Local analytic sector subtraction for final state radiation at NNLO

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We present recent developments of the *local analytic sector* subtraction of infrared singularities for final state real radiation at NNLO in QCD.

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1. Introduction

The Large Hadron Collider (LHC) is entering its high-precision phase, and theoretical predictions need to achieve similar degree of accuracy, in order to have the Standard Model background under control and be able to disentangle possible signals of new physics. As the LHC is a hadron machine, basically all processes are essentially QCD-based, and precise theoretical predictions must take into account higher-order effects in QCD perturbation theory. To this end, many ingredients are necessary: an accurate determination of parton distribution functions, a proper description of final-state hadronic jets, as well as resummations to all orders of large fixed-order contributions. But of course the main ingredient to have accurate predictions is the computation of QCD corrections to the relevant partonic processes, at a sufficiently high order, which actually means at next-to-next-to-leading order (NNLO) for the most common processes. Furthermore, because of the variety and complexity of scattering processes, it would be desirable that these NNLO QCD computations could be automated, at the same level presently achieved at next-to-leading order (NLO). To reach this ambitious goal, one needs not only automated computations of two-loop corrections, but also a universal framework to deal with the cancellation of soft and collinear singularities, arising both in virtual corrections and in the phase space integration of unresolved real radiation of massless particles.

The most precise way to treat these cancellation is by means of a *subtraction procedure*, which basically consists in subtracting from the real squared matrix elements one or more simple local counterterms, mimicking its singular behaviour in the entire phase space, and adding them back, integrated in the extra radiations, in order to cancel the singularities of the virtual matrix element. There is a lot of freedom in defining these counterterms and in the way the integration of the radiated phase space is performed, giving raise to many possible subtraction procedures. At NLO, the most successful general algorithms are the *Frixione-Kunzst-Signer (FKS)* [1], the *Catani-Seymour (CS)* [2] and the *Nagy-Soper* [3] subtraction methods. At NNLO, the overlapping of singular regions increases the complexity of the problem, and several different methods, not always based on a subtraction procedure, have been developed, however, so far, without reaching the desired degree of generality and automation. The first subtraction procedure to be developed at NNLO was the *Antenna* subtraction [4], which is essentially a generalisation of the NLO CS subtraction. A different framework, based on the known singular limits of the squared matrix elements with double real radiation, is the *CoLoRFulNNLO* subtraction [5]. A complete numerical approach, extending the FKS subtraction at NNLO, is the *Sector-improved residue* subtraction [6], which basically generalises the subtraction procedures based on the sector decomposition technique [7, 8], and was the first method to be successfully applied to a hadronic scattering process (top pair production) at NNLO. A recent analytical development of this approach is the *Nested Soft-Collinear* subtraction [9]. Successful methods not based on a local subtraction procedure are the q_T slicing [10] and *N-Jettiness* slicing [11]. Finally, new methods, or refinements of existing ones, are also being introduced [12, 13, 14].

Recently, we presented a new approach [15], which we called *local analytic sector subtraction*: it attempts to take maximal advantage of the available freedom in the definition of the local infrared counterterms, essentially combining ideas that have been successfully implemented at NLO. The first crucial element is the partition of phase space in sectors, as done in the *FKS* subtraction [1], by means of sector functions obeying a set of sum rules which allow to simplify the analytic integration of counterterms when sectors are appropriately recombined. A second key ingredient is the remapping of momenta to Born kinematics, following CS factorisation of the radiative phase space, which is particularly suitable for a straightforward integration of counterterms at NLO. Finally, we use the known expressions for the 2-unresolved singular limits [16] and take maximal advantage of the simple structure of multiple limits, which follows from the factorisation properties of scattering amplitudes [17].

2. Local analytic sector subtraction for final state radiation at NLO

At NLO the differential cross sections $d\sigma_{\rm NLO}/dX$ with respect to any infrared-safe observable X can schematically be written as

$$\frac{d\sigma_{\rm NLO}-d\sigma_{\rm LO}}{dX} = \int d\Phi_n V\,\delta_n(X) + \int d\Phi_{n+1} R\,\delta_{n+1}(X)\,. \tag{2.1}$$

where R and V denote the real and virtual squared matrix elements respectively, the latter renormalised in the $\overline{\rm MS}$ scheme. We have also introduced $\delta_i(X) \equiv \delta(X - X_i)$, with X_i representing the observable X computed with i-body kinematics. In dimensional regularisation, with $d = 4 - 2\varepsilon$ space-time dimensions, the virtual contribution features poles in ε, while the real contribution is characterised by singularities in the radiation phase space, which are of soft and collinear nature. When computed in d dimensions, the phase space integration in $d\Phi_{n+1}$ results in explicit ε poles, which cancel those of virtual origin [18, 19].

Any local subtraction procedure at NLO consists in adding and subtracting a counterterm K to Eq. (2.1), and exploiting the factorisation of the $(n+1)$ phase space $d\Phi_{n+1} = d\Phi_n\, d\Phi_1$, getting

$$\frac{d\sigma_{\rm NLO}-d\sigma_{\rm LO}}{dX} = \int d\Phi_n \Big[V + I\Big]\,\delta_n(X) + \int d\Phi_{n+1} \Big[R\,\delta_{n+1}(X) - K\,\delta_n(X)\Big]\,, \qquad I = \int d\Phi_1\, K\,. \tag{2.2}$$

The counterterm K must reproduce all the singular limits of the real-radiation contribution R, so that the combination $R - K$ does not present any phase space singularities. Its integral I in the radiative phase space $d\Phi_1$ features poles in ε, which exactly cancel those of the virtual squared matrix element V. The choice of the counterterm K and of the phase space factorisation $d\Phi_{n+1} = d\Phi_n\, d\Phi_1$ defines the subtraction scheme.

