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The Polarized Three-Loop Anomalous Dimensions from a Massive Calculation

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We present results on the calculation of the polarized 2- and 3-loop anomalous dimensions in a massive computation of the associated operator matrix element. We also discuss the treatment of γ_5 and derive results in the M-scheme.

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1. Introduction

The next-to-next-to leading order anomalous dimensions $\gamma_{ij}^{(2)}$ and splitting functions $P_{ij}^{(2)}$ govern the scale evolution of the polarized parton distributions in Quantum Chromodynamics to order α_s^3. They are important ingredients for precision measurements at ep and hadron colliders, different fixed-target experiments and future experiments like the EIC [1] and at RHIC. Furthermore they are instrumental for the precise measurement of the strong coupling constant $\alpha_s(M_Z)$ [2] at these facilities and key processes like the polarized Drell-Yan cross section, jet production cross sections and the analysis of deep-inelastic scattering data [3]. Precision analyses of polarized processes are also important to resolve the spin-composition of polarized nucleons [4, 5].

The calculation of the 2-loop polarized splitting functions has been performed already in 1995 [6, 7]. The complete 3-loop results have been obtained in [8] within the so-called M-scheme. In the flavor non-singlet case the 3-loop splitting function $P_{qq}^{(2),\rm NS}$ are the same as in the unpolarized case [9] and have also been obtained in [10]. The unpolarized splitting functions have been obtained in [9, 11] and the terms $\propto T_F$ have been confirmed in independent massive calculations in [10, 12–15]. In the present paper we review the corresponding calculation in the polarized case. For the complete results see Ref. [16]. Already in 2010 the odd moments $N = 1$–7 of the polarized OMEs $A_{Qg}^{(3)}$ and $A_{qg,Q}^{(3)}$ and in 2013 also for $A_{gg,Q}^{(3)}$ have been calculated. Most recently we also added the moment $N = 9$. The set of moments remained unpublished since an important detail in the definition of the Larin scheme had to be understood first.

The paper is organized as follows. In Section 2 we review the general structure of the massive OMEs and their evaluation using the Larin scheme. The methods used for the calculation of the massive OMEs are described in Section 3. Section 4 contains the conclusions.

2. The polarized massive Operator Matrix Elements

There are seven massive operator matrix elements $A_{qq,Q}^{(3),\rm NS}$, $A_{qq,Q}^{(3),\rm PS}$, $A_{Qq}^{(3),\rm PS}$, $A_{qg,Q}^{(3)}$, $A_{Qg}^{(3)}$, $A_{gq,Q}^{(3)}$ and $A_{gg,Q}^{(3)}$. The non-singlet OME has already been calculated in [10]. Due to a Ward identity it has to agree in the polarized and unpolarized case and can be given in the $\overline{\rm MS}$ scheme. The principle structure of the OMEs in Mellin-N space has been given in Ref. [17], i.e. one obtains

$$
\begin{aligned}
\hat{\hat{A}}_{Qg}^{(3)} = {} & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3} \Biggl((N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Bigl[\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 6\beta_0 - 8\beta_{0,Q}\Bigr] + 8\beta_0^2 \\
& +28\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 + \gamma_{gg}^{(0)}\Bigl[\gamma_{gg}^{(0)} + 6\beta_0 + 14\beta_{0,Q}\Bigr]\Biggr) + \frac{1}{6\varepsilon^2}\Biggl(\hat{\gamma}_{qg}^{(1)}\Bigl[2\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 8\beta_0 \\
& -10\beta_{0,Q}\Bigr] + \hat{\gamma}_{qg}^{(0)}\Bigl[\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\{1-2N_F\} + \gamma_{qq}^{(1),\mathsf{NS}} + \hat{\gamma}_{qq}^{(1),\mathsf{NS}} + 2\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} - 2\beta_1 - 2\beta_{1,Q}\Bigr] \\
& +6\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}\Bigl[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 3\beta_0 + 5\beta_{0,Q}\Bigr]\Biggr) + \frac{1}{\varepsilon}\Biggl(\frac{\hat{\gamma}_{qg}^{(2)}}{3} - N_F\frac{\hat{\hat{\gamma}}_{qg}^{(2)}}{3} + \hat{\gamma}_{qg}^{(0)}\Bigl[a_{gg,Q}^{(2)} \\
& -N_F a_{Qq}^{(2),\mathsf{PS}}\Bigr] + a_{Qg}^{(2)}\Bigl[\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q}\Bigr] + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl[\gamma_{gg}^{(0)}\Bigl\{2\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 6\beta_0 \\
& +2\beta_{0,Q}\Bigr\} - (N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Bigl\{-\gamma_{qq}^{(0)} + 6\beta_0\Bigr\} - 8\beta_0^2 + 4\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2\Bigr]
\end{aligned}
$$

