

How $\mathcal{N} = 1, D = 4$ SYM domain walls look like

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We review main features of the pure $\mathcal{N} = 1, D = 4$ SYM and its effective description by the Veneziano-Yankielowicz generalized sigma-model. We then indicate that the construction of BPS domain walls interpolating between different SYM vacua requires the presence of a dynamical membrane source. We will show how such a membrane is coupled to the SYM and present the explicit form of BPS domain walls which it creates in the Veneziano-Yankielowicz effective theory. In particular, we will describe 1/2 BPS domain wall configurations with $|k| \leq N/3$, where k is the membrane charge that sets the “distance” between two distinct SUSY vacua.

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1. Introduction

The $\mathcal{N} = 1, D = 4$ supersymmetric Yang-Mills was constructed in 1974 [1, 2, 3] and studied intensively over 45 years. In spite of its seeming simplicity, this theory has revealed a rich quantum structure. For instance, since early '80s it has been known that e.g. pure $\mathcal{N} = 1, D = 4$ SYM with a gauge group $SU(N)$ has N degenerate susy vacua distinguished by different vacuum expectation values of the gluino condensate [4, 5, 6] which are related to each other by discrete R-symmetry transformations,

$$\langle \text{Tr} \lambda^\alpha \lambda_\alpha \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}}, \quad n = 0, 1, \dots, N-1, \quad (1.1)$$

where Tr denotes the trace with respect to gauge symmetry indices and Λ is a dynamical scale at which the condensate of the gluino λ_α is formed by non-perturbative effects.

On general grounds, it was then suggested that there should exist BPS domain walls interpolating between different SYM vacua (labeled by l and n) and having the following tension saturating the BPS bound [7]

$$T_{\text{DW}} = \frac{N}{8\pi^2} |\langle \lambda \lambda \rangle_n - \langle \lambda \lambda \rangle_l|. \quad (1.2)$$

Since the '90s, domain walls in $\mathcal{N} = 1, D = 4$ SYM and Supersymmetric QCD have been extensively studied with the use of different approaches (for a recent review and latest developments see e.g. [8]). However, explicit solitonic solutions of a low-energy effective field theory describing domain walls in the pure SYM have not been found until recently. A reason for this is that SYM BPS domain walls are not smooth solitonic field configurations. Their existence requires the presence of a source which has been lacking in pure $\mathcal{N} = 1, D = 4$ SYM. In [9] it was suggested that such an object (associated with integrated heavy modes of the theory) should contribute to the BPS value (1.2) of the domain wall tension. In [10] we have shown that this object is a dynamical membrane. The purpose of this contribution is to show how to couple the membrane to $\mathcal{N} = 1$ SYM and its Veneziano-Yankielowicz effective field theory formulation [11], how the membrane creates BPS domain walls and what is their shape.

2. Pure $\mathcal{N} = 1$ $SU(N)$ SYM

To set up the stage and point out essential blocks of our construction, let us briefly review the main features of the simple SYM theory with the gauge group $SU(N)$. Its field content is a vector supermultiplet in the adjoint representation of $SU(N)$

$$A_m^I, \lambda_\alpha^I, \bar{\lambda}_{\dot{\alpha}}^I, D^I, \quad (2.1)$$

where $A_m^I(x)$ ($m = 0, 1, 2, 3$) is the vector gauge field, $\lambda_\alpha^I(x)$ ($\alpha = 1, 2$) and its complex conjugate $\bar{\lambda}_{\dot{\alpha}}^I$ ($\dot{\alpha} = 1, 2$) are the Weyl spinor gluino, $D^I(x)$ is a space-time scalar auxiliary field and $I = 1, \dots, N^2 - 1$ is the index of the adjoint representation of $SU(N)$. In the rest of the paper we will skip these $SU(N)$ indices over the fields, i.e. we will consider $su(N)$ algebra valued fields $A_m(x) = A_m^I(x) T_I$, etc. where T_I are the $su(N)$ generators.

The building block of the SYM action is chiral spinor superfield which in the chiral superspace basis, parametrized by complex Grassmann-odd coordinates θ_α and $x_L^m = x^m + i\theta \sigma^m \bar{\theta}$, has the

following form

$$\mathcal{W}_\alpha(x_L, \theta) = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2} F_{mn} \sigma^{mn} \alpha^\beta \theta_\beta + \theta^2 \sigma_{\alpha\beta}^m \nabla_m \bar{\lambda}^\beta, \quad (2.2)$$

where F_{mn} is the gauge field strength, $\nabla_m = \partial_m - iA_m$ is the gauge covariant derivative and $\sigma_{\alpha\beta}^m$ are the relativistic Pauli matrices.

The $\mathcal{N} = 1$ SYM Lagrangian is an integral of the square of \mathcal{W}_α over θ

$$\mathcal{L}_{\text{SYM}} = \frac{1}{4g^2} \int d^2\theta \text{Tr} \mathcal{W}^\alpha \mathcal{W}_\alpha + \text{c.c.}, \quad (2.3)$$

where g is the SYM coupling constant.

Note that the Lagrangian is invariant under the $U(1)$ R-symmetry $\mathcal{W}_\alpha \rightarrow e^{i\varphi} \mathcal{W}_\alpha$, which is broken by quantum chiral anomalies to a discrete subgroup \mathbb{Z}_{2N} . Namely, the R-symmetry current conservation becomes anomalous

$$\partial_m J^m := \partial_m \text{Tr} (\lambda \sigma^m \bar{\lambda}) = \frac{2N}{32\pi^2} \varepsilon^{mnpq} \text{Tr} F_{mn} F_{pq}. \quad (2.4)$$

The factor of $2N$ appears on the right hand side of the above equation since λ are in the adjoint of $SU(N)$. As a result the generating functional of the quantum theory is invariant under a residual \mathbb{Z}_{2N} symmetry. The latter is however further broken down to \mathbb{Z}_2 ($\lambda \rightarrow -\lambda$) due to the formation of the gluino condensate (1.1) by non-perturbative effects.