In our local analytic sector subtraction scheme for final state radiation, we first introduce the FKS sector functions $\mathcal{W}_{ij}$, forcing the projection $R\,\mathcal{W}_{ij}$ to approach a singular configuration only if the final-state particle i becomes soft, or particles i and j become collinear. Requiring for the sector functions the sum rule

$$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1\,, \tag{2.3}$$

we can construct the counterterm K as

$$K = \sum_{i,\,j\neq i} \Big[\Big(\overline{\mathbf{S}}_i R\,\mathcal{W}_{ij}\Big) + \Big(\overline{\mathbf{C}}_{ij} R\,\mathcal{W}_{ij}\Big) - \Big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R\,\mathcal{W}_{ij}\Big)\Big]\,, \tag{2.4}$$

where we have introduced the operators $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$, which act on all objects to their right in the following way: $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$ extract the leading behaviour for particle i becoming soft and for particles i and j becoming collinear, respectively; when acting on matrix elements, they also define implicitly a remapping of momenta (to be specified), such that the resolved particles of all matrix elements are on the mass-shell and satisfy four-momentum conservation. Concretely

$$\overline{\mathbf{S}}_i R = -\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \frac{s_{cd}}{s_{ic}s_{id}} B_{cd}\Big(\{\bar{k}\}^{(icd)}\Big)\,, \qquad \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij} R = 2\mathcal{N}_1 C_{f_j} \frac{s_{jr}}{s_{ij}s_{ir}} B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,$$
$$\overline{\mathbf{C}}_{ij} R = \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij} B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu} B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big]\,, \tag{2.5}$$

where $s_{ab} = 2k_a \cdot k_b$, $\mathcal{N}_1 = 8\pi\alpha_s(\mu^2 e^{\gamma_E}/(4\pi))^{\epsilon}$, B_{cd} is the colour-connected Born-level squared matrix element, and $B_{\mu\nu}$ is the spin-connected Born-level squared matrix element. The spin-averaged Altarelli-Parisi kernels P_{ij} and the azimuthal kernels $Q_{ij}^{\mu\nu}$ are functions of $x_i = s_{ir}/(s_{ir}+s_{jr})$ and $x_j = s_{jr}/(s_{ir}+s_{jr})$, defined by

$$\begin{aligned} P_{ij} &= \delta_{f_i g}\delta_{f_j g}\, 2C_A \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} + x_i x_j\right) + \delta_{\{f_i f_j\}\{q\bar{q}\}}\, T_R \left(1 - \frac{2x_i x_j}{1-\epsilon}\right) \\ &\quad + \delta_{f_i\{q,\bar{q}\}}\delta_{f_j g}\, C_F \left(\frac{1+x_i^2}{x_j} - \epsilon x_j\right) + \delta_{f_i g}\delta_{f_j\{q,\bar{q}\}}\, C_F \left(\frac{1+x_j^2}{x_i} - \epsilon x_i\right)\,, \\ Q_{ij}^{\mu\nu} &= \left[-\delta_{f_i g}\,\delta_{f_j g}\, 2C_A\, x_i x_j + \delta_{\{f_i f_j\}\{q\bar{q}\}}\, T_R \frac{2x_i x_j}{1-\epsilon}\right]\left[-g^{\mu\nu} + (d-2)\,\frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2}\right]. \end{aligned} \tag{2.6}$$

The next important step is the choice of the remappings $\{\bar{k}\}^{(ijr)}$ and $\{\bar{k}\}^{(icd)}$ that, in our approach, are not referred to the specific sector, as in FKS, but depend on the IR kernels of Eq. (2.5). We decided to use CS remappings, which are particularly suited for an easy analytic integration of the counterterms, defined by

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}} k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac}+s_{bc}} k_c\,, \qquad \bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a,b,c\,, \tag{2.7}$$

where $s_{abc} = s_{ab}+s_{ac}+s_{bc}$. Under these remappings, the $(n+1)$-particle phase space factorises as

$$d\Phi_{n+1} = d\Phi_n\Big(\{\bar{k}\}^{(abc)}\Big)\, d\Phi_1\Big(\bar{s}_{bc}^{(abc)}; y, z, \phi\Big)$$
$$\int d\Phi_1(s; y, z, \phi) \equiv N_1\, s^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y)^2 z(1-z)\Big]^{-\epsilon} (1-y)\,,$$

where the invariants are given by

$$s_{ab} = y\,\bar{s}_{bc}^{(abc)}, \quad s_{ac} = z(1-y)\,\bar{s}_{bc}^{(abc)}, \quad s_{bc} = (1-z)(1-y)\,\bar{s}_{bc}^{(abc)}, \quad \bar{s}_{bc}^{(abc)} = 2\bar{k}_b^{(abc)}\cdot\bar{k}_c^{(abc)}\,, \tag{2.8}$$

and $N_1 = (4\pi)^{\epsilon-2}\pi^{-1/2}/\Gamma(1/2-\epsilon)$. The integral I of the counterterm K in the $d\Phi_1$ phase space can then be computed analytically, after having summed away the sector functions, obtaining

$$I = \frac{\alpha_s}{2\pi}\left[\sum_{c,d\neq c} J_s^{cd} B_{cd} + \sum_p J_{hc}^{pr} B\right], \tag{2.9}$$

where

$$\begin{aligned} J_s^{cd} &= -\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - 6 + \frac{7}{2}\zeta_2 + \ln\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{1}{\epsilon} + 2 - \frac{1}{2}\ln\frac{\bar{s}_{cd}}{\mu^2}\right) + \mathcal{O}(\epsilon)\,, \\ J_{hc}^{pr} &= -\delta_{f_p g}\frac{C_A + 4T_R N_f}{6}\left(\frac{1}{\epsilon} + \frac{8}{3} - \ln\frac{\bar{s}_{pr}}{\mu^2}\right) - \delta_{f_p\{q,\bar{q}\}}\frac{C_F}{2}\left(\frac{1}{\epsilon} + 2 - \ln\frac{\bar{s}_{pr}}{\mu^2}\right) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{2.10}$$