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$$
+\frac{\delta m_1^{(-1)}}{2}\left[-2\hat{\gamma}_{qg}^{(1)}+3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}+2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\right]+\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\left[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+2\beta_0+4\beta_{0,Q}\right]
$$
$$
\left.\left.-\delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)}\right)+a_{Qg}^{(3)}\right]. \tag{2.1}
$$

In this expression all dependencies on N have been dropped for brevity. μ denotes the factorization and renormalization scale, $\bar{m}$ the bare heavy quark mass, $\varepsilon = D-4$ the dimensional regulator, $\zeta_l, l \in \mathbb{N}, l \geq 2$ the values of the Riemann ζ function at integer argument. β_i are the expansion coefficients of the QCD β-function, $\beta_{i,Q}$ are related expansion coefficients associated to heavy quark effects, $\gamma_{ij}^{(k)}$ the expansion coefficients of the anomalous dimensions, and $\delta m_k^{(l)}$ the expansion coefficients of the unrenormalized quark mass. The above quantities depend on the color factors $C_A = N_C, C_F = (N_C^2-1)/(2N_C), T_F = 1/2$ for $SU(N_C)$ and $N_C = 3$ for QCD, cf. e.g. Ref. [17] and the number of massless quark flavors N_F. The coefficients $a_{ij}^{(k)}$ denote the constant terms of the OMEs at k–loop order and $\bar{a}_{ij}^{(k)}$ the corresponding terms at $O(\varepsilon)$, cf. [18–23]. Furthermore, we use

$$
\hat{f}(N_F) = f(N_F+1) - f(N_F) \tag{2.2}
$$
$$
\tilde{f}(N_F) = \frac{f(N_F)}{N_F}\,. \tag{2.3}
$$

From the poles $O(1/\varepsilon^3)$ one can obtain the one-loop anomalous dimensions, from the poles $O(1/\varepsilon^2)$ the full two-loop anomalous dimensions while from the poles $O(1/\varepsilon)$ the contributions $\propto T_F$ of the three-loop anomalous dimensions can be extracted.

We work in the Larin scheme to describe γ_5 in D dimensions [24]. In this scheme γ^5 is described by

$$
\gamma^5 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{2.4}
$$
$$
\not{\Delta}\gamma^5 = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\Delta^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{2.5}
$$

and two Levi-Civita symbols are contracted in D dimensions using

$$
\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\lambda\tau\gamma} = -\mathrm{Det}[g_\omega^\beta], \quad \beta = \alpha,\lambda,\tau,\gamma; \;\; \omega = \mu,\nu,\rho,\sigma. \tag{2.6}
$$

We use the projectors P_g and P_q for the amplitudes $\hat{G}_{\mu\nu}^{ab}$ and $\hat{G}_l^{ij}$ with external gluonic or quarkonic states respectively (it turns out that external ghost states always lead to vanishing traces). The gluonic projector is given by

$$
P_g\hat{G}_{\mu\nu}^{ab} = \frac{\delta^{ab}}{N_C^2-1}\frac{1}{(D-2)(D-3)}(\Delta p)^{-N-1}\varepsilon^{\mu\nu\rho\sigma}\Delta_\rho p_\sigma \hat{G}_{\mu\nu}^{ab}, \tag{2.7}
$$

while we use

$$
P_q\hat{G}_l^{ij} = -\delta_{ij}\frac{i(\Delta.p)^{-N-1}}{4N_C(D-2)(D-3)}\varepsilon_{\mu\nu p\Delta}\mathrm{tr}\left[\not{p}\gamma^\mu\gamma^\nu\hat{G}_l^{ij}\right] \tag{2.8}
$$

for external light quarks. This quarkonic projector differs from the one proposed in Ref. [18] but yields the proper definition of the massive OMEs with massless external quark lines in the Larin scheme, unlike the one in Ref. [18]. For more details see [16, 21]. The existence of a single projector (2.8) allows the use of the methods already developed for the unpolarized case. This also applies to the calculation of a series of fixed moments using MATAD [25].

