac{\alpha}{\pi}\right)^2+A_3\left(\frac{\alpha}{\pi}\right)^3+A_4\left(\frac{\alpha}{\pi}\right)^4+\dots\ . \tag{1.4}$$

The coefficients known in analytical form are [1–3]:

$$\begin{aligned}
A_1&=-\frac{1}{8}-\frac{1}{6\varepsilon}\ .\\
A_2&=-\frac{4819}{5184}-\frac{49}{432}\pi^2+\frac{1}{2}\pi^2\ln 2-\frac{3}{4}\zeta(3)=0.469\,941\,487\,459\,992\ldots\ ,\\
A_3&=-\frac{17}{24}\pi^2\zeta(3)+\frac{25}{8}\zeta(5)-\frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{\ln^4 2}{24}\right)-\frac{103}{1080}\pi^2\ln^2 2+\frac{3899}{25920}\pi^4\\
&\quad-\frac{2929}{288}\zeta(3)+\frac{41671}{2160}\pi^2\ln 2-\frac{454979}{38880}\pi^2-\frac{77513}{186624}=0.171\,720\,018\,909\,775\ldots\ .
\end{aligned} \tag{1.5}$$

loop	$F_1'(0)$	$F_2(0)$
1	∞	0.5
2	0.469941487459	−0.328478965579
3	0.171720018909	1.181241456587
4	0.886545673946	−1.912245764926
5		6.737(159)
	positive	alternating signs

Table 2: Values of the known contributions to $F_1'(0)$ and $F_2(0)$

Recently, in Ref. [4] we have presented the result of the calculation of A_4 with a precision of 1100 digits. The first digits of the result are

$$A_4 = 0.886545673946443145836821730610315 3\ldots\,. \tag{1.6}$$

The full-precision result is shown in table 1. In table 2 we have listed the known values of the slope and g-2; we see that A_2, A_3 and A_4 are all positive, in contrast with the alternating signs so far observed in the g-2. The distribution of the digits of A_4 is shown in Fig.1.

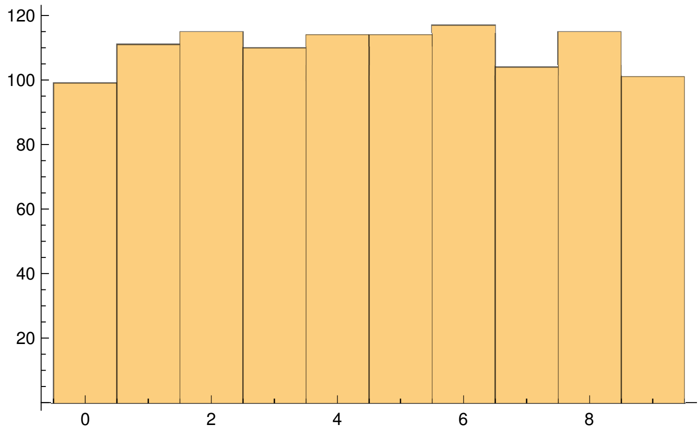

Figure 1: Distribution of the first 1100 digits of A_4.

We note that the sum of A_4 and the 4-loop contribution C_4 to g-2 [5] (see table 2) has a curious decimal pattern with 8 zeros in the first 14 digits:

$$A_4 + C_4 = -1.025700090980002428315825436829\ldots\,. \tag{1.7}$$

1.2 Shift to the hydrogen levels

Let us now consider the shift to the hydrogen energy levels due to A_4. We express the energy shift in terms of the frequency shift $\Delta f = \Delta E/h$. For the level nS the frequency shift is [6, 7]

$$\Delta f_{\mathrm{slope}}(n\mathrm{S},\, 4\text{-loop}) = \frac{4(Z\alpha)^4 mc^2}{h\, n^3} \left(\frac{m_r}{m}\right)^3 \left[\left(\frac{\alpha}{\pi}\right)^4 A_4\right] , \tag{1.8}$$