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3. Local analytic sector subtraction for final state radiation at NNLO

At NNLO the structure of the differential cross section contains three contributions,

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_n(X) + \int d\Phi_{n+1} RV\, \delta_{n+1}(X) + \int d\Phi_{n+2} RR\, \delta_{n+2}(X)\,. \quad (3.1)$$

where RR, RV and VV denote the double-real, real-virtual and double-virtual squared matrix elements respectively, the latter two renormalised in the $\overline{\rm MS}$ scheme. The double-virtual contribution has only poles in ε, while the real-virtual contribution features both poles in ε and phase space singularities, and the double-real term is characterised only by singularities in the radiation phase space. When computed in d dimensions, the phase space integrations in $d\Phi_{n+1}$ and $d\Phi_{n+2}$ result in explicit poles in ε, which cancel those arising from virtual corrections [18, 19].

In this case the structure of the counterterms is more involved than at NLO and we construct it step by step. Following the same strategy described in the previous section, we first introduce new sector functions $\mathcal{W}_{ijkl}$ for RR, while we use the NLO sector functions $\mathcal{W}_{ij}$ for RV. The singular behaviour of RV for soft and/or collinear emission is similar to R at NLO, and we build the corresponding counterterm in the same way, according to

$$K^{(\mathbf{RV})} = \sum_{i,\, j\neq i} \Big[\Big(\overline{\mathbf{S}}_i\, RV\, \mathcal{W}_{ij} \Big) + \Big(\overline{\mathbf{C}}_{ij}\, RV\, \mathcal{W}_{ij} \Big) - \Big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij}\, RV\, \mathcal{W}_{ij} \Big) \Big]\,, \quad (3.2)$$

The soft and/or collinear singular behaviour of RV is known [20], and the explicit expressions for $\overline{\mathbf{S}}_i RV$, $\overline{\mathbf{C}}_{ij} RV$, and $\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RV$ is obtained by introducing proper remappings in the matrix elements, analogous to those introduced at NLO.

The new sector functions $\mathcal{W}_{ijkl}$ for RR are defined to minimize the number of singular regions of $RR\,\mathcal{W}_{ijkl}$, and must of course sum to 1, according to

$$\sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1\,. \quad (3.3)$$

Notice that in the previous formula we allow the last two indices k and l to be equal to the second index j. This is done to catch specific collinear limits of RR: in $RR\,\mathcal{W}_{ijjk}$ and $RR\,\mathcal{W}_{ijkj}$ $(k \neq j)$, the only singular double collinear limit is when the three particles i, j, k become simultaneously collinear (which we represent by the operator $\mathbf{C}_{ijk}$); on the other hand, in $RR\,\mathcal{W}_{ijkl}$ $(k, l \neq j)$ the surviving singular double collinear limit is when the two pairs (i, j) and (k, l) become collinear separately (which we represent by the operator $\mathbf{C}_{ijkl}$). It is possible to define the sector functions $\mathcal{W}_{ijkl}$ such that in the three mentioned topologies just the following singular limits survive:

$$\begin{aligned}
&\mathcal{W}_{ijjk} RR \;:\quad \mathbf{S}_i, \quad \mathbf{C}_{ij}, \quad \mathbf{S}_{ij}, \quad \mathbf{C}_{ijk}, \quad \mathbf{SC}_{ijk}, \qquad\qquad\quad j\neq i,\; k\neq i,j; \\
&\mathcal{W}_{ijkj} RR \;:\quad \mathbf{S}_i, \quad \mathbf{C}_{ij}, \quad \mathbf{S}_{ik}, \quad \mathbf{C}_{ijk}, \quad \mathbf{SC}_{ijk}, \quad \mathbf{SC}_{kij}, \quad j\neq i,\; k\neq i,j; \\
&\mathcal{W}_{ijkl} RR \;:\quad \mathbf{S}_i, \quad \mathbf{C}_{ij}, \quad \mathbf{S}_{ik}, \quad \mathbf{C}_{ijkl}, \quad \mathbf{SC}_{ikl}, \quad \mathbf{SC}_{kij}, \quad j\neq i,\; k\neq i,j,\; l\neq i,j,k.
\end{aligned} \quad (3.4)$$

where, besides the operators for the single soft limit $\mathbf{S}_i$ and for the single collinear limit $\mathbf{C}_{ij}$, and the already mentioned operators for the double collinear limits $\mathbf{C}_{ijk}$ and $\mathbf{C}_{ijkl}$, we have defined the double-soft operator for particles i and j, $\mathbf{S}_{ij}$, and the soft-collinear operator $\mathbf{SC}_{ijk}$, which extract