2.1 SYM Lagrangian and the special chiral superfield

For the possibility of coupling the membrane to the SYM multiplet it is important to notice that $\text{Tr} \mathcal{W}^\alpha \mathcal{W}_\alpha$ is a chiral scalar superfield (which is special as we will see in a minute)

$$S = \text{Tr} \mathcal{W}^\alpha \mathcal{W}_\alpha = s + \sqrt{2} \theta^\alpha \chi_\alpha + \theta^2 F, \quad (2.5)$$

where

$$s = -\text{Tr} \lambda^\alpha \lambda_\alpha, \quad (2.6)$$

$$\chi_\alpha = \sqrt{2} \text{Tr} \left(\frac{1}{2} F_{mn} \sigma_\alpha^{mn} \lambda_\beta - i \lambda_\alpha D \right), \quad (2.7)$$

and

$$F = \text{Tr} \left(-2i\lambda \sigma^m \nabla_m \bar{\lambda} - \frac{1}{2} F_{mn} F^{mn} + D^2 - \frac{i}{4} \varepsilon_{mnpq} F^{mn} F^{pq} \right). \quad (2.8)$$

Note that the real part of the F-term is the SYM Lagrangian, while its imaginary part is an instanton density which is (locally) a differential of a Chern-Simons three-form, namely

$$\begin{aligned} F_4 = d^4x \text{Im} F &= -\text{Tr} F_2 \wedge F_2 - d^4x \partial_m (\text{Tr} \lambda \sigma^m \bar{\lambda}) \\ &= -d\text{Tr} \left(AdA + \frac{2i}{3} A^3 + \frac{1}{3!} dx^k dx^l dx^m \varepsilon_{mnpq} \text{Tr} \lambda \sigma^l \bar{\lambda} \right) \equiv dC_3. \end{aligned} \quad (2.9)$$

(For shortness, we have omitted the wedge product symbols in the second line). Therefore, the complex field F in (2.5) has the following form

$$F = \hat{D} + i * dC_3 = \hat{D} + i \partial_m C^m, \quad (2.10)$$

where \hat{D} is a scalar field and C^m is the Hodge dual of the three-form C_3 ($C_1 = *C_3$). Hence, the F-term is gauge invariant under $C_3 \rightarrow C_3 + d\Lambda_2$ with $\Lambda_2(x)$ being a two-form gauge parameter.

This structure of the F-term makes the chiral superfield (2.5) special, as was noticed e.g. in [12, 13]. Superfields of this kind were first considered by Gates in [14]. They can always be expressed as a second super-covariant derivative of a real scalar ‘prepotential’ $U(x, \theta, \bar{\theta})$

$$S = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}U. \quad (2.11)$$

In the case of the conventional generic chiral superfields the prepotential U is complex.

The superfield U contains the components of the real one-form $C_1 = dx^m C_m$ dual to C_3 among its independent bosonic components

$$\begin{aligned} U|_{\theta=\bar{\theta}=0} &= u, \\ -\frac{1}{8}\bar{\sigma}_m^{\dot{\alpha}\alpha}[D_{\alpha}, \bar{D}_{\dot{\alpha}}]U|_{\theta=\bar{\theta}=0} &= C_m, \\ \frac{1}{4}D^2U|_{\theta=\bar{\theta}=0} &= -\bar{s} = \text{Tr } \bar{\lambda}\bar{\lambda}, \\ \frac{1}{16}D^2\bar{D}^2U|_{\theta=\bar{\theta}=0} &= \hat{D} + i\partial^m C_m \equiv F. \end{aligned} \quad (2.12)$$

Note that the superfield S in (2.11) is invariant under the superfield transformation

$$U' = U + L, \quad (2.13)$$

where L is a real linear superfield, i.e. the superfield satisfying

$$\bar{D}^2L = 0 = D^2L. \quad (2.14)$$

This transformation is the superfield extension of the gauge variation of the three-form $C_3 \rightarrow C_3 + d\Lambda_2$ under which the F-term in (2.12) is invariant. Hence, only gauge-invariant combinations of the components of U appear in S .

As we will see later, the presence of the three-form C_3 in U allows one to couple it to a membrane. Since we will look for domain walls of SYM induced by the membranes in its effective Veneziano-Yankielowicz field theory description, let us now revisit the structure of the latter.

3. Veneziano-Yankielowicz Lagrangian and potential

The VY Lagrangian [11] provides an effective description of colorless bound states of the SYM multiplet (like glueballs and gluinoballs), and demonstrates the formation of the gluino condensate and the N-degeneracy of the $SU(N)$ SYM vacuum. The form of the VY Lagrangian is (almost) fixed by anomalous superconformal Ward identities of the SYM. Its building block is the colorless chiral superfield (2.5) whose components (2.6)-(2.10) are now regarded as independent fields, rather than composites of the SYM multiplet. The VY Lagrangian has the following form

$$\mathcal{L}_{\text{VY}} = \frac{1}{16\pi^2\rho} \int d^2\theta d^2\bar{\theta} (S\bar{S})^{\frac{1}{3}} + \int d^2\theta W(S) + \text{c.c.} \quad (3.1)$$

The first term in (3.1) contains the Kähler potential

$$K(S, \bar{S}) = \frac{1}{16\pi^2 \rho} (S\bar{S})^{\frac{1}{3}} \quad (3.2)$$

with a priori arbitrary dimensionless positive constant ρ . Its simplest form is chosen due to the mass dimension 3 of the superfield S .

In general, the kinetic part of the Lagrangian is not fixed by anomalous symmetries and can also include higher order terms, however only for the above choice of the Kähler potential the scalar field potential is bounded from below [15].