The $O(\varepsilon^0)$ and $O(\varepsilon)$ contributions to the 2-loop OMEs are needed for the renormalization. They are given in [21] for $a_{Qg}^{(2)}$, [20, 21] for $a_{qq,Q}^{(2),\rm PS}$, [22] for $a_{gg,Q}^{(2)}$, [23] for $a_{gq,Q}^{(2)}$. Although the non-singlet OMEs can be given in the $\overline{\rm MS}$ scheme without a recomputation, it has to be supplied in the Larin-scheme for the consistent calculation in one scheme. The contributions are given by

$$a_{qq,Q}^{(2),\rm NS} = C_F T_F \left\{ \frac{R_1}{54N^3(N+1)^3} + \left(\frac{2(2+3N+3N^2)}{3N(N+1)} - \frac{8}{3}S_1\right)\zeta_2 - \frac{224}{27}S_1 + \frac{40}{9}S_2 - \frac{8}{3}S_3 \right\} \tag{2.9}$$

$$\overline{a}_{qq,Q}^{(2),\rm NS} = C_F T_F \left\{ \frac{R_2}{648N^4(1+N)^4} + \left(\frac{2(2+3N+3N^2)}{9N(N+1)} - \frac{8}{9}S_1\right)\zeta_3 + \left(\frac{R_3}{18N^2(N+1)^2} - \frac{20}{9}S_1 + \frac{4}{3}S_2\right)\zeta_2 - \frac{656}{81}S_1 + \frac{112}{27}S_2 - \frac{20}{9}S_3 + \frac{4}{3}S_4 \right\}, \tag{2.10}$$

$$R_1 = 72 + 240N + 344N^2 + 379N^3 + 713N^4 + 657N^5 + 219N^6, \tag{2.11}$$

$$R_2 = -432 - 1872N - 3504N^2 - 3280N^3 + 1407N^4 + 7500N^5 + 9962N^6 + 6204N^7 + 1551N^8, \tag{2.12}$$

$$R_3 = -12 - 28N - N^2 + 6N^3 + 3N^4. \tag{2.13}$$

3. Calculation Methods and Results

For the calculation of the pole parts of the OMEs $A_{qq,Q}^{(3),\rm PS}, A_{Qq}^{(3),\rm PS}, A_{qg,Q}^{(2)}, A_{gq,Q}^{(3)}$ and $A_{gg,Q}^{(3)}$ we were able to use standard techniques like the method of hypergeometric functions [26, 27], the method of hyperlogarithms [28–30], the solution of ordinary differential equation systems [31–33] and the Almkvist–Zeilberger algorithm [34, 35], see [36] for a survey of these methods, since no elliptic integrals contribute even in higher orders in the dimensional regulator ε. The master integrals necessary for the calculation have been already available from the calculation of the unpolarized three–loop anomalous dimensions in Ref. [15], only a few additional integrals had to be solved using the method of differential equations. In all of the above methods corresponding sum representations have been derived which were solved using the difference–field techniques [37–44] of the packages Sigma [45, 46], EvaluateMultiSums, SumProduction [47], and using HarmonicSums [35, 48–53]. The reduction to master integrals has been performed using Reduze2 [54, 55]; for more details see Ref. [16].

These methods however do not work for the OME $A_{Qg}^{(3)}$. Using standard techniques one encounters elliptic contributions [56, 57], which cannot be handled in the same automated way. We therefore apply the method of arbitrarily large moments [58] in this case. Using this method one can recursively generate higher and higher moments of the master integrals and thereby the complete OME. These moments are used to derive a difference equation by the method of guessing [59] implemented in Sage [60, 61].

Due to the structure of the IBP relations some higher expansion in ε is necessary also to extract the term $\propto 1/\varepsilon$. Here one would encounter elliptic terms by using the above techniques.