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the singular behaviour when particle i becomes soft and simultaneously particles j and k become collinear. Of course this list of limits depends on the actual form of the sector functions.
Since all previous limits commute when acting on both RR and sector functions, we can easily build expressions which, by construction have no phase space singularties. Indeed

$$\begin{aligned}
(1-\overline{\mathbf{S}}_i)(1-\overline{\mathbf{C}}_{ij})(1-\overline{\mathbf{S}}_{ij})(1-\overline{\mathbf{C}}_{ijk})(1-\overline{\mathbf{SC}}_{ijk})\,RR\,\mathscr{W}_{ijjk} &= \text{finite}\,, \\
(1-\overline{\mathbf{S}}_i)(1-\overline{\mathbf{C}}_{ij})(1-\overline{\mathbf{S}}_{ik})(1-\overline{\mathbf{C}}_{ijk})(1-\overline{\mathbf{SC}}_{ijk})(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathscr{W}_{ijkj} &= \text{finite}\,, \\
(1-\overline{\mathbf{S}}_i)(1-\overline{\mathbf{C}}_{ij})(1-\overline{\mathbf{S}}_{ik})(1-\overline{\mathbf{C}}_{ijkl})(1-\overline{\mathbf{SC}}_{ikl})(1-\overline{\mathbf{SC}}_{kij})\,RR\,\mathscr{W}_{ijkl} &= \text{finite} \quad (k,l\neq j)\,,
\end{aligned} \tag{3.5}$$

where the bar denotes again an implicit remapping on matrix elements, to preserve mass-shell conditions and momuntum conservation. The explicit expressions for $\overline{\mathbf{S}}_i RR$ and $\overline{\mathbf{C}}_{ij} RR$ are analogous to the ones at NLO, while $\overline{\mathbf{C}}_{ijkl} RR$ and $\overline{\mathbf{SC}}_{ijk} RR$ are essentially products of two single operators and can be obtained again from the NLO case (see Ref. [15] for the explicit expressions). The only non-trivial limits are $\overline{\mathbf{S}}_{ij} RR$ and $\overline{\mathbf{C}}_{ijk} RR$, which are given by

$$\begin{aligned}
\overline{\mathbf{S}}_{ij}\,RR &= \frac{\mathscr{N}_1^{\,2}}{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) + 4\sum_{e\neq i,j,c,d} I_{cd}^{(i)} I_{ed}^{(j)} B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) \\
&\qquad + 2 I_{cd}^{(i)} I_{cd}^{(j)} B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(I_{cd}^{(ij)} - \frac{1}{2} I_{cc}^{(ij)} - \frac{1}{2} I_{dd}^{(ij)}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg] \\
\overline{\mathbf{C}}_{ijk}\,RR &= \frac{\mathscr{N}_1^{\,2}}{s_{ijk}^2}\Big[P_{ijk}\,B\Big(\{\bar{k}\}^{(ijkr)}\Big) + Q_{ijk}^{\mu\nu} B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big)\Big]
\end{aligned} \tag{3.6}$$

where $I_{cd}^{(i)} = s_{cd}/(s_{ic}s_{id})$ is the NLO eikonal factor, while $I_{cd}^{(ij)}$, P_{ijk}, and $Q_{ijk}^{\mu\nu}$ are pure NNLO soft and collinear kernels, which have been computed explicitly in Ref. [16], and will be analysed in more details in the next section. The remappings introduced implicitly in Eq. (3.6) are again chosen to simplify the analytical integration procedure, and are basically double CS remappings, given by

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad}+s_{bd}+s_{cd}}\,k_d\,, \qquad \bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad}+s_{bd}+s_{cd}}\,k_d\,, \tag{3.7}$$

while $\bar{k}_i^{(abcd)} = k_i$ if $i \neq a,b,c,d$.
From the finite expressions of Eq. (3.5), we construct the counterterms which cancel the phase space singularities of RR. To this end we introduce the 1- and 2-unresolved limits $\overline{\mathbf{L}}_{ij}^{(\mathbf{1})}$ and $\overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}$, as

$$\begin{aligned}
1-\overline{\mathbf{L}}_{ij}^{(\mathbf{1})} &\equiv (1-\overline{\mathbf{S}}_i)(1-\overline{\mathbf{C}}_{ij})\,, \\
1-\overline{\mathbf{L}}_{ijjk}^{(\mathbf{2})} &\equiv (1-\overline{\mathbf{S}}_{ij})(1-\overline{\mathbf{C}}_{ijk})(1-\overline{\mathbf{SC}}_{ijk}) \\
1-\overline{\mathbf{L}}_{ijkj}^{(\mathbf{2})} &\equiv (1-\overline{\mathbf{S}}_{ik})(1-\overline{\mathbf{C}}_{ijk})(1-\overline{\mathbf{SC}}_{ijk})(1-\overline{\mathbf{SC}}_{kij})\,, \\
1-\overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})} &\equiv (1-\overline{\mathbf{S}}_{ik})(1-\overline{\mathbf{C}}_{ijkl})(1-\overline{\mathbf{SC}}_{ikl})(1-\overline{\mathbf{SC}}_{kij})\,, \qquad \text{for} \quad k,l\neq j\,.
\end{aligned} \tag{3.8}$$

In this way we can rewrite the three equations (3.5) in one formula (for $k \neq i, l \neq i,k$)

$$(1-\overline{\mathbf{L}}_{ij}^{(\mathbf{1})})(1-\overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})})\,RR\,\mathscr{W}_{ijkl} = \Big[RR - \overline{\mathbf{L}}_{ij}^{(\mathbf{1})} - \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})} + \overline{\mathbf{L}}_{ij}^{(\mathbf{1})}\,\overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})}\Big]\,\mathscr{W}_{ijkl} = \text{finite}\,, \tag{3.9}$$

and define the three counterterms

$$K^{(\mathbf{1})} = \sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\, RR\, \mathcal{W}_{ijkl}\,, \qquad K^{(\mathbf{2})} = \sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}\,,$$
$$K^{(\mathbf{12})} = -\sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{1})}_{ij}\,\overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, \mathcal{W}_{ijkl}\,. \tag{3.10}$$