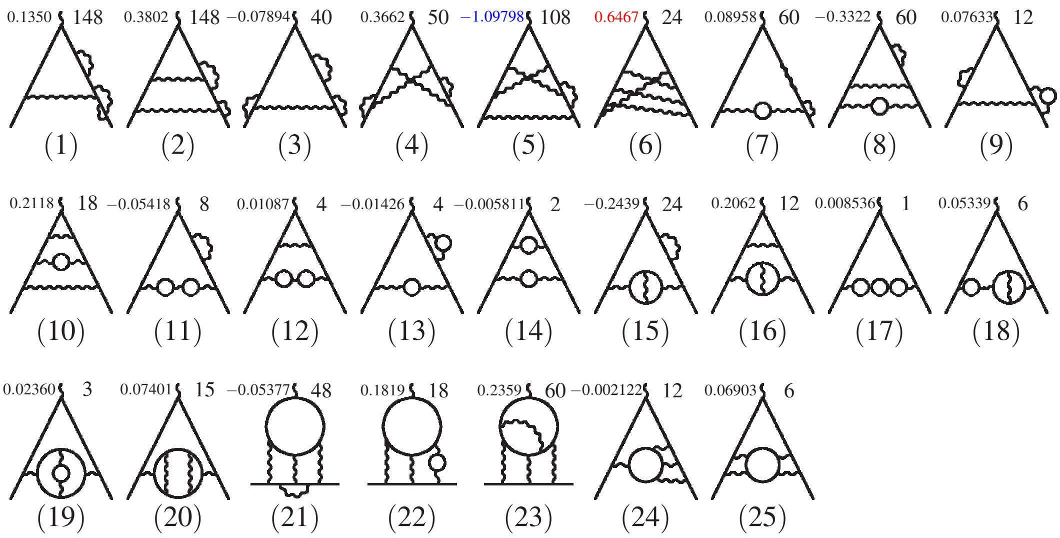

Figure 2: Typical representative diagrams of gauge-invariant sets. For each set only one diagrams is shown. Top left: contribution of the set to the slope; top right: number of diagrams of the set

where m_r is the reduced mass $m_r = mM/(m+M)$ and M is the proton mass. Inserting the values of m, M, c, h and $Z = 1$, the correction due to A_4 is

$$\Delta f_{\text{slope}}(n\text{S, 4-loop}) = \frac{36.11}{n^3}\ \text{Hz}\ , \tag{1.9}$$

and is comparable with the experimental error of the extremely precise measurement of $1\text{S}-2\text{S}$ transition [8]

$$f(1\text{S}-2\text{S}) = 2466\,061\,413\,187\,018 \pm 11\ \text{Hz}\ . \tag{1.10}$$

Eq.(1.9) is the first calculated four-loop correction to the energy levels, of the kind $\left(\frac{\alpha}{\pi}\right)^4 (Z\alpha)^4$.

2. Gauge-invariant sets

There are 891 vertex diagrams contributing to A_4. These vertex diagrams can be arranged in 25 gauge-invariant sets (Fig.2). The sets are classified according to the number of photon corrections on the same side of the main electron line and the insertions of electron loops (see Ref. [9]). The numerical contributions of each set, truncated to 40 digits, are listed in the table 3. The separate contributions to the slope from diagrams without or with internal loops are listed in table 4.

3. The analytical fit

By building systems of integration-by-parts identities [10, 11] and solving them [12], the contributions to A_4 of all the diagrams are expressed as linear combinations of 334 master integrals, the same ones as appeared in the calculation of 4-loop g-2 [5]. In Ref. [5] these master integrals were calculated numerically with precision ranging from 1100 to 9600 digits; analytical expressions were fit to all these master integrals (single or in particular combinations) by using the PSLQ algorithm [13, 14]. We have used those results [4].