We therefore apply the method of arbitrarily large moments [58] in this case.[1] Here one works in moment–space and the IBP relations are expressed in terms of recurrences for the master integrals. Using these relations one generates systematically higher and higher moments both for the master integrals and the operator matrix elements. We generated 2000 Mellin moments, which allowed to find most of the recurrences for all seventeen color–ζ projections. To determine the recurrences of the projections $C_F C_A T_F$ and $C_A^2 T_F$ we used 4000 moments, out of which 2640 turned out to be sufficient. Here we refer to representations in terms of even and odd moments, with the even moments being unphysical. The analytic continuation is finally performed from the odd moments only. The characteristics of the recurrences for the different color–ζ factors contributing to the $1/\varepsilon$ term of the unrenormalized massive OME $A_{Qg}^{(3)}$ are summarized in Table 1. For all the pole terms these recurrences are first–order factorizable and can be solved by applying the package `Sigma`. Here some color–ζ structures contribute for technical reasons, which cancel in the final expression.

All anomalous dimensions can be expressed by nested harmonic sums [52, 53]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_\emptyset = 1 \quad ,b,a_i \in \mathbb{Z}\backslash\{0\}. \tag{3.1}$$

To provide comparisons on a diagram-by-diagram basis we have calculated the first few Mellin moments for $N = 1,3,5,7,9$ using `MATAD` [25].

We would like to compare to the results obtained in Ref. [8] which are given in the M–scheme. This scheme was defined in implicit form in Ref. [64]. Up to two–loop order it is the same as the one in which the results of Refs. [6, 7] were obtained. At leading order, the anomalous dimensions are scheme–invariant. The finite renormalizations between the Larin and the M–scheme to three–loop order can be obtained following [64], see also [8]. For the explicit transformations and the results on the complete polarized 2–loop and T_F-contributions to the polarized 3–loop anomalous dimensions, see [16]. At 3–loop order the complete anomalous dimensions $\gamma_{qq}^{(2),\mathrm{PS}}$ and $\gamma_{qg}^{(2)}$ are obtained. Ref. [16] also contains the results for the anomalous dimensions and splitting in computer-readable form. We fully agree with the results given in [8].

4. Conclusions

Since the QCD anomalous dimensions are universal quantities one can compute them within various setups. In the present case they have been obtained from the pole structure of massive polarized OMEs at three loop order. The calculation of these OMEs is part of an ongoing project with the final goal to compute the massive polarized Wilson coefficients for deep–inelastic scattering in the region $Q^2 \gg m^2$. Through this calculation we got the contributions $\propto T_F$ to the polarized 3–loop anomalous dimension $\gamma_{ij}^{(2)}(N)$ and the associated splitting functions in a massive calculation. This calculation is fully independent of the earlier computation in Ref. [8]. We completely agree with the previous results. To use conventional IBP reduction techniques we resum the local operator into a formal Taylor series using the auxiliary parameter x. As in the unpolarized case [15] before, we had to use the method of arbitrarily high moments [58] to deal with potential elliptic contributions in the necessary deeper expansions in the dimensional parameter ε in the case of the OME $A_{Qg}^{(3)}$.

[1] This method has been successfully applied also in a series of other calculations, cf. [15, 62, 63].

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color/ζ	order	degree
$C_F T_F^2$	7	68
$C_F T_F^2 \zeta_2$	3	17
$C_F T_F^2 N_F$	7	68
$C_F T_F^2 N_F \zeta_2$	3	17
$C_F^2 T_F$	22	283
$C_F^2 T_F \zeta_2$	6	32
$C_F^2 T_F \zeta_3$	2	10
$C_A T_F^2$	10	85
$C_A T_F^2 \zeta_2$	3	12
$C_A T_F^2 N_F$	14	131
$C_A T_F^2 N_F \zeta_2$	4	16
$C_F C_A T_F$	30	484
$C_F C_A T_F \zeta_2$	8	46
$C_F C_A T_F \zeta_3$	3	19
$C_A^2 T_F$	30	472
$C_A^2 T_F \zeta_2$	10	57
$C_A^2 T_F \zeta_3$	4	19

Table 1: Characteristics of the recurrences contributing to the anomalous dimension $\gamma_{qg}^{(2)}$.

Using the method of high Mellin moments [58], the moments are calculated recursively using the system of difference equation associated with the differential equations given by the IBP relations. Individual master integrals are only calculated in terms of moments. In all other contributions, standard techniques, cf. [36], are used in the calculation of the master integrals.

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