Finally, we can build our subtraction formula, which we write as

$$\begin{aligned}\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} &= \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_n(X)\\ &+ \int d\Phi_{n+1}\left[\left(RV + I^{(\mathbf{1})}\right)\delta_{n+1}(X) - \left(K^{(\mathbf{RV})} - I^{(\mathbf{12})}\right)\delta_n(X)\right]\\ &+ \int d\Phi_{n+2}\left[RR\,\delta_{n+2}(X) - K^{(\mathbf{1})}\delta_{n+1}(X) - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right)\delta_n(X)\right].\end{aligned} \tag{3.11}$$

where $I^{(\mathbf{1})}$, $I^{(\mathbf{2})}$, $I^{(\mathbf{12})}$, and $I^{(\mathbf{RV})}$ are given by

$$I^{(\mathbf{1})} = \int d\Phi_1\, K^{(\mathbf{1})}, \quad I^{(\mathbf{12})} = \int d\Phi_1\, K^{(\mathbf{12})}, \quad I^{(\mathbf{2})} = \int d\Phi_2\, K^{(\mathbf{2})}, \quad I^{(\mathbf{RV})} = \int d\Phi_1\, K^{(\mathbf{RV})}. \tag{3.12}$$

The calculation of $I^{(\mathbf{1})}$ and $I^{(\mathbf{12})}$ needs the integrations in a single radiation phase space $d\Phi_1$, and can be readily performed following the NLO case, obtaining

$$\begin{aligned} I^{(\mathbf{1})} &= \frac{\alpha_s}{2\pi}\sum_{k,l\neq k}\left[\sum_{c,d\neq c} J_s^{cd}\, R_{cd} + \sum_p J_{\text{hc}}^{pr}\, R\right]\mathcal{W}_{kl}\,,\\ I^{(\mathbf{12})} &= -\frac{\alpha_s}{2\pi}\sum_{k,l\neq k}\left[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl}(1-\overline{\mathbf{S}}_k)\right]\left[\sum_{c,d\neq c} J_s^{cd}\, R_{cd} + \sum_p J_{\text{hc}}^{pr}\, R\right]\mathcal{W}_{kl}\,.\end{aligned} \tag{3.13}$$

As one can see, $I^{(\mathbf{12})}$ corresponds to the IR limit of $I^{(\mathbf{1})}$, with opposite sign. The second line of Eq. (3.11) is therefore free from phase space singularities, exactly as the third one. Explicit calculations show that $I^{(\mathbf{1})}$ cancels the ε poles of RV and $I^{(\mathbf{12})}$ cancels those of $K^{(\mathbf{RV})}$.
Of course, because of the KLN theorem [18, 19], $I^{(\mathbf{2})}$ and $I^{(\mathbf{RV})}$ cancel the ε poles of VV. Their integration is the most difficult part of the calculation, but can be performed following the procedure sketched in the next session.

3.1 Integration of $I^{(2)}$ and $I^{(\mathbf{RV})}$

The integrals $I^{(\mathbf{2})}$ and $I^{(\mathbf{RV})}$ consist of many terms. A large fraction of these terms are convolutions of integrals of the NLO type and their integration is trivial, but we refrain from showing them here for the sake of brevity. We describe instead the method we used to integrate the most intricate parts of these counterterms, namely those which depend also on the azimuthal angle of the unresolved particle(s). To make the integration of such terms feasible, the choice of remappings and the factorisation of phase space are crucial. Single and double CS remappings seem the best choice in view of the analytical integration, because they involve only the invariants that are actually present in the singular kernels and moreover generate a simple radiative phase space.

For $I^{(\mathbf{RV})}$, involving at most 5 invariants s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad} (where a is the unresolved particle), the CS remappings of Eq. (2.7) give:

$$
\begin{aligned}
d\Phi_{n+1} &= d\Phi_n\Big(\{\bar{k}\}^{(abc)}\Big)\, d\Phi_1\Big(\bar{s}_{bc}^{(abc)}; y,z,x\Big)\,,\\
\int d\Phi_1\Big(s;y,z,x\Big) &= 2^{-2\epsilon} N_1\, s^{1-\epsilon}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,[x(1-x)]^{-\epsilon-\frac12}\Big[y(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y)\,,\\
s_{ab} = y\,\bar{s}_{bc}^{(abc)}\,, \quad s_{ac} &= z(1-y)\,\bar{s}_{bc}^{(abc)}\,, \quad s_{bc} = (1-z)(1-y)\,\bar{s}_{bc}^{(abc)}\,, \quad s_{dc} = (1-y)\,\bar{s}_{cd}^{(abc)}\,,\\
s_{ad} &= y(1-z)\,\bar{s}_{cd}^{(abc)} + z\,\bar{s}_{bd}^{(abc)} - 2(1-2x)\sqrt{yz(1-z)\,\bar{s}_{bd}^{(abc)}\,\bar{s}_{cd}^{(abc)}}\,.
\end{aligned}
\tag{3.14}
$$

For $I^{(\mathbf{2})}$, involving at most 6 invariants s_{ab}, s_{ac}, s_{bc}, s_{cd}, s_{ad}, s_{bd} (where a and b are the unresolved particles), the double CS remappings of Eq. (3.7) give

