

Analytic multi-loop results using finite fields and dataflow graphs with FiniteFlow

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FINITEFLOW is a public framework for defining and executing numerical algorithms over finite fields and reconstructing multivariate rational functions. The framework allows to build complex algorithms by combining basic building blocks into graphical representations of the calculation, known as dataflow graphs. It offers an easy-to-use MATHEMATICA interface for implementing efficient custom algorithms without any low-level coding. We report on some new features of FINITEFLOW which have been published after its initial release, give some simple example of usage for common tasks and review recent cutting-edge applications to two-loop five-point scattering and the four-loop cusp anomalous dimension.

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1. Introduction

State-of-the-art phenomenological and theoretical problems in high-energy physics demand high-precision calculations for complex processes. These are characterized by either a high number of loops, for higher precision, or many external legs and physical scales, for processes involving more complicated combinations of external and internal states. Modern collider experiments at the LHC are sensitive to these kinds of theoretical predictions, which therefore can be an essential ingredient for validating the Standard Model in wider regions of the phase-space and distinguishing possible signals of new or interesting physics from the large Standard Model background.

In recent years, great effort has been put into the development of efficient methods for complex multi-loop calculations. A recent breakthrough stemmed from the idea of reconstructing analytic results from numerical evaluations over finite fields, where each arithmetic operation is performed modulo a prime integer. While some of these ideas had already been used in other fields, such as computer algebra, their introduction into high-energy physics [1, 2] yielded several impressive results (see e.g. refs. [3, 4, 5, 6, 7, 8, 9, 10, 11]) which are beyond the capabilities of traditional computational tools.

A number of these results [12, 4, 7, 11, 13] have been obtained using FINITEFLOW [14], a framework for defining and executing numerical algorithms over finite fields and reconstructing multivariate rational functions. In these proceedings, we briefly review some of its main features and applications and report on some new procedures and algorithms implemented and published after its initial release.

2. The FiniteFlow framework

In this section, we give a brief summary of the most important features of the FINITEFLOW framework. FINITEFLOW consists of three main components

- An efficient low-level implementation of a number of common basic algorithms (evaluation of functions, linear solvers and many more).
- A framework for combining these basic algorithms, used as building blocks, into more complex ones, by means of computational graphs known as *dataflow graphs*. The basic algorithms are used as nodes of the graph and their output can be used as input by other nodes.
- Functional reconstruction techniques yielding analytic expressions from numerical evaluations.

In FINITEFLOW we use a simplified implementation of dataflow graphs where *nodes*, i.e. our basic algorithms, can have one or more lists of numerical values, each represented by an *arrow*, as input, and exactly one list, i.e. one arrow, as output. A special node called *input node* represents the input variables of the numerical algorithm defined by the graph. A node is then identified as the *output node*, whose output is identified as the output of the graph itself.

These computational graphs can be easily defined by combining basic algorithms into computational graphs, from computer algebra systems and high-level programming languages (at the

moment we provide a MATHEMATICA interface). This allows to implement a wide number of algorithms without any low-level coding or compilation step.

The `finiteflow` repository contains the source code of a proof-of-concept implementation of this framework, as well as instructions and tutorials for its installation and usage. The `finiteflow-mathtools` repository contains a number of MATHEMATICA packages and examples using FINITEFLOW for the solution of a number of extremely relevant problems in high-energy physics. Both repositories can be accessed at the URL <https://github.com/peraro>.

A more comprehensive description of the FINITEFLOW framework and its implementation is given in [14].

3. New features and algorithms

In this section, we report on some new features added to FINITEFLOW after its initial release. These include new algorithms which can be used as nodes of computational graphs as well as additional procedures and options.

3.1 Arbitrary rational expressions

Most calculations start from some analytic input which can be cast in the form of one or more lists of rational functions. In FINITEFLOW, the preferred method for using these lists of functions is embedding them in a node which evaluates them efficiently using the so-called Horner's scheme (in the MATHEMATICA interface, this is done via the procedure `FFAlgRatFunEval`). While this works extremely well in most cases, there are a few situations where it is not ideal. Indeed, in order to use the Horner scheme for a rational function, one must first write it as a ratio of two polynomials (not necessarily GCD-simplified) and expand its numerator and denominator. In some cases, however, performing such operations may be difficult, due to the complexity of the analytic input, or may yield an expression which is considerably more complicated than the original one, thus negatively affecting performance.