The second term in (3.1) is the VY superpotential. It is uniquely fixed by anomalous superconformal Ward identities of the SYM theory and has the following form

$$W(S) = \frac{N}{16\pi^2} S \left(\ln \frac{S}{\Lambda^3} - 1 \right), \quad W_S := \partial_S W(S) = \frac{N}{16\pi^2} \ln \frac{S}{\Lambda^3}. \quad (3.3)$$

However, the superpotential and hence the Lagrangian are not single valued under an identical phase transformation of S . Indeed, they shift as

$$S \rightarrow S e^{2\pi i}, \quad \mathcal{L}_{VY} \rightarrow \mathcal{L}_{VY} - \frac{N}{4\pi} \partial_m C^m. \quad (3.4)$$

Another (related) issue is that the F-term in S is not a complex auxiliary scalar field but contains the dual four-form field strength $\partial_m C^m$ associated with the SYM instanton density $\text{Tr} F_2 \wedge F_2$. So the integration of the auxiliary field F out of the VY action requires caution. A recipe of how one can take care of these subtleties by modifying the VY superpotential was proposed in [16]. Instead, we will follow a somewhat different procedure of augmenting the VY Lagrangian [10] prompted by general requirements of the consistent construction of $4D$ Lagrangians containing three-form gauge fields (see e.g. [17, 18] and [19] for a recent review). Namely, the special form of the chiral superfield S (2.11) requires the variation of the VY Lagrangian with respect to the independent real superfield U . The variation principle is well-defined only with the addition of the boundary (total derivative) term which for the case under consideration is ¹

$$\mathcal{L}_{\text{bd}} = -\frac{1}{128\pi^2} \left(\int d^2\theta \bar{D}^2 - \int d^2\bar{\theta} D^2 \right) \left[\left(\frac{1}{12\rho} \bar{D}^2 \frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \ln \frac{\Lambda^{3N}}{S^N} \right) U \right] + \text{c.c.} \quad (3.5)$$

It is not hard to see that the Lagrangian

$$\mathcal{L} = \mathcal{L}_{VY} + \mathcal{L}_{\text{bd}} \quad (3.6)$$

is invariant not only under the identical phase transformation (3.4) but under a continuous $U(1)$ symmetry. To break this symmetry down to Z_{2N} , as it occurs in the SYM, we will require that the term $X(S, \bar{S}) \equiv \frac{1}{16\pi^2} \left(\frac{1}{12\rho} \bar{D}^2 \frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \ln \frac{\Lambda^{3N}}{S^N} \right)$ in the Lagrangian (3.5) satisfies the following boundary conditions

$$X(S, \bar{S})|_{\text{bd}} = -\frac{in}{8\pi}, \quad \text{where} \quad n = 0, 1, \dots, (N-1) \pmod{N} \quad (3.7)$$

¹A general prescription for constructing such terms was given in [18].

which characterizes the asymptotic vacua of the theory. Note that with this choice of the boundary conditions the Lagrangian (3.6) is gauge invariant under the superfield transformation (2.13).

We are interested in studying classical configurations of fields in the VY model with no fermionic excitations. Thus we set $\chi_\alpha = 0$ in (3.1) and (3.5), and get the following bosonic Lagrangian

$$\mathcal{L}_{\text{VY}}^{\text{bos}} = K_{s\bar{s}}(-\partial_m s \partial^m \bar{s} + (\partial_m C^m)^2 + \hat{D}^2) + (W_s(\hat{D} + i\partial_m C^m) + \text{c.c.}) + \mathcal{L}_{\text{bd}}^{\text{bos}} \quad (3.8)$$

where the boundary term has the following form

$$\mathcal{L}_{\text{bd}}^{\text{bos}} = -2\partial_m [C^m (K_{s\bar{s}} \partial_n C^n - \text{Im} W_s)] \quad (3.9)$$

and

$$K_{s\bar{s}} \equiv \partial_s \partial_{\bar{s}} K(s, \bar{s}), \quad W_s \equiv \partial_s W(s).$$

We will now eliminate the fields \hat{D} and C^m from the Lagrangian by solving their equations of motion. For the field \hat{D} we have

$$K_{s\bar{s}} \hat{D} + \text{Re} W_s = 0 \quad \rightarrow \quad \hat{D} = -\frac{\text{Re} W_s}{K_{s\bar{s}}} \quad (3.10)$$

and for C^m

$$\partial_m (K_{s\bar{s}} \partial_n C^n - \text{Im} W_s) = 0 \quad \rightarrow \quad \partial_m C^m = \frac{\text{Im} W_s - \frac{n}{8\pi}}{K_{s\bar{s}}}. \quad (3.11)$$

So

$$F \equiv \hat{D} + i\partial_m C^m = -\frac{\bar{W}_{\bar{s}} + i\frac{n}{8\pi}}{K_{s\bar{s}}} = -\frac{\partial_{\bar{s}}(\bar{W} + i\frac{n}{8\pi}\bar{s})}{K_{s\bar{s}}}, \quad (3.12)$$

where $\frac{n}{8\pi}$ with $(n = 0, 1, \dots, N-1) \pmod{N}$ is the integration constant compatible with the choice (3.7) of the boundary conditions.

The right hand side of (3.11) implies that the original VY superpotential gets effectively shifted by a term linear in S , i.e.

$$W(S) \rightarrow W(S) - i\frac{n}{8\pi} S. \quad (3.13)$$

Substituting the expression (3.12) for F into the action (3.8) we get the scalar field potential first derived in [16]

$$V(s, \bar{s}) = \frac{9\rho N}{16\pi^2} |s|^{\frac{4}{3}} \left(\ln^2 \frac{|s|}{\Lambda^3} + (\arg s - 2\pi \frac{n}{N})^2 \right), \quad n = 0, 1, 2, \dots, N-1 \pmod{N}. \quad (3.14)$$

In this potential the parameter n should be considered as a discrete variable. This makes the potential single-valued and multi-branched (periodic in n with the period N). It has cusps at $\arg s = \frac{\pi(k+1)}{N}$, at which n changes its value from k to $k+1$, and becomes zero (namely, it has absolute SUSY minima) at $\langle s \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}}$. The latter effectively reproduce the gluino condensate (1.1) of the $SU(N)$ SYM.

For instance, for $|s| = \Lambda^3$ and $N = 3$ for which $\frac{\pi}{N} \simeq 1$, the potential has the dependence on $\arg s$ and n that is depicted on Figure 1.

The presence of cusps indicates that at the corresponding points of space there sits an object that causes n to change its value and correspondingly the superpotential to “jump” as in (3.13). We will now show that this object is a dynamical membrane which carries a quantized charge that couples it to the three-form gauge field.