$$
\begin{aligned}
d\Phi_{n+2} &= d\Phi_n\Big(\{\bar{k}\}^{(abcd)}\Big)\, d\Phi_2\Big(\bar{s}_{cd}^{(abcd)}; y',z',x',y,z,\phi\Big)\,,\\
\int d\Phi_2(s;y',z',x',y,z,\phi) &= N_2\, s^{2-2\epsilon}\int_0^1 dx'\int_0^1 dy'\int_0^1 dz'\int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon}\int_0^1 dy\int_0^1 dz\,\big[x'(1-x')\big]^{-\epsilon-\frac12}\\
&\quad\times\Big[y'(1-y')^2 z'(1-z')\,y^2(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y')\,y(1-y)\,,\\
s_{ab} &= y'\,y\,\bar{s}_{cd}^{(abcd)}\,, \qquad s_{ac} = z'(1-y')\,y\,\bar{s}_{cd}^{(abcd)}\,, \qquad s_{bc} = (1-y')(1-z')\,y\,\bar{s}_{cd}^{(abcd)}\,,\\
s_{bd} &= (1-y)\Big[y'z'(1-z) + (1-z')z + 2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\Big]\,\bar{s}_{cd}^{(abcd)}\,,\\
s_{cd} &= (1-y')(1-y)(1-z)\,\bar{s}_{cd}^{(abcd)}\,, \qquad s_{ad} = (y'+z-y'z)(1-y)\,\bar{s}_{cd}^{(abcd)} - s_{bd}\,,
\end{aligned}
\tag{3.15}
$$

with $N_2 = 2^{-2\epsilon} N_1^2$.

In order to explain how the integration is performed, we restrict the analysis to the azimuth-dependent terms of $I^{(\mathbf{2})}$, namely to the integration in $d\Phi_2$ of $I_{cd}^{(ij)}$ and P_{ijk} of Eq. (3.6) (but note that with the techniques shown here we were able to integrate $I^{(\mathbf{2})}$ and $I^{(\mathbf{RV})}$ completely).

The explicit expression of $I_{cd}^{(ij)}$ is

$$
I_{cd}^{(ij)} = \delta_{\{f_i f_j\}\{q\bar{q}\}}\,\frac{T_R}{2}\, I_{ijcd}^{(q\bar{q})} - \delta_{f_i g}\,\delta_{f_j g}\,\frac{C_A}{2}\, I_{ijcd}^{(gg)} \tag{3.16}
$$

where $\delta_{\{f_a f_b\}\{q\bar{q}\}} = \delta_{f_a q}\delta_{f_b \bar{q}} + \delta_{f_a \bar{q}}\delta_{f_b q}$, while $I_{ijcd}^{(q\bar{q})}$ and $I_{ijcd}^{(gg)}$ are taken from Ref. [16],

$$
\begin{aligned}
I_{ijcd}^{(q\bar{q})} &= I_{cd}(k_i,k_j) \qquad \text{(eq. (96) of [16])}\,,\\
I_{ijcd}^{(gg)} &= S_{cd}(k_i,k_j) \qquad \text{(eq. (110) of [16])}\,.
\end{aligned}
$$

On the other hand, for P_{ijk} we have

$$
\begin{aligned}
P_{ijk} &= P_{ijk}^{(qq'\bar{q}')}\,\delta_{\{f_i f_j\}\{q\bar{q}\}}\,\delta_{f_k\{q'\bar{q}'\}} + P_{jki}^{(qq'\bar{q}')}\,\delta_{\{f_j f_k\}\{q\bar{q}\}}\,\delta_{f_i\{q'\bar{q}'\}} + P_{kij}^{(qq'\bar{q}')}\,\delta_{\{f_k f_i\}\{q\bar{q}\}}\,\delta_{f_j\{q'\bar{q}'\}}\\
&\quad + P_{ijk}^{(qq\bar{q})}\,\delta_{\{f_i\{f_j f_k\}\}\{q\bar{q}\}} + P_{jki}^{(qq\bar{q})}\,\delta_{\{f_j\{f_k f_i\}\}\{q\bar{q}\}} + P_{kij}^{(qq\bar{q})}\,\delta_{\{f_k\{f_i f_j\}\}\{q\bar{q}\}}\\
&\quad + P_{ijk}^{(gq\bar{q})}\,\delta_{\{f_i f_j\}\{q\bar{q}\}}\,\delta_{f_k g} + P_{jki}^{(gq\bar{q})}\,\delta_{\{f_j f_k\}\{q\bar{q}\}}\,\delta_{f_i g} + P_{kij}^{(gq\bar{q})}\,\delta_{\{f_k f_i\}\{q\bar{q}\}}\,\delta_{f_j g}\\
&\quad + P_{ijk}^{(ggq)}\,\delta_{f_i g}\,\delta_{f_j g}\,\delta_{f_k\{q,\bar{q}\}} + P_{jki}^{(ggq)}\,\delta_{f_j g}\,\delta_{f_k g}\,\delta_{f_i\{q,\bar{q}\}} + P_{kij}^{(ggq)}\,\delta_{f_k g}\,\delta_{f_i g}\,\delta_{f_j\{q,\bar{q}\}}\\
&\quad + P_{ijk}^{(ggg)}\,\delta_{f_i g}\,\delta_{f_j g}\,\delta_{f_k g}\,,
\end{aligned}
\tag{3.17}
$$

where q and q' are quarks of equal or different flavours and

$$\delta_{f_a\{q\bar{q}\}} = \delta_{f_a q} + \delta_{f_a\bar{q}}\,, \qquad \delta_{\{f_a\{f_b f_c\}\}\{q\bar{q}\}} = \delta_{f_a q}\,\delta_{f_b\bar{q}}\,\delta_{f_c\bar{q}} + \delta_{f_a\bar{q}}\,\delta_{f_b q}\,\delta_{f_c q}\,. \tag{3.18}$$