We stress that, in realistic problems, one can almost always sidestep these issues by implementing larger portions of an algorithm using FINITEFLOW, rather than analytically (see e.g. section 4 and the examples in the public repositories). However, for the few cases where this is impossible or inconvenient, we have implemented an algorithm which allows to evaluate arbitrary *rational expressions* in a list of variables \mathbf{z} , without the need of preliminarily casting them into any special form.

More in detail, rational expressions are “compiled” into *bytecode*, i.e. an array of bytes representing a sequence of rational operations needed to evaluate the functions numerically (this is all done at runtime and no actual compilation to machine code is performed). In the MATHEMATICA interface, this is done via the procedure `FFAlgRatExprEval`. The bytecode is currently generated directly in MATHEMATICA by recursively inspecting the list of expressions and all their subexpressions. The evaluation of the function in C++ is done via a simple *virtual machine*, i.e. a procedure which reads the bytecode in a loop and executes the corresponding instructions. At the moment no optimization (such as reuse of common subexpressions) is performed, although these can be implemented directly in MATHEMATICA by combining several nodes of this type.

As we mentioned, this algorithm can be extremely useful when dealing with functions which are too hard to manipulate analytically or when the functions are significantly simpler in a form which is *not* the ratio of two expanded polynomials. It may also be more efficient for very sparse functions.

As an example, we applied this to a complicated function in three variables, which is about 1.5 GB in plain text form. The definition of the node, including the generation of the bytecode in MATHEMATICA, takes about 10 minutes on a modern machine. The numerical evaluation of the list of functions then takes about 0.5 seconds per point.

3.2 Rational functions with parametric coefficients

Consider a generic rational function in the variables $\mathbf{z} = \{z_1, z_2, \dots\}$ with the form

$$f(\mathbf{z}) = \frac{\sum_{\alpha} n_{\alpha} \mathbf{z}^{\alpha}}{\sum_{\alpha} d_{\alpha} \mathbf{z}^{\alpha}}. \quad (3.1)$$

As already mentioned, FINITEFLOW provides routines for evaluating functions from their analytic expressions, cast as in the previous equation where n_{α} and d_{α} are rational numbers, using Horner's method. However, there are cases where the coefficients n_{α} and d_{α} are not known rational numbers but they can be computed numerically as the output of a node in a computational graph. We have therefore added an alternative to the standard evaluation algorithms, called `FFAlgRatFunEvalFromCoeffs` in the MATHEMATICA interface. This node takes two lists as input, i.e. a list of values for the variables \mathbf{z} and a list of values for the coefficients $\{n_{\alpha}, d_{\alpha}\}$.

3.3 Partial reconstruction of subgraphs

FINITEFLOW can define some type of nodes in a graph called *subgraph* nodes, that are nodes which evaluate another graph (a subgraph) a number of times in order to produce their output (more details can be found in [14]). We added a new type of subgraph node, which performs a partial functional reconstruction of the output of another graph.

Let $G = G(\mathbf{z}, \mathbf{y})$ be a graph representing a rational function in two sets of variables \mathbf{z} and \mathbf{y} . The *subgraph reconstruction* algorithm is a node of another graph, which takes a list of values for \mathbf{y} as input, evaluates G a number of times, and reconstructs its output as a function of \mathbf{z} only, as on the right-hand side of eq. (3.1). It returns a list of numerical coefficients n_{α} and d_{α} as functions of the numerical values of \mathbf{y} which are the input of the node. Hence, the (numerical) coefficients $\{n_{\alpha}, d_{\alpha}\}$ may subsequently be used as inputs for other nodes of the graph.