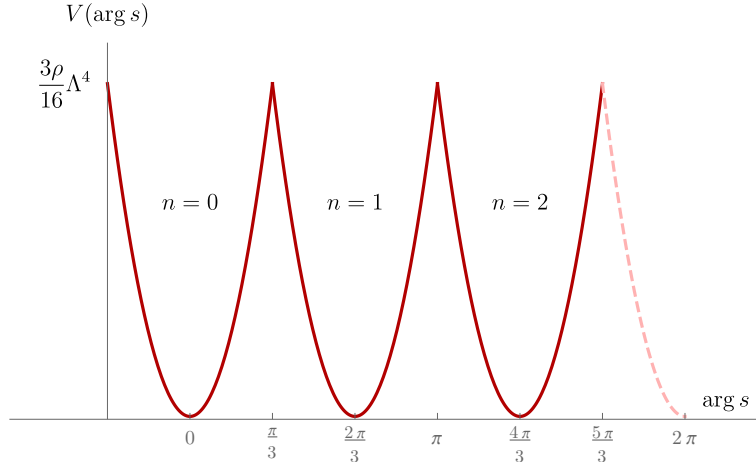


Figure 1: The VY potential as a function of $\arg s$ for $|s| = \Lambda^3$ and $N = 3$. The dashed line to the right should be identified with that to the very left. The potential is zero (has the absolute minima) at $\arg s = \frac{2\pi n}{3}$ ($n = 0, 1, 2$) and cusps at $\arg s = \frac{\pi(n+1)}{3}$ at which the variable n changes its value.

4. Coupling the membrane to the SYM and VY model

Membranes in flat and supergravity superspaces of various dimensions, and their effects in string/M-theory have been studied at length since the 80's after a seminal paper [20]. It was soon realized [21, 22] that in $D = 4$ supersymmetric membranes have to do with BPS-saturated domain walls. In this respect it is somewhat surprising that the problem of coupling dynamical membranes to $\mathcal{N} = 1$, $D = 4$ SYM has been addressed only recently in [10], though back in the early '90s generic constructions of couplings of p -branes to bosonic Young-Mills fields in various dimension were proposed in [23, 24] and a bosonic membrane coupled to $4D$ gauge fields via the Chern-Simons term (which is closely related to our supersymmetric construction) was considered in [25].

The action describing the dynamics of a membrane coupled to the $\mathcal{N} = 1$ SYM or its VY effective field model is a generalization of the supermembrane action of [26] coupled to a special chiral superfield (2.11). It has the following form

$$S_{\text{membrane}} = -\frac{1}{4\pi} \int_{\mathcal{M}_3} d^3 \xi \sqrt{-\det h_{ij}} |kS + c| - \frac{k}{4\pi} \int_{\mathcal{M}_3} \mathcal{E}_3 - \left(\frac{\bar{c}}{4\pi} \int_{\mathcal{M}_3} \mathcal{E}_3^0 + c.c. \right), \quad (4.1)$$

where $c = k_1 + ik_2$, and k , k_1 and k_2 are real constant charges characterizing the membrane coupling to a real three-form gauge superfield \mathcal{E}_3 and a complex super three-form \mathcal{E}_3^0 to be defined below. The normalization factor $\frac{1}{4\pi}$ has been chosen to have the canonical form of the Chern-Simons term in the static membrane action which forces the charge k be quantized $k = \pm 1, \pm 2 \dots$

In the Nambu-Goto part of the action (4.1) the bulk superfield $S(x_L, \theta)$ is either a composite special chiral superfield (2.5) or its Veneziano-Yankielowicz counterpart.² It is evaluated on the membrane worldvolume $z^M = z^M(\xi)$ parametrized by ξ^i ($i = 0, 1, 2$). The constant c added to S insures that when $S = \text{Tr} W^\alpha W_\alpha$ (which is a nilpotent superfield, $S^{2N} \equiv 0$) the module $|S + c|$ is

²The coupling of a membrane to an Abelian gauge field supermultiplet (associated with R-symmetry) within $\mathcal{N} = 1$, $D = 4$ supergravity, which enters a special chiral superfield in a different way has been recently considered in [27].

well defined. In the VY model in which S is a fully-fledged special chiral superfield, we will for simplicity set $c = 0$.

The induced metric on the membrane worldvolume is

$$h_{ij}(\xi) = \eta_{ab} E_i^a(\xi) E_j^b(\xi), \quad \text{with} \quad E_i^a(\xi) := \partial_i z^M(\xi) E_M^a(z(\xi)), \quad (4.2)$$

and

$$E^a(\xi) = dz^M(\xi) E_M^a(z(\xi)) =: d\xi^i E_i^a(\xi) = dx^a(\xi) + i\theta \sigma^a d\bar{\theta}(\xi) - id\theta \sigma^a \bar{\theta}(\xi) \quad (4.3)$$

is the worldvolume pull-back of the flat superspace vector supervielbein.

The super three-form \mathcal{C}_3 is constructed in terms of the real prepotential U (see eqs. (2.11) and (2.12))

$$\begin{aligned} \mathcal{C}_3 &= iE^a \wedge d\theta^\alpha \wedge d\bar{\theta}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}} U \\ &\quad - \frac{1}{4} E^b \wedge E^a \wedge d\theta^\alpha \sigma_{ab\alpha}{}^\beta D_\beta U - \frac{1}{4} E^b \wedge E^a \wedge d\bar{\theta}^{\dot{\alpha}} \bar{\sigma}_{ab}{}^\beta{}_{\dot{\alpha}} \bar{D}_\beta U \\ &\quad - \frac{1}{48} E^c \wedge E^b \wedge E^a \varepsilon_{abcd} \bar{\sigma}^{d\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] U. \end{aligned} \quad (4.4)$$

Note that the last, purely tensorial, term in (4.4) coincides, at $\theta = \bar{\theta} = 0$, with the three-form dual of the vector component of U in (2.12). Specifically, in the SYM case this term is nothing but the Chern-Simons term (2.9). So the leading bosonic component of the Wess-Zumino term \mathcal{C}_3 is

$$\mathcal{C}_3|_{\theta=0} = C_3 = -\text{Tr} \left(AdA + \frac{2i}{3} A^3 + dx^k dx^n dx^m \varepsilon_{mnl} \text{Tr} \lambda \sigma^l \bar{\lambda} \right).$$

