Discovery of Chromomagnetic Gluon Condensation

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I reexamine the phenomena of the chromomagnetic gluon condensation in Yang-Mills theory. The extension of the Heisenberg-Euler Lagrangian to the Yang-Mills theory allows to predict the chromomagnetic gluon condensation. The energy density curve crosses the zero energy level of the perturbative vacuum state at nonzero angle and continuously enters into the negative energy density region. At the crossing point and further down the effective coupling constant is small and demonstrate that the true vacuum state of the Yang-Mills theory is below the perturbative vacuum state and is described by the nonzero chromomagnetic gluon condensate. The renormalisation group analyses allows to express the energy momentum tensor, its trace and the vacuum magnetic permeabilities in terms of effective coupling constant and Callan-Symanzik beta function. In the vacuum the energy-momentum tensor is proportional to the space-time metric and induces a negative cosmological constant.

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1. Effective Action in QED and QCD

In this presentation I will reexamine the effective action in QED and QCD by using the perturbative loop expansion and renormalisation group equations and discuss the physical consequences which can be derived from their explicit expressions. The discovery of the chromomagnetic gluon condensation in Yang-Mills (YM) theory and the new results will be discussed and analysed [11, 12, 13, 14, 15, 15].

The Heisenberg-Euler effective Lagrangian in QED [1, 2, 3, 4, 5, 6] is a sum of the one loop diagrams with a vacuum electron-positron pair circulating in the loop and the gluons and quarks in case of QCD [7, 8, 9, 10, 11, 12, 13, 14, 15]. The effective action $\Gamma[A]$ has the following representation:

$$\Gamma[A] = \int \mathcal{L}dx = \sum_n \int dx_1...dx_n \Gamma^{(n)a_1...a_n}_{\mu_1...\mu_n}(x_1,...,x_n)A^{a_1}_{\mu_1}(x_1)...A^{a_n}_{\mu_n}(x_n) = S + W^{(1)} + ..., \quad (1.1)$$

where $\mathcal{L}$ is the effective Lagrangian, $\Gamma^{(n)}$ is a one-particle irreducible (1PI) n-point vertex function, $A^a_\mu(x) \equiv \langle 0|\mathcal{A}^a_\mu(x)|0\rangle$ is the vacuum expectation value of the field operator $\mathcal{A}^a_\mu$ and $W^{(n)}$, $n = 1, 2, ..$ represent the terms of the loop expansion.

Considering the limit of massless electrons and quarks one can demonstrate that the proper time integral in the Heisenberg-Euler Lagrangian can be integrated explicitly [15] by using covariant renormalisation condition [11, 13, 14]

$$\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\Big|_{t=\frac{1}{2}\ln(\frac{2e^2|\mathcal{F}|}{\mu^4})=\mathcal{G}=0} = -1, \quad (1.2)$$

where $\mathcal{F} = \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$ is the Lorentz and gauge invariant form of the YM field strength tensor $G^a_{\mu\nu}$ and μ^2 is the renormalisation scale parameter. In the massless limit the QED effective Lagrangian [15] has the exact logarithmic dependence as a function of the invariant $\mathcal{F}$ shown on Fig.1:

$$\mathcal{L}_e = -\mathcal{F} + \frac{e^2\mathcal{F}}{24\pi^2}\Big[\ln(\frac{2e^2\mathcal{F}}{\mu^4}) - 1\Big], \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}^2 - \vec{\mathcal{E}}^2}{2}, \quad \mathcal{G} = \vec{\mathcal{E}}\vec{\mathcal{H}} = 0, \quad (1.3)$$

where $\vec{\mathcal{H}}$ and $\vec{\mathcal{E}}$ are magnetic and electric fields. This expression should be compared with the one-loop effective Lagrangian in pure SU(N) gauge field theory, which has the form [11, 13] (see Fig.2):

$$\mathcal{L}_g = -\mathcal{F} - \frac{11N}{96\pi^2}g^2\mathcal{F}\Big(\ln\frac{2g^2\mathcal{F}}{\mu^4} - 1\Big), \qquad \mathcal{F} = \frac{\vec{\mathcal{H}}_a^2 - \vec{\mathcal{E}}_a^2}{2} > 0, \quad \mathcal{G} = \vec{\mathcal{E}}_a\vec{\mathcal{H}}_a = 0. \quad (1.4)$$