An example where this might be useful is when \mathbf{z} are loop propagators and \mathbf{y} are kinematic variables. One may want to reconstruct the dependence of the integrand on the propagators \mathbf{z} and only compute the kinematic coefficients numerically so that they can be used in other steps of the calculation (such as Integration By Parts) *before* the full analytic reconstruction. In other words, this can be used, in some cases, as an implementation of *integrand reduction* [15, 16, 17, 18, 19, 20] which employs the functional reconstruction algorithms of FINITEFLOW rather than evaluations on cuts and system solving.

3.4 Sparse output for sparse solvers

We added the option to return a *sparse output* from a sparse linear solver node. By default, linear solvers, in the general case where there are fewer independent equations than unknowns,

return a dense matrix of coefficients which convert the linearly dependent unknowns into linearly independent ones (e.g. a matrix converting a list of integrals to a linear combination of master integrals, in the case of Integration By Parts). There are cases where one may wish to return a sparse representation of this matrix, i.e. only the non-vanishing coefficients of the solution. This effect could already be achieved by combining a linear solver with the `NonZero` algorithm (see ref. [14]) but it is now possible to return such a representation from a sparse solver node directly, as an option (this is done using `FFSolverSparseOutput` from the MATHEMATICA interface).

3.5 Parallel evaluation of user-defined lists of points

We added a procedure for evaluating a graph for a user-specified list of sample points. The procedure is called `FFGraphEvaluateMany` in the MATHEMATICA interface. By default, it evaluates the points in parallel, using multi-threading. The output should be equivalent to calling the `FFGraphEvaluate` procedure (which has been present since the first release of FINITEFLOW) on each individual point, except that the evaluations can be parallelized with the new procedure. This can be useful any time one needs to evaluate a graph for a large number of sample points, e.g. for using custom reconstruction algorithms or finding linear relations between smaller subsets of the full output.

As an example, consider a process whose result might be too hard or inconvenient to reconstruct analytically. If, for the purposes of a particular study, it is sufficient to evaluate an amplitude numerically for a number of phase-space points which is not too large, one may proceed by building a graph and reconstructing its numerical output for these points.¹ More in detail, for each phase-space point, one can consider a numerical rational approximation of the independent kinematic invariants which is correct up to a given accuracy. Then we evaluate the graph at these points over several prime fields, which can be done in parallel using `FFGraphEvaluateMany`, and use them to reconstruct the output over the rational field. From the MATHEMATICA interface, one can use the `FFRatMod` to convert a rational number to an integer over a finite field. The rational reconstruction can be done (see e.g. [2, 22] for a pedagogical description) by combining the Chinese remainder theorem (using e.g. the builtin `ChineseRemainder` routine of MATHEMATICA) and Wang's reconstruction (using e.g. the `FFRatRec` procedure of FINITEFLOW). A similar approach can obviously be implemented directly in the C++ interface as well.

While this is not as performant as a floating-point calculation, it is free of numerical instabilities and provides an exact result (up to the accuracy of the rationalization one started with). Moreover, this strategy allows to exploit the computational tools and high-level interface of the FINITEFLOW framework, which can drastically reduce development time compared to a custom implementation of the same algorithm in a low-level language.

4. Simple examples

In this section, we illustrate some examples of usage of FINITEFLOW for simple but common tasks. Some prior familiarity with the FINITEFLOW code might be beneficial to the reader. For an

¹Here we refer to the part of the calculation which can be performed via rational operations and thus implemented over finite fields, for instance the reduction of an amplitude to master integrals. If the master integrals themselves are not known, we understand that one also needs a method to compute those numerically, e.g. using sector decomposition [21].

introduction to the usage of the code, we refer to the MATHEMATICA tutorial in the main repository and to the examples in the `finiteflow-mathtools` repository.

4.1 Linear substitutions

Linear substitutions are among the most ubiquitous algebraic operations in multi-loop calculations. Performing such substitutions analytically sometimes results in an explosion of the algebraic complexity of the expressions, before the final GCD simplification of the result. These substitutions can, however, be reinterpreted as matrix multiplications, which in turn can be easily performed using FINITEFLOW.