Finally, the complex three-form \mathcal{C}_3^0 has the following form

$$\mathcal{C}_3^0 = iE^a \wedge d\theta^\alpha \wedge d\bar{\theta}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}} \theta^2 - \frac{1}{2} E^b \wedge E^a \wedge d\theta^\alpha \sigma_{ab\alpha}{}^\beta \theta_\beta. \quad (4.5)$$

4.1 Kappa-symmetry and a supersymmetric static membrane in SYM

The action (4.1) is invariant under the fermionic kappa-symmetry (a worldvolume counterpart of supersymmetry) under which the imbedding super-coordinates of the membrane are transformed as follows

$$\delta\theta^\alpha = \kappa^\alpha(\xi), \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\kappa}^{\dot{\alpha}}(\xi), \quad \delta x^m = i\kappa \sigma^m \bar{\theta} - i\theta \sigma^m \bar{\kappa}. \quad (4.6)$$

The fermionic parameters κ_α and $\bar{\kappa}_{\dot{\alpha}} = (\kappa_\alpha)^*$ are restricted by the following condition

$$\kappa_\alpha = -i \frac{kS + c}{|kS + c|} \Gamma_{\alpha\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}} \Leftrightarrow \bar{\kappa}_{\dot{\alpha}} = -i \frac{k\bar{S} + \bar{c}}{|kS + c|} \Gamma_{\alpha\dot{\alpha}} \kappa^\alpha, \quad (4.7)$$

where

$$\Gamma_{\alpha\dot{\alpha}} := \frac{i \varepsilon^{ijk}}{3! \sqrt{-\det h}} \varepsilon_{abcd} E_i^b E_j^c E_k^d \sigma_{\alpha\dot{\alpha}}^a, \quad \Gamma_{\alpha\dot{\alpha}} \Gamma^{\dot{\alpha}\beta} = \delta_\alpha^\beta. \quad (4.8)$$

Therefore, kappa-symmetry gauges away 2 of 4 fermionic modes $\theta^\alpha(\xi)$, $\bar{\theta}^{\dot{\alpha}}(\xi)$ of the membrane. Since the membrane action is manifestly invariant under worldvolume diffeomorphisms, the latter allow one to gauge fix 3 of 4 bosonic modes $x^m(\xi)$. The remaining $3d$ scalar mode $\phi(\xi)$ and a two-component $SL(2, \mathbb{R})$ Majorana spinor $\nu_\alpha(\xi)$ form a Goldstone $\mathcal{N} = 1$, $d = 3$ supermultiplet associated with partial breaking of $\mathcal{N} = 1$, $d = 3$ supersymmetry by the membrane.

It can be shown [10] that for a static membrane configuration for which the goldstone fields are equal to zero the membrane action reduces to that of an $\mathcal{N} = 1, d = 3$ $SU(N)$ Chern-Simons theory of level $-k$ (in the conventions of [8]) induced on the membrane worldvolume by its coupling to the SYM

$$S_{\text{static}} = -\frac{ik}{4\pi} \int d^3\xi \text{Tr} \psi^\alpha \psi_\alpha + \frac{k}{4\pi} \int \text{Tr} \left(A dA + \frac{2i}{3} A^3 \right) - \frac{|c|}{4\pi} \int d^3\xi, \quad (4.9)$$

where $A_i(\xi)$ is the induced CS vector field and $\psi_\alpha(\xi)$ is a 2-component Majorana spinor composed of the real and imaginary component $\lambda_1 = \frac{1}{2}(\psi_1 + i\psi_2)$ of the gluino. The integrand in the last term in (4.9) is constant and can be consistently removed (e.g. by sending $c \rightarrow 0$ at this stage).

5. BPS domain wall solutions sourced by the membrane in the VY effective theory

Let us now consider how the presence of the membrane modifies the equations of motion of the VY theory and induces BPS domain wall solutions [10]. We will consider a static membrane whose worldvolume is extended along three space-time directions $\xi^i = x^i$ ($i=0,1,2$) and sitting at $x^3 = 0 = \theta^\alpha = \bar{\theta}^{\dot{\alpha}}$. We are interested in solutions for which the fermionic VY field χ_α vanishes. Then the membrane action (4.1), in which we set $c = 0$ reduces to

$$S_{\text{static}} = -\frac{1}{4\pi} \int d^3\xi (|ks(\xi, 0)| + kC^3), \quad (5.1)$$

where $C^3(\xi, 0) = \frac{1}{6} \varepsilon^{ijk} C_{ijk}(\xi, 0)$.

Adding this action to the bosonic VY action (3.8), we get

$$S = \int d^3\xi dx^3 \left(\mathcal{L}_{\text{VY}}^{\text{bos}} - \frac{1}{4\pi} \delta(x^3) (|ks| + kC^3) \right). \quad (5.2)$$

Varying this action with respect to \hat{D} and C^m we get modified equations of motions whose solution is (compare with (3.12))

$$F \equiv \hat{D} + i\partial_m C^m = -\frac{\partial_{\bar{s}}(\bar{W} + i\frac{n+k\Theta(x^3)}{8\pi}\bar{s})}{K_{s\bar{s}}}, \quad (5.3)$$

where $\Theta(x^3)$ is the step function at the point $x^3 = 0$ at which the membrane sits. We see that the value of n gets shifted by k units when we cross the membrane. This indicates that the membrane of charge k separates two vacua labeled by n and $n+k$, respectively.

The equation of motion of the scalar field $s(\xi)$ has the following form

$$\square s K_{s\bar{s}} + \partial_m s \partial^m s K_{s\bar{s}} + F \bar{F} K_{s\bar{s}} + \bar{F} W_{\bar{s}\bar{s}} = \frac{k}{8\pi} \delta(x^3) \frac{ks}{|ks|}, \quad (5.4)$$

where it is understood that F is given by (5.3).