From (1.3) it follows that the corresponding quark contribution considered in the chiral limit is

$$\mathcal{L}_q = -\mathcal{F} + \frac{N_f}{48\pi^2}g^2\mathcal{F}\Big[\ln(\frac{2g^2\mathcal{F}}{\mu^4}) - 1\Big], \quad (1.5)$$

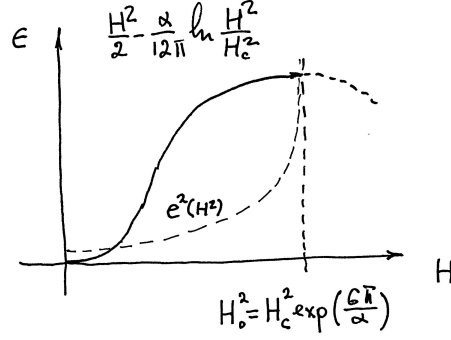where N_f is the number of quark flavours.

Figure 1: The graph shows the qualitative behaviour of the QED vacuum energy density $\epsilon(\vec{\mathcal{H}}^2)$ (1.12) and of the effective coupling constant $\bar{e}^2(\vec{\mathcal{H}}^2)$ (2.1) as the functions of magnetic field. The effective coupling constant is singular at $\vec{\mathcal{H}}_0^{\,2}$, the "Moscow zero" [25, 26].

The effective Lagrangian technique allows to calculate the magnetic induction $\vec{\mathcal{B}}$ of the vacuum through the derivative of the effective Lagrangian [11]:

$$\vec{\mathcal{B}}_a = -\frac{\partial \mathcal{L}}{\partial \vec{\mathcal{H}}_a} = \mu_{vac}\, \vec{\mathcal{H}}_a. \qquad (1.6)$$

From (1.3), (1.4) and (1.5) it follows that in QED the vacuum responds to the background magnetic field as diamagnet and in QCD as paramagnet with the magnetic permeabilities of the following form [11]:

$$\mu_{QED} = 1 - \frac{e^2}{24\pi^2}\log\left(\frac{e^2\vec{\mathcal{H}}^2}{\mu^4}\right) < 1, \qquad\qquad diamagnetic, \qquad (1.7)$$

$$\mu_{QCD} = 1 + \frac{g^2}{96\pi^2}(11N - 2N_f)\log\frac{g^2\vec{\mathcal{H}}_a^2}{\mu^4} > 1, \qquad paramagnetic, \quad N > \frac{2}{11}N_f. \ (1.8)$$

The diamagnetism of the QED vacuum (1.7) means that it repels the magnetic fields by forming induced magnetic field in the direction opposite to that of the applied magnetic field. This phenomenon is similar to the Landau orbital diamagnetism of free electron gas when the counteracting field is formed when the electron trajectories are curved due to the Lorentz force [17].

The paramagnetism of the QCD vacuum (1.8) means that it amplifies the applied chromomagnetic field by generating induced chromomagnetic field in the direction of the applied field. In QCD the large polarisation of the gluon spins is responsible for the amplification of the background field. This phenomenon is similar to the Pauli paramagnetism, an effect associated with the polarisation of the electron spins [16].

Using the effective Lagrangian approach one can calculate the quantum-mechanical corrections to the energy momentum tensor by using the formula derived by Schwinger [5]:

$$T_{\mu\nu} = (F_{\mu\lambda}F_{\nu\lambda} - g_{\mu\nu}\frac{1}{4}F_{\lambda\rho}^2)\frac{\partial \mathcal{L}}{\partial \mathcal{F}} - g_{\mu\nu}(\mathcal{L} - \mathcal{F}\frac{\partial \mathcal{L}}{\partial \mathcal{F}} - \mathcal{G}\frac{\partial \mathcal{L}}{\partial \mathcal{G}}). \qquad (1.9)$$

In case of the Heisenberg-Euler effective Lagrangian Schwinger presented the expression for the $T_{\mu\nu}$ in terms of the fine structure constant $\alpha = e^2/4\pi$ expansion:

$$T_{\mu\nu} = T_{\mu\nu}^M \Big(1 - \frac{16}{45m^4}\alpha^2 \mathcal{F}\Big) + g_{\mu\nu}\frac{2}{45m^4}\alpha^2\Big(4\mathcal{F}^2 + 7\mathcal{G}^2\Big) + ... \tag{1.10}$$

with its nonzero trace

$$T = T_{\mu\mu} = \frac{8}{45m^4}\alpha^2\Big(4\mathcal{F}^2 + 7\mathcal{G}^2\Big) + ... \tag{1.11}$$

In massless QED using the one-