Let us consider a list of quantities A_j , such as a list of amplitudes or form factors for a process, which can be written as linear combinations of some objects f_k , representing e.g. Feynman integrals or other special functions,

$$A_j = \sum_j a_{jk} f_k, \quad (4.1)$$

where the coefficients $a_{jk} = a_{jk}(\mathbf{z})$ are *rational functions* of some variables $\mathbf{z} = \{z_1, z_2, \dots\}$. Let us consider also a set of linear relations between the f_j and another set of objects g_k ,

$$f_j = \sum_k c_{jk} g_k, \quad (4.2)$$

where the coefficients $c_{jk} = c_{jk}(\mathbf{z})$ are also *rational functions* of \mathbf{z} . The previous equation may represent e.g. the reduction of a set of Feynman integrals f_j to master integrals g_k , or the explicit analytic solution for the integrals f_j in terms of a list of special functions g_k , or any other relation with a similar form.

We thus wish to substitute eq. (4.2) into eq. (4.1) in order to express A_j in terms of g_k . As already mentioned, we can use the matrix multiplication routines of FINITEFLOW to perform such substitution numerically and reconstruct the final result analytically. In other words, we want to reconstruct the matrix of coefficients $x_{jk} = x_{jk}(\mathbf{z})$ such that

$$A_j = \sum_{jk} x_{jk} g_k \quad (4.3)$$

which are given by the matrix multiplication

$$x_{jk} = \sum_l a_{jl} c_{lk}. \quad (4.4)$$

In fig. 4.1 we show a possible implementation of this in FINITEFLOW, assuming that the coefficients a_{jk} and c_{jk} above are known analytically. A similar approach can, of course, be used when the coefficients are known numerically and computed by other nodes of the graph.

4.2 Changes of variables

Changes of variables are another common operation. This is trivially implemented in FINITEFLOW by concatenating nodes that evaluate rational functions.

More explicitly, suppose we have a list of functions

$$\mathbf{f}(\mathbf{y}) = \{f_1(\mathbf{y}), f_2(\mathbf{y}), \dots\} \quad (4.5)$$

```

(* Input variables *)
zs = {z1,z2,...};

(* Analytic input matrices and their dimension *)
ajkmat = {{a11(z),a12(z),...},...};
cjkmat = {{c11(z),c12(z),...},...};
{n1,n2} = Dimensions[ajkmat];
{m1,m2} = Dimensions[cjkmat]; (* must satisfy n2 == m1 *)

(* Implement multiplication in FiniteFlow *)
FFNewGraph[graph,input,zs];
FFAlgRatFunEval[graph,ajk,{input},zs,Join@@ajkmat];
FFAlgRatFunEval[graph,cjk,{input},zs,Join@@cjkmat];
FFAlgMatMul[graph,xjk,{ajk,cjk},n1,n2,m2];
FFGraphOutput[graph,xjk];

(* Reconstruct and reshape the result *)
result = FFReconstructFunction[graph,zs];
result = ArrayReshape[result,{n1,m2}];

```

Figure 1: An implementation of linear substitutions in FINITEFLOW.

```

zs = {z1,z2,...};
FFNewGraph[graph,input,zs];
FFAlgRatFunEval[graph,y,{input},zs,{y1(z),y2(z),...}];
FFAlgRatFunEval[graph,f,{y},{y1,y2,...},{f1(y),f2(y),...}];
FFGraphOutput[graph,f];
result = FFReconstructFunction[graph,zs];

```

Figure 2: A change of variables in FINITEFLOW.

of the variables \mathbf{y} and that we want to perform a change of variables from \mathbf{y} to \mathbf{z}

$$\mathbf{y} = \mathbf{y}(\mathbf{z}) = \{y_1(\mathbf{z}), y_2(\mathbf{z}), \dots\}. \quad (4.6)$$

In other words, we want to compute a list of functions $\mathbf{g} = \mathbf{g}(\mathbf{z})$ such that

$$\mathbf{g}(\mathbf{z}) = \{g_1(\mathbf{z}), g_2(\mathbf{z}), \dots\} = \mathbf{f}(\mathbf{y}(\mathbf{z})). \quad (4.7)$$

Assuming the initial list of functions \mathbf{f} and the change of variables $\mathbf{y} = \mathbf{y}(\mathbf{z})$ are known analytically, one can implement this, as in fig. 2, simply by creating a node which evaluates $\mathbf{y} = \mathbf{y}(\mathbf{z})$ and use it as input for a node which evaluates $\mathbf{f} = \mathbf{f}(\mathbf{y})$.