We look for solutions of (5.4) which describe BPS domain walls that preserve 1/2 supersymmetry and interpolate between two vacua, which are reached as $x^3 \rightarrow -\infty$ and $x^3 \rightarrow +\infty$ and are separated by the membrane, i.e.

$$\langle s \rangle_{-\infty} = \Lambda^3 e^{\frac{2\pi i n}{N}} \quad \text{and} \quad \langle s \rangle_{+\infty} = \Lambda^3 e^{\frac{2\pi i (n+k)}{N}}.$$

To this end we follow the well known method (see e.g. [22, 7, 28]). Namely, the field s is assumed to depend only on the coordinate x^3 orthogonal to the membrane and the supersymmetry variation of the fermionic field χ_α vanishes

$$\delta\chi_\alpha = i s \sigma_{\alpha\dot{\alpha}}^3 \bar{\epsilon} + \varepsilon_\alpha F = 0, \quad (5.5)$$

where $\dot{s} \equiv \partial_3 s \equiv \frac{\partial}{\partial x^3} s$. On the other hand, for the static membrane we have $\theta^\alpha = \bar{\theta}^{\dot{\alpha}} = 0$. These conditions are preserved by a combined supersymmetry and kappa-symmetry transformations of θ

$$\delta\theta^\alpha = \varepsilon^\alpha + \kappa^\alpha = 0 \quad \Rightarrow \quad \varepsilon^\alpha = -\kappa^\alpha. \quad (5.6)$$

Now remember that the kappa-symmetry parameters are restricted by the condition (4.7) (with $c = 0$) which reduces the number of independent real components to two. Therefore, also the supersymmetry parameters should satisfy the very same condition (4.7). For the static membrane case under consideration, this reduces to

$$\varepsilon_\alpha = e^{i\alpha} \sigma_{\alpha\dot{\alpha}}^3 \bar{\epsilon}^{\dot{\alpha}}, \quad e^{i\alpha} := \frac{ks(0)}{|ks(0)|}. \quad (5.7)$$

Substituting this relation into (5.5) we get the BPS equation for the field s

$$\dot{s} = ie^{i\alpha} F \quad (5.8)$$

in which it is understood that F is given by (5.3). One can check that this BPS relation solves the field equation (5.4). From (5.8) it also follows that

$$\frac{d}{dx^3} \text{Re}(\hat{W} e^{-i\alpha}) = 0, \quad \hat{W}(s) \equiv W(s) - \frac{i}{8\pi} (n + k\Theta(x^3)) s \quad (5.9)$$

that is

$$\text{Re}(\hat{W} e^{-i\alpha}) = \text{const} \quad (5.10)$$

at each point along x^3 including the position of the membrane ($x^3 = 0$), which is compatible with the worldvolume field equation of $x^3(\xi)$ that for the static membrane has the following form

$$(\partial_3 |ks| + k\partial_m C^m)|_{x^3=0} = 0.$$

Substituting into (5.8) the form of K and W of the Veneziano-Yankielowiz model, eqs. (3.2) and (3.3), we get the explicit form of the BPS equation

$$\dot{s} = 9i\rho N |s|^{\frac{4}{3}} e^{i\alpha} \left(\ln \frac{\Lambda^3}{|s|} + i \arg s - \frac{2\pi i}{N} (n + k\Theta(x^3)) \right). \quad (5.11)$$

Using this equation we can now compute the BPS value of the on-shell action (5.2) of the VY model coupled to the membrane. The result [10] gives the correct value (1.2) of the BPS domain wall, namely

$$S_{\text{BPS}} = S_{\text{VY}} + S_{\text{membr}} = -2 \int d^3 \xi |W_{x^3 \rightarrow +\infty} - W_{x^3 \rightarrow -\infty}| \quad (5.12)$$

and

$$T_{\text{DW}} = T_s + T_{\text{membr}} = 2 |W_{+\infty} - W_{-\infty}| = \frac{N}{8\pi^2} \Lambda^3 \left| e^{2\pi i \frac{n+k}{N}} - e^{2\pi i \frac{n}{N}} \right|. \quad (5.13)$$

It should be stressed that this value of the tension comprises the contribution of the energy density of the scalar field s and the membrane tension

$$T_M = \frac{|ks(0)|}{4\pi}. \quad (5.14)$$

Without the latter the domain wall tension would not saturate the BPS bound, as the one estimated in [9].

We are now ready to present the explicit form of 1/2 BPS domain walls in the VY description of the $SU(N)$ SYM.

6. Shape of 1/2 BPS domain walls

In [10] continuous $s(x)$ -field solutions of the BPS equation (5.11) were found for the following values of the membrane charge k

$$|k| \leq \frac{N}{3}.$$

For $|k| \leq \frac{N}{3}$ the domain walls have the profiles given in Figures 2, 3 and 4. The behaviour of the modulus $|s(x^3)|$ and the phase $\beta(x^3) = \arg s - \frac{2\pi n}{N}$ are given in Figures 2 and 3, and the behaviour of "jumping" superpotential is given in Figure 4. The profiles of these SYM BPS domain walls are similar to those obtained in $\mathcal{N} = 1$ $SU(N)$ super-QCD with $N_f \leq \frac{N}{3}$ (where N_f is the number of flavours) in the limit $m \rightarrow \infty$ of the mass of the flavour multiplets [29]. From this perspective the membrane may be viewed as an artefact of integrated-out massive flavour modes.

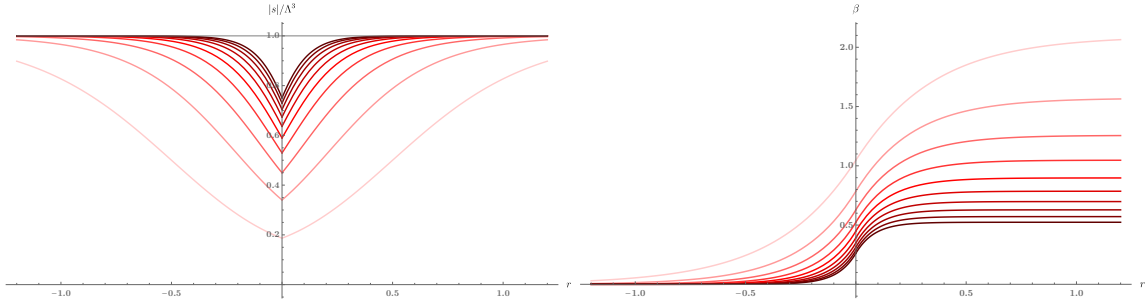


Figure 2: Flow of $\frac{|s|}{\Lambda^3}$ (on the left) and the phase $\beta(x^3)$ (on the right) along $r = 9\rho\Lambda x^3$ for different values of N , with fixed $k = 1$. N are chosen in the interval $[3, 12]$, with darker colors corresponding to larger N (alternatively, one might keep N fixed and vary k). $\frac{|s|}{\Lambda^3}$ takes the vacuum value 1 at $x^3 = \pm\infty$, decreases and has a cusp at $x^3 = 0$ where the membrane is sitting. The flow of β , starts from $\beta_{-\infty} = 0$ on the left, passes through $\beta(0) = \frac{\pi}{N}$ on the membrane and reaches $\beta_{+\infty} = \frac{2\pi}{N}$ on the right. Thickness of the domain wall solutions decreases when N increases. This can be fixed by choosing $\rho = \frac{1}{N}$ in the Kähler potential.