1	0.1350531726346435372674724541103838371038
2	0.3802929165240844585552528298843579658371
3	- 0.0789488893676831608109628366941799823079
4	0.3662786736588470044584250527325325702299
5	- 1.0979832148317652705103820073196531832520
6	0.6467871429585372084492391789800382619165
7	0.0895891170440342216099366534902414320652
8	- 0.3322086225106643608126657791889571079890
9	0.0763376479373933425961220467893817339605
10	0.2118669010888818123786340161652003594809
11	- 0.0541837571893361764657206136746826299854
12	0.0108761535582321058694530867351119912448
13	- 0.0142646608196830116628021692409901716905
14	- 0.0058117416010420357833143542203438251011
15	- 0.2439068506475319592123409557076293747890
16	0.2062012570841125786262218639260170000956
17	0.0085366428673036656037790352019835488011
18	0.0533927095302949341276880145918233326838
19	0.0236058911191014021135877461122766184082
20	0.0740163162205724051338179043210727390276
21	- 0.0537711607064956999082765338567906834199
22	0.1819474273966664016975772159395176159307
23	0.2359289294543601921365690660148707901595
24	- 0.0021225895319909487365222280699442649666
25	0.0690362620755704991160330435886767859471

Table 3: Contribution to A_4 of the 25 gauge-invariant sets of Fig.2.

loop	$F_1'(0)$ no elec. loop	$F_1'(0)$ only elec. loop
2	0.438352514986	0.031588972473
3	-0.002437444568	0.174157463478
4	0.351479801576	0.535065872369

Table 4: Separate contributions to the slope from diagrams without and with electron loops.

Therefore, the analytical expression of A_4 contains the same transcendentals appeared in the g-2 result: values of harmonic polylogarithms [15] with argument 1, $\frac{1}{2}$, $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$ [16, 17] , a family of one-dimensional integrals of products of elliptic integrals, and the finite terms of the $\varepsilon-$expansions of six master integrals belonging to topologies 24 and 25 of Fig.2. The expression of the analytical fit is written as follows:

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U \ , \tag{3.1}$$

,

$$
\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}t_4 + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}t_5 - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&+ \frac{402152509}{189000}t_4\zeta(2) - \frac{18215}{27}t_{61} + \frac{26062}{27}t_{62} - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{971827}{128}\zeta(6)\ln 2 - \frac{6242389}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{427145}{504}t_4\zeta(3) + \frac{1420289}{180}t_5\zeta(2) \\
&+ \frac{256321}{756}t_{71} - \frac{116987}{63}t_{72} + \frac{104041}{20}t_{73} \,,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
W =& -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 472v_{73} \,, \\
V_a =& -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}v_{61} + \frac{844343}{28350}v_{62} + \frac{178619489}{3980340}v_{63} - \frac{263673944}{295245}v_{64} \,, \\
V_b =& \frac{212671}{2400}v_{65} - \frac{1031987}{14400}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) - \frac{507}{4}v_{71} - \frac{295}{4}v_{72} \,,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
E_a =& \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) - \frac{365478661}{24494400}e_{61} + \frac{119022487}{5443200}e_{62} \\
&- \frac{98285}{248832}e_{71} - \frac{157753}{497664}e_{72} \,, \\
E_b =& -\frac{751}{729}\zeta(2)f_1(0,0,1) + \frac{157753}{41472}e_{73} - \frac{99731}{1944}e_{74} \,,
\end{aligned}
\tag{3.4}
$$

$$
U = \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c} \,.
\tag{3.5}
$$

We made use of the constants

$$
t_4 = a_4 + \frac{1}{24}\ln^4 2 \,, \qquad t_5 = a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2 \,, \qquad t_{62} = a_6 - \frac{1}{48}\zeta(2)\ln^4 2 + \frac{1}{720}\ln^6 2 \,, \tag{3.6}
$$

$$
t_{61} = b_6 - a_5\ln 2 + \zeta(5)\ln 2 + \frac{1}{6}\zeta(3)\ln^3 2 - \frac{1}{12}\zeta(2)\ln^4 2 + \frac{1}{144}\ln^6 2 \,, \tag{3.7}
$$

$$
\begin{aligned}
t_{71} =& d_7 - 2b_6\ln 2 + 4a_6\ln 2 + 2a_5\ln^2 2 - \frac{49}{32}\zeta^2(3)\ln 2 - \frac{95}{32}\zeta(5)\ln^2 2 + \frac{1}{8}\zeta(4)\ln^3 2 - \frac{1}{3}\zeta(3)\ln^4 2 \\
&+ \frac{1}{12}\zeta(2)\ln^5 2 - \frac{1}{120}\ln^7 2 \,,
\end{aligned}
\tag{3.8}
$$