4.3 GCD simplification

All the analytic results reconstructed by FINITEFLOW are automatically GCD-simplified, i.e. numerator and denominator have minimal degrees and no common factor, as a feature of the functional reconstruction algorithm. Hence, one can perform a GCD-simplification simply by asking FINITEFLOW to reconstruct the analytic expression of the corresponding rational function.

If one has an analytic expression which is sufficiently simple or such that the number of operations required for GCD simplification is not too large, then the traditional simplification procedures of computer algebra systems (such as MATHEMATICA's `Together`) are likely to be more efficient than a full functional reconstruction. However for very complicated expressions, where algebraic manipulations are very expensive, one may consider building a numerical algorithm evaluating that expression and reconstruct its simplified result out of numerical evaluations with FINITEFLOW.

In order to build such an algorithm, one must combine the builtin basic algorithms of FINITEFLOW in a suitable computational graph. For this purpose, the examples above and the ones in the public repositories might be useful as an inspiration. The `TakeAndAdd` algorithm of FINITEFLOW can also be useful when the expression is a sum of simple functions.

A simple (albeit not always optimal) strategy that can always be applied consists in using the full expression to be simplified, together with the new algorithm described in section 3.1, to quickly and effectively build the required numerical algorithm. This can be immediately followed by a call to the reconstruction procedures. For the convenience of the users, we already provide a wrapper routine, called `FFTogether`, which does this automatically for any expression and can simply be used with

$$\text{FFTogether}[\text{expression}]$$

where `expression` is a rational function (or a list, in which case `FFTogether` is applied to each element). The output is a collected and GCD-simplified form of the input expression, i.e. roughly equivalent to MATHEMATICA's builtin `Together` function.

5. Cutting-edge applications

In this section, we briefly summarize some cutting edge applications of the FINITEFLOW framework. These show the effectiveness of this framework as well as its versatility since it has been applied to a variety of methods and techniques.

5.1 Two-loop five-point scattering

FINITEFLOW, before and after its public release, has been used in a number of published results for two-loop five-point scattering. These processes are extremely complicated, not only because of the number of diagrams and the size of the systems of Integration By Parts identities to be solved, but also for the presence of at least five independent invariants. In the following, we briefly summarize the usage of FINITEFLOW in the context of these calculations, while we refer the reader to the corresponding papers for further details.

In [12] the first numerical results for a full set of two-loop five-gluon helicity amplitudes have been published. We used an implementation of integrand reduction in FINITEFLOW and sector decomposition to compute the integrals.

In [4] we reconstructed the first analytic results for the single-minus helicity configuration. We used a custom C++ implementation of unitarity cuts and linear fits based on [2] (the implementation of unitarity cuts is not public since it is no longer actively developed). Using FINITEFLOW, we used this building block to implement a computational graph containing a complete set of unitarity cuts. This graph, for any phase-space point, reconstructs a complete integrand representation. This

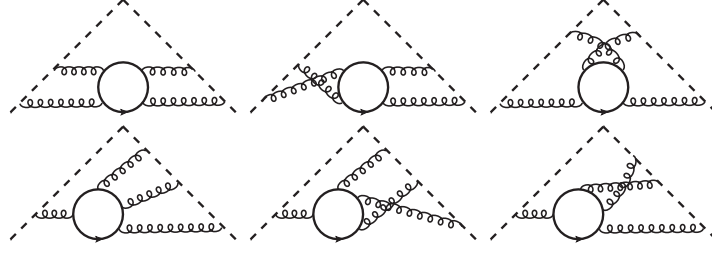


Figure 3: Fermion-loop diagrams contributing to the quartic Casimir part of the four-loop light-like cusp anomalous dimension.