As one can see from the plot of $\frac{|s|}{\Lambda^3}$ in Fig. 2, it tends to reach zero for smaller $N \geq 3$. As a result the solution breaks down for $N = 2$, or equivalently for $k = \frac{N}{2}$. This indicates that the domain walls induced by the membranes with a large three-form charge $|k| \geq N/2$ should be regarded as strongly coupled systems, whose internal structure is not captured by the VY effective theory. Note that the cases with $k \leftrightarrow N - k$ are dual to each other since the sum of the charges of the membranes with charge k and $N - k$ is N , i.e. equal to the periodicity of the SYM vacua. If $k \leq \frac{N}{3}$ then $N - k \geq \frac{2N}{3}$ and the corresponding dual configurations carry large three-form charges, and are strongly coupled in this sense.

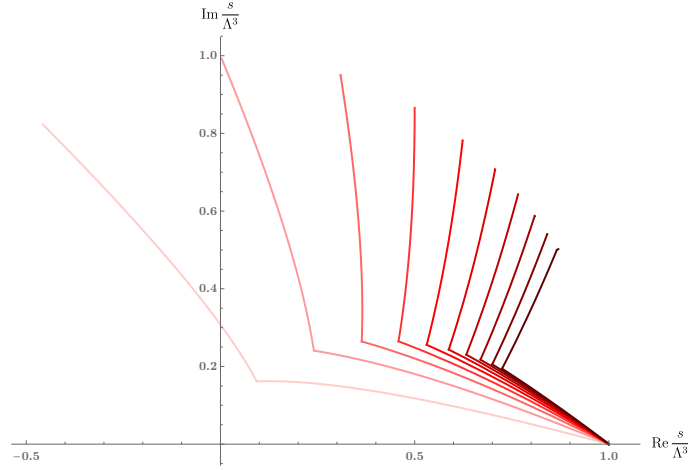


Figure 3: Behaviour of s along $x^3 \in (-\infty, +\infty)$ in the complex plane (for $k = 1$ and N varying from 3 to 12). Darker colors correspond to larger N . At the point where the membrane is located, $s(x^3)$ has a cusp.

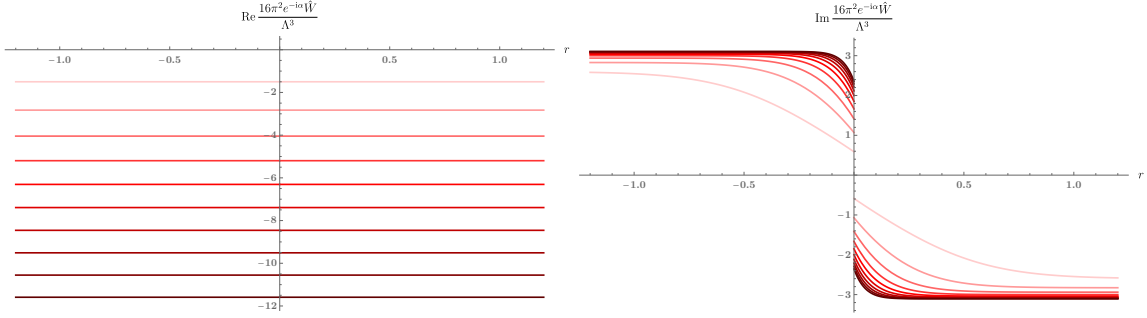


Figure 4: Behaviour (for $k = 1$ and N varying from 3 to 12) of the real and imaginary part of $\frac{16\pi^2\hat{W}e^{-i\alpha}}{\Lambda^3}$ ($\alpha = \frac{\pi(1+2n)}{N}$) along $r = 9\rho\Lambda x^3$. Darker colors correspond to larger N . The ‘jump’ of $\Im \frac{16\pi^2\hat{W}e^{-i\alpha}}{\Lambda^3}$ depicted on the right is proportional to the membrane tension (5.14).

7. Conclusion

We have reviewed the construction of the supersymmetric and kappa-invariant action which describes the coupling of a membrane to $\mathcal{N} = 1$, $D = 4$ SYM and its Veneziano-Yankielowicz effective generalized sigma-model. We have shown that in the framework of the VY model the presence of the dynamical membrane is required for the formation of 1/2 BPS domain walls interpolating between different SYM gluino condensate vacua and obtained explicit continuous domain wall configurations for $|k| \leq \frac{N}{3}$.

More details on the construction and solutions described in this contribution may be found in [10], where it was also shown that a BPS equation similar to (5.11) does not have continuous s -field BPS domain wall solutions which might be formed by separated parallel membranes. This indicates that to form a BPS domain wall the membranes should form a stack of coincident membranes, i.e. a composite membrane of a total charge k effectively described by the supermembrane action (4.1). Other problems considered in [10] include the construction of a system of an open membrane with a string attached to its boundary. This dynamical brane system was coupled to a massive three-form superfield extension [12, 30] of the Veneziano-Yankielowicz theory, which may be applied to the

study of domain-wall junctions.

An interesting issue which requires further study is the relation of the membrane worldvolume action (4.1) (for the membrane charge $k > 1$) to $3d$ gauge theories associated with domain walls in SYM and SQCD (see [8] for a review and an exhaustive list of references). As was discussed in [10], for the case $k = 1$ our supermembrane action (with the Goldstone fields switched off), eq. (4.9), is level-rank dual to a corresponding Acharya-Vafa theory [31] which was constructed with the use of a stack of k D4-branes wrapped on an internal 2-cycle with N RR fluxes in type IIA string theory.