The expressions for $P_{ijk}^{(qq'\bar{q}')}$, $P_{ijk}^{(qq\bar{q})}$, $P_{ijk}^{(gq\bar{q})}$, $P_{ijk}^{(ggq)}$ and $P_{ijk}^{(ggg)}$ can again be found in [16]:

$$\begin{aligned}
P_{ijk}^{(qq'\bar{q}')} &= \langle \hat{P}_{\bar{q}_i q_j q'_k} \rangle && \text{(eq. (57) of [16])}\,,\\
P_{ijk}^{(qq\bar{q})} &= \langle \hat{P}^{(\mathrm{id})}_{\bar{q}_i q_j q_k} \rangle && \text{(eq. (59) of [16])}\,,\\
P_{ijk}^{(gq\bar{q})} &= C_F\,T_R\,\langle \hat{P}^{(\mathrm{ab})}_{g_k q_i \bar{q}_j} \rangle + C_A\,T_R\,\langle \hat{P}^{(\mathrm{nab})}_{g_k q_i \bar{q}_j} \rangle && \text{(eqs. (68) and (69) of [16])}\,,\\
P_{ijk}^{(ggq)} &= C_F^2\,\langle \hat{P}^{(\mathrm{ab})}_{g_i g_j q_k} \rangle + C_F\,C_A\,\langle \hat{P}^{(\mathrm{nab})}_{g_i g_j q_k} \rangle && \text{(eqs. (61) and (62) of [16])}\,,\\
P_{ijk}^{(ggg)} &= \langle \hat{P}_{g_i g_j g_k} \rangle && \text{(eq. (70) of [16])}\,.
\end{aligned}$$

From the expressions of $I_{cd}^{(ij)}$ and P_{ijk} we see that they are symmetric under the permutation of some of the involved momenta. However, when integrating in the two-body radiative phase space $d\Phi_2$, we have a larger freedom of choosing the outgoing momenta k_a, k_b, k_c, k_d, depending on the symmetries of their four-body phase space, which is invariant under

- any permutations of the four momenta k_a, k_b, k_c, k_d;
- any of the following permutations of invariants: $s_{ab} \leftrightarrow s_{cd}$, $s_{ac} \leftrightarrow s_{bd}$, and $s_{ad} \leftrightarrow s_{bc}$.

These symmetries reflect in the factorisation of phase space: in fact, when reparametrising the four body phase space from (k_a,k_b,k_c,k_d) to $(\bar{k}_c^{(abcd)},\bar{k}_d^{(abcd)},y,z,\phi,y',z',x')$, we have the freedom of performing any one of the permutations listed above. This is of crucial importance in simplifying the analytical computation of the $d\Phi_2$ integration of $I_{cd}^{(ij)}$ and P_{ijk}. Exploiting this freedom in a systematic way, it is possible to rearrange $I_{cd}^{(ij)}$ and P_{ijk} so that in the denominators of each term only the following combinations of invariants appear:

$$s_{ab}\,,\quad s_{ac}\,,\quad s_{bc}\,,\quad s_{bd}\,,\quad s_{cd}\,,\quad s_{ac}+s_{bc}\,,\quad s_{ad}+s_{bd}\,,\quad s_{ab}+s_{bc}\,.$$

Among these denominators, only s_{bd} depends on the azimuthal angle (parametrised by the variable x'): therefore all terms without s_{bd} in the denominator can be trivially integrated in dx', and those with $1/s_{bd}$ can be integrated using the integral relation

$$\begin{aligned}
I_b(A,B) &\equiv \int_0^1 dx'\,\left[x'(1-x')\right]^{\frac{1}{2}-b}\frac{1}{A^2+B^2+2(1-2x')AB} \qquad (3.19)\\
&= \frac{\Gamma^2\left(\frac{3}{2}-b\right)}{\Gamma(3-2b)}\left[\frac{1}{B^2}\,{}_2F_1\!\left(1,b,2-b,\frac{A^2}{B^2}\right)\Theta(B^2-A^2)+\frac{1}{A^2}\,{}_2F_1\!\left(1,b,2-b,\frac{B^2}{A^2}\right)\Theta(A^2-B^2)\right].
\end{aligned}$$

In addition to the integration in x', we could perform those in ϕ (there is no ϕ dependence in the integrands) and y (giving just Beta functions). The z and z' integrations are performed by using known properties of the hypergeometric function ${}_2F_1$, and by introducing integrals in a new variable t (with no direct physical meaning). The remaining integrations are then of the following types:

$$\int_0^1 dt\int_0^1 dy'\,(1-t)^{\mu}\,t^{\nu}\,(1-y')^{\rho}\,(y')^{\sigma}\,{}_2F_1(n_1,n_2-\varepsilon,n_3-2\varepsilon,1-ty')\,, \tag{3.20}$$

$$\int_0^1 dt\,(1-t)^{\mu}\,t^{\nu}\,{}_2F_1(n_1,n_2-\varepsilon,n_3-2\varepsilon,1-t)\,, \qquad n_1,n_2,n_3\in\mathbb{N},\quad n_1\geq 1,\quad n_3\geq n_1+1,n_2\,,$$

with $\mu,\nu,\rho,\sigma = n + m\varepsilon$ ($n,m \in \mathbb{Z}$, $n \geq -1$). These integrals can of course be written in terms of hypergemetric functions ${}_2F_1$, ${}_3F_2$, ${}_4F_3$. Since however we are not interested in the full ε dependence, we first expanded in ε and then integrated in t and y', obtaining the following compact results

$$\int d\Phi_2(s;y',z',x',y,z,\phi)\, I^{(X)}_{ijcd} = A(s)\, \mathscr{I}^{(X)}_{ijcd}\,, \qquad \int d\Phi_2(s;y',z',x',y,z,\phi)\, P^{(X)}_{ijk} = A(s)\, \mathscr{P}^{(X)}_{ijk}\,, \tag{3.21}$$

with $A(s) = s^{-2\varepsilon} e^{-2\varepsilon\gamma_E}/(4\pi)^{4-2\varepsilon}$ and

$$\begin{aligned}
\mathscr{I}^{(q\bar{q})}_{ijcd} &= \frac{2}{3}\frac{1}{\varepsilon^3} + \frac{28}{9}\frac{1}{\varepsilon^2} + \left[\frac{416}{27} - \frac{7}{9}\pi^2\right]\frac{1}{\varepsilon} + \frac{5260}{81} - \frac{104}{27}\pi^2 - \frac{76}{9}\zeta(3) \qquad (c \neq d)\,, \qquad (3.22)\\
\mathscr{I}^{(q\bar{q})}_{ijcc} &= -\frac{2}{3}\frac{1}{\varepsilon^2} - \frac{16}{9}\frac{1}{\varepsilon} - \frac{212}{27} + \pi^2\,,\\
\mathscr{I}^{(gg)}_{ijcd} &= \frac{2}{\varepsilon^4} + \frac{35}{3}\frac{1}{\varepsilon^3} + \left[\frac{481}{9} - \frac{8}{3}\pi^2\right]\frac{1}{\varepsilon^2} + \left[\frac{6218}{27} - \frac{269}{18}\pi^2 - \frac{154}{3}\zeta(3)\right]\frac{1}{\varepsilon} + \frac{76912}{81} - \frac{3775}{54}\pi^2 - \frac{2050}{9}\zeta(3) - \frac{23}{60}\pi^4 \qquad (c \neq d)\,,\\
\mathscr{I}^{(gg)}_{ijcc} &= -\frac{2}{3}\frac{1}{\varepsilon^2} - \frac{10}{9}\frac{1}{\varepsilon} - \frac{164}{27} + \pi^2\,,
\end{aligned}$$

$$\begin{aligned}
\mathscr{P}^{(qq'\bar{q}')}_{ijk} &= -\frac{1}{3}\frac{1}{\varepsilon^3} - \frac{31}{18}\frac{1}{\varepsilon^2} + \left[-\frac{889}{108} + \frac{\pi^2}{2}\right]\frac{1}{\varepsilon} - \frac{23941}{648} + \frac{31}{12}\pi^2 + \frac{80}{9}\zeta(3)\,, \qquad (3.23)\\
\mathscr{P}^{(qq\bar{q})}_{ijk} &= \left[-\frac{13}{8} + \frac{1}{4}\pi^2 - \zeta(3)\right]\frac{1}{\varepsilon} - \frac{227}{16} + \pi^2 + \frac{17}{2}\zeta(3) - \frac{11}{120}\pi^4\,,\\
\mathscr{P}^{(gq\bar{q})}_{ijk} &= C_A T_R \left\{-\frac{2}{3}\frac{1}{\varepsilon^3} - \frac{41}{12}\frac{1}{\varepsilon^2} + \left[-\frac{1675}{108} + \frac{17}{18}\pi^2\right]\frac{1}{\varepsilon} - \frac{5404}{81} + \frac{1063}{216}\pi^2 + \frac{139}{9}\zeta(3)\right\}\\
&\quad + C_F T_R \left\{-\frac{2}{3}\frac{1}{\varepsilon^3} - \frac{31}{9}\frac{1}{\varepsilon^2} + \left[-\frac{889}{54} + \pi^2\right]\frac{1}{\varepsilon} - \frac{23833}{324} + \frac{31}{6}\pi^2 + \frac{160}{9}\zeta(3)\right\},\\
\mathscr{P}^{(ggq)}_{ijk} &= C_F C_A \left\{\frac{1}{2}\frac{1}{\varepsilon^4} + \frac{8}{3}\frac{1}{\varepsilon^3} + \left[\frac{905}{72} - \frac{2}{3}\pi^2\right]\frac{1}{\varepsilon^2} + \left[\frac{11773}{216} - \frac{89}{24}\pi^2 - \frac{65}{6}\zeta(3)\right]\frac{1}{\varepsilon} + \frac{295789}{1296} - \frac{845}{48}\pi^2 - \frac{2191}{36}\zeta(3) + \frac{19}{240}\pi^4\right\}\\
&\quad + C_F^2 \left\{\frac{2}{\varepsilon^4} + \frac{7}{\varepsilon^3} + \left[\frac{251}{8} - 3\pi^2\right]\frac{1}{\varepsilon^2} + \left[\frac{2125}{16} - \frac{21}{2}\pi^2 - \frac{154}{3}\zeta(3)\right]\frac{1}{\varepsilon} + \frac{17607}{32} - \frac{753}{16}\pi^2 - \frac{548}{3}\zeta(3) + \frac{13}{20}\pi^4\right\},\\
\mathscr{P}^{(ggg)}_{ijk} &= \frac{5}{2}\frac{1}{\varepsilon^4} + \frac{21}{2}\frac{1}{\varepsilon^3} + \left[\frac{853}{18} - \frac{11}{3}\pi^2\right]\frac{1}{\varepsilon^2} + \left[\frac{5450}{27} - \frac{275}{18}\pi^2 - \frac{188}{3}\zeta(3)\right]\frac{1}{\varepsilon} + \frac{180739}{216} - \frac{1868}{27}\pi^2 - \frac{1555}{6}\zeta(3) + \frac{41}{60}\pi^4\,.
\end{aligned}$$

As a cross check, all these integrals have been computed also numerically, using sector decomposition and without using the symmetries of the phase space.

4. Summary

We have presented the latest developments of the *local analytic sector* subtraction. The method takes advantage of the partition of phase space through sector functions. This, in turn, allows to easily identify counterterms by using the known singular limits of the real matrix elements, and by introducing proper remappings of the momenta, in order to preserve mass-shell conditions and momentum conservation at each step of the calculation. We exploit the freedom in the choice of remappings, and obtain a simple factorisation of phase space, that gives us the possibility to integrate analytically all counterterms in the radiation phase space. We have shown the application of the method to final state radiation, sketching the procedure developed for integrating the counterterms, and giving analytic results for the integrated double-soft and double-collinear kernels.

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