$$
t_{72} = b_7 - 3a_7 - a_6\ln 2 - \frac{1}{2}\zeta(5)\ln^2 2 + \frac{1}{48}\zeta(4)\ln^3 2 - \frac{1}{24}\zeta(3)\ln^4 2 + \frac{1}{120}\zeta(2)\ln^5 2 - \frac{1}{1680}\ln^7 2 \,, \tag{3.9}
$$

$$
t_{73} = \left(a_4 - \frac{1}{4}\zeta(2)\ln^2 2 + \frac{7}{16}\zeta(3)\ln 2 + \frac{1}{24}\ln^4 2\right)\zeta(2)\ln 2 \,, \tag{3.10}
$$

$$v_{61} = \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{27}{26}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+\frac{207}{104}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10}{3}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7}{4}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{8}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$-\frac{5}{72}\zeta(3)\zeta(2)\pi - \frac{5}{6}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{5}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 - \frac{27413}{67392}\zeta(5)\pi + \frac{4975}{11583}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \quad (3.11)$$

$$v_{62} = \zeta(2)\left(\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{1}{6}\zeta(3)\pi + \frac{1}{108}\zeta(2)\pi\ln 2 + \frac{25}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2\right.$$
$$\left.-\frac{5}{2}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\ln 2 - \frac{15}{4}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\ln 2 - \frac{661}{1188}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\right), \quad (3.12)$$

$$v_{63} = \mathrm{Cl}_6\left(\frac{\pi}{3}\right) - \frac{3}{4}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \qquad v_{64} = \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{91}{66}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \quad (3.13)$$

$$v_{65} = \mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right), \quad (3.14)$$

$$v_{71} = \mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 4\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{27}{8}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{135}{16}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$-\frac{27}{2}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{145}{132}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi, \quad (3.15)$$

$$v_{72} = \zeta(2)\left(\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 2\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{9}{4}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{9}{2}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\right.$$
$$\left.+\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right), \quad (3.16)$$

$$v_{73} = \zeta(2)\left(\mathrm{Re}H_{0,1,0,1,1}(i) + \mathrm{Cl}_2\left(\frac{\pi}{2}\right)\mathrm{Im}H_{0,1,1}(i) - \frac{1}{2}\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\pi + \frac{1}{4}\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\right), \quad (3.17)$$

$$e_{61} = \pi\left(f_2(0,2,0) - \frac{9}{4}\ln 2 f_2(0,0,1)\right), \qquad e_{62} = \pi\left(f_2(0,1,1) - \frac{3}{8}f_2(0,0,2) - \frac{3}{2}\ln 2 f_2(0,0,1)\right), \quad (3.18)$$

$$e_{71} = \pi\left(f_2(2,1,0) + \frac{7}{3}f_2(1,2,0) - 2f_2(1,1,1) + \frac{40}{27}f_2(0,3,0) - \frac{7}{3}f_2(0,2,1) + f_2(0,1,2)\right.$$
$$\left.-30\ln 2 f_2(0,2,0) + 45\ln 2 f_2(0,1,1) - \frac{135}{8}\ln 2 f_2(0,0,2)\right), \quad (3.19)$$

$$e_{72} = \pi\left(f_2(2,0,1) + \frac{14}{3}f_2(1,2,0) - 2f_2(1,1,1) - 2f_2(1,0,2) - \frac{370}{27}f_2(0,3,0) + \frac{85}{3}f_2(0,2,1) - 22f_2(0,1,2)\right.$$
$$\left.+7f_2(0,0,3) + 11\zeta(2)f_2(0,0,1) - 20\ln 2 f_2(0,2,0) + 30\ln 2 f_2(0,1,1) - \frac{45}{4}\ln 2 f_2(0,0,2)\right), \quad (3.20)$$

$$e_{73} = \zeta(2)\left(f_1(1,0,1) - f_1(0,1,1) + \frac{1}{4}f_1(0,0,2)\right), \quad (3.21)$$

$$e_{74} = \zeta(2)\left(f_1(0,2,0) - \frac{3}{2}f_1(0,1,1) + \frac{9}{16}f_1(0,0,2)\right). \quad (3.22)$$

T	- 2191.2354696575117884131629228588288 5509
$\sqrt{3}V_a$	- 648.74441479274053140037234999290048941
V_b	- 400.66449515766079257160481868291283752
W	1539.3291991668164535098127610875690 5937
$\sqrt{3}E_a$	- 266.54091710106238111286732183079933994
E_b	1928.2225364884424154854165537906612 3429
U	40.520106727663067648614920217821 54366

Table 5: Numerical values of the addends appearing in Eq.3.1.

In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i}\, i^{-n}$, $b_6 = H_{0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $b_7 = H_{0,0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$. $H_{i_1,i_2,\ldots}(x)$ are the harmonic polylogarithms [15]. The integrals f_j are defined as follows:

$$f_m(i,j,k) = \int_1^9 ds\, D_1(s) \mathrm{Re}\left(\sqrt{3^{m-1}} D_m(s)\right) \left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s) \;, \qquad (3.23)$$

$$D_m(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(m - 1 - (2m-3)\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \;; \qquad (3.24)$$

$K(x)$ is the complete elliptic integral of the first kind. The constants B_3 and C_3 have the following hypergeometric representations [5, 18]:

$$B_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\,\frac{1}{3}\,\frac{1}{3}\,\frac{1}{2} \atop \frac{5}{6}\,\frac{5}{6}\,\frac{2}{3}};1\right) - {}_4\tilde{F}_3\left({\frac{5}{6}\,\frac{2}{3}\,\frac{2}{3}\,\frac{1}{2} \atop \frac{7}{6}\,\frac{7}{6}\,\frac{4}{3}};1\right)\right] \;, \qquad (3.25)$$

$$C_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\,\frac{1}{3}\,\frac{4}{3}\,-\frac{1}{2} \atop -\frac{1}{6}\,\frac{5}{6}\,\frac{5}{3}};1\right) - {}_4\tilde{F}_3\left({-\frac{7}{6}\,-\frac{1}{3}\,\frac{2}{3}\,-\frac{1}{2} \atop -\frac{5}{6}\,\frac{1}{6}\,\frac{1}{3}};1\right)\right] \;, \qquad (3.26)$$

$${}_4\tilde{F}_3\left({a_1\,a_2\,a_3\,a_4 \atop b_1\,b_2\,b_3};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\, {}_4F_3\left({a_1\,a_2\,a_3\,a_4 \atop b_1\,b_2\,b_3};x\right). \qquad (3.27)$$

The numerical values of the constants appearing in Eq.(3.1) are listed in Table 5. Note the strong numerical cancellations in Eq.(3.1): the largest term is $-\frac{2749470791}{387072}\zeta(2)\zeta(5) = -12115.862$.

4. Method of calculation

We sketch the method used to obtain A_4. It is the same used in Ref. [5].

1. Generation of 891 vertex diagrams (*C* program) from 104 self-mass diagrams. These are the same of the 4-loop *g*-2 calculation.

2. Extraction of the contribution to A_4 from the amplitude of each diagram by using projectors [19, 20] with a `FORM` program [21, 22].

3. Algebraic reduction to master integrals, obtained by building and solving *large* systems of integration-by-parts identities [10, 11] by using the program `SYS` [12].

4. For the sake of checks we generate a different system for each group of vertex diagrams obtained from the same self-mass diagram.

5. The smallest system contains 10^8 identities, with size of 90GB. The system with the largest number of identities contains 5×10^8, with a size of 170GB. The largest system has 3×10^8 identities with a size of 1.2TB.

6. The ratio between number of independent identities and total number of generated identities is in the range $0.2 - 0.3$. The dependent identities become trivial zeroes when substituted into the system, and have been used to check the reliability of hardware and software. No hardware errors were detected. Instead, software errors have been detected in this way (frequency: one every 2-3 weeks), caused by a bug in the OpenMPI message passing library used with the highest level of threads support.

7. We algebraically check that the contribution from a diagram is invariant to the changes in the particular internal routing of the momentum of the external photon.

8. The renormalization is carried out by subtracting suitable counterterms, which are generated with C and `FORM` programs and calculated numerically with `SYS`.

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