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References

- [1] J. Wess and B. Zumino, *Supergauge Invariant Extension of Quantum Electrodynamics*, *Nucl. Phys.* **B78** (1974) 1.
- [2] S. Ferrara and B. Zumino, *Supergauge Invariant Yang-Mills Theories*, *Nucl. Phys.* **B79** (1974) 413.
- [3] A. Salam and J. A. Strathdee, *Supersymmetry and Nonabelian Gauges*, *Phys. Lett.* **51B** (1974) 353.
- [4] E. Witten, *Constraints on Supersymmetry Breaking*, *Nucl. Phys.* **B202** (1982) 253.
- [5] M. A. Shifman and A. I. Vainshtein, *On Gluino Condensation in Supersymmetric Gauge Theories. $SU(N)$ and $O(N)$ Groups*, *Nucl. Phys.* **B296** (1988) 445.
- [6] N. M. Davies, T. J. Hollowood, V. V. Khoze and M. P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys.* **B559** (1999) 123 [hep-th/9905015].
- [7] G. R. Dvali and M. A. Shifman, *Domain walls in strongly coupled theories*, *Phys. Lett.* **B396** (1997) 64 [hep-th/9612128].
- [8] V. Bashmakov, F. Benini, S. Benvenuti and M. Bertolini, *Living on the walls of super-QCD*, *SciPost Phys.* **6** (2019) 044 [1812.04645].
- [9] I. I. Kogan, A. Kovner and M. A. Shifman, *More on supersymmetric domain walls, N counting and glued potentials*, *Phys. Rev.* **D57** (1998) 5195 [hep-th/9712046].
- [10] I. Bandos, S. Lanza and D. Sorokin, *Supermembranes and domain walls in $\mathcal{N} = 1$, $D = 4$ SYM*, *JHEP* **12** (2019) 021 [1905.02743].
- [11] G. Veneziano and S. Yankielowicz, *An Effective Lagrangian for the Pure $N=1$ Supersymmetric Yang-Mills Theory*, *Phys. Lett.* **113B** (1982) 231.
- [12] C. P. Burgess, J. P. Derendinger, F. Quevedo and M. Quiros, *Gaugino condensates and chiral linear duality: An Effective Lagrangian analysis*, *Phys. Lett.* **B348** (1995) 428 [hep-th/9501065].

- [13] P. Binetrui, F. Pillon, G. Girardi and R. Grimm, *The Three form multiplet in supergravity*, *Nucl. Phys.* **B477** (1996) 175 [[hep-th/9603181](#)].
- [14] S. J. Gates, Jr., *Super P-Form Gauge Superfields*, *Nucl. Phys.* **B184** (1981) 381.
- [15] G. M. Shore, *Constructing Effective Actions for $N = 1$ Supersymmetry Theories. I. Symmetry Principles and Ward Identities*, *Nucl. Phys.* **B222** (1983) 446.
- [16] A. Kovner and M. A. Shifman, *Chirally symmetric phase of supersymmetric gluodynamics*, *Phys. Rev.* **D56** (1997) 2396 [[hep-th/9702174](#)].
- [17] K. Groh, J. Louis and J. Sommerfeld, *Duality and Couplings of 3-Form-Multiplets in $N=1$ Supersymmetry*, *JHEP* **05** (2013) 001 [[1212.4639](#)].
- [18] F. Farakos, S. Lanza, L. Martucci and D. Sorokin, *Three-forms in Supergravity and Flux Compactifications*, *Eur. Phys. J.* **C77** (2017) 602 [[1706.09422](#)].
- [19] S. Lanza, *Exploring the Landscape of effective field theories*, Ph.D. thesis, Padua U., 2019. [1912.08935](#).
- [20] E. Bergshoeff, E. Sezgin and P. K. Townsend, *Supermembranes and Eleven-Dimensional Supergravity*, *Phys. Lett.* **B189** (1987) 75.
- [21] P. K. Townsend, *Supersymmetric extended solitons*, *Phys. Lett.* **B202** (1988) 53.
- [22] E. R. C. Abraham and P. K. Townsend, *Intersecting extended objects in supersymmetric field theories*, *Nucl. Phys.* **B351** (1991) 313.
- [23] J. A. Dixon, M. J. Duff and E. Sezgin, *The Coupling of Yang-Mills to extended objects*, *Phys. Lett.* **B279** (1992) 265 [[hep-th/9201019](#)].
- [24] J. A. Dixon and M. J. Duff, *Chern-Simons forms, Mickelsson-Faddeev algebras and the p-branes*, *Phys. Lett.* **B296** (1992) 28 [[hep-th/9205099](#)].
- [25] P. K. Townsend, *Effective description of axion defects*, *Phys. Lett.* **B309** (1993) 33 [[hep-th/9303171](#)].
- [26] I. A. Bandos and C. Meliveo, *Superfield equations for the interacting system of $D=4$ $N=1$ supermembrane and scalar multiplet*, *Nucl. Phys.* **B849** (2011) 1 [[1011.1818](#)].
- [27] N. Cribiori, F. Farakos and G. Tringas, *Three-forms and Fayet-Iliopoulos terms in Supergravity: Scanning Planck mass and BPS domain walls*, [2001.05757](#).
- [28] M. Shifman and A. Yung, *Supersymmetric solitons*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009, [10.1017/CBO9780511575693](#).
- [29] A. V. Smilga, *Tenacious domain walls in supersymmetric QCD*, *Phys. Rev.* **D64** (2001) 125008 [[hep-th/0104195](#)].
- [30] G. R. Farrar, G. Gabadadze and M. Schwetz, *On the effective action of $N=1$ supersymmetric Yang-Mills theory*, *Phys. Rev.* **D58** (1998) 015009 [[hep-th/9711166](#)].
- [31] B. S. Acharya and C. Vafa, *On domain walls of $N=1$ supersymmetric Yang-Mills in four-dimensions*, [hep-th/0103011](#).