was then multiplied by the numerical output of the solution of Integration-By-Parts, computed with the sparse linear solver of FINITEFLOW. Finally, the result has been cast in terms of known pentagon functions [23] and the known divergence structure has been subtracted. For each numerical phase-space point, we performed a full reconstruction with respect to the dimensional regulator ϵ and used it to extract the finite part of its Laurent expansion around $\epsilon = 0$. This can be easily done using the tools provided by FINITEFLOW, directly from the MATHEMATICA interface. Hence, the finite part has been reconstructed analytically from these numerical evaluations, as a function of kinematic invariants. More details on the procedure can be found in [4, 14] and explicit examples of similar calculations are available in the public repositories.

In [11] we computed analytically the two-loop five-gluon all-plus helicity amplitude, including for the first time the full-colour dependence. The main difference with respect to the previous results was the availability of an analytic integrand representation [24]. We used FINITEFLOW to reduce the linear combinations of integrals contributing to the amplitude to master integrals. Although this required rank five reductions (and rank six for one-loop squared diagrams), since the results were relatively simple, these were reconstructed analytically in a few hours using a modern computing node with 32 threads. From this, assembling the full amplitude, summed over the 120 permutations of the integrand in [24], and reconstructing it in terms of iterated integrals (which we derived in the physical s_{12} -scattering region) took only a few minutes on a modern laptop.

In [13] we computed numerically a complete set of two-loop helicity amplitudes involving four-partons and a W boson. The setup was similar to the one in [4], with some key differences. The integrand was generated by applying integrand reduction to Feynman diagrams, rather than using unitarity cuts. We reconstructed numerically the coefficients of the master integrals. The master integrals which are unknown have been computed using sector decomposition. In particular, we chose as masters some combinations of integrals which have a simpler divergence structure, thus improving performance and reliability of the sector decomposition method.

FINITEFLOW has also been used in a number of unpublished multi-leg and multi-scale calculations and tests, as well as for simplifying complicated analytic expressions (see e.g. ref. [25]).

5.2 Four-loop cusp anomalous dimension

The light-like cusp anomalous dimension is a universal quantity in QCD. It determines infrared divergences of scattering amplitudes and also appears in the resummation of soft and collinear Sudakov double logarithms. The matter-dependent four-loop contributions coming from both scalar and fermion loops have been computed analytically in [7] (the fermionic contributions have also

been independently computed in [6]). The quartic Casimir terms of the cusp anomalous dimension have been computed from the double pole of a form factor, defined by a composite operator inserted into on-shell scalar states (the other contributions have been derived using a conjecture in a parallel publication [26]). The main computational bottleneck of the calculation was the reduction of the relevant four-loop diagrams to master integrals. The reduction was performed using the sparse solver of *FINITEFLOW*. Rather than producing large and complex reduction tables, we reconstructed analytically the reduction of each individual diagram to a set of master integrals, which have been chosen for their analytic properties, namely uniform and maximal transcendental weight and simpler pole structure (see ref. [7] for more details). We also used *FINITEFLOW* for producing some of the differential equations satisfied by off-shell generalizations of these integrals, which have been used for their analytic calculation.

Unlike the five-point case, where the complexity is mostly a consequence of the large number of scales, in this application, there is only one scale and the complexity is due to the huge size of the systems of equations to be solved for the reduction to master integrals. This shows that *FINITEFLOW* can be effectively used for a wide and diverse range of complex multi-loop calculations.

6. Conclusions

FINITEFLOW is a powerful framework for performing complex algebraic calculations using modern techniques. It allows to define complicated numerical algorithms over finite fields via a high-level interface based on dataflow graphs and reconstruct full analytic results using functional reconstruction techniques. A number of recent applications [12, 4, 7, 11, 13] showed the capability of this framework of pushing the state-of-the-art for theoretical predictions in high-energy physics.

In these proceedings, we reported on new features published after its initial release, we provided simple examples of its usage and we reviewed some of its most relevant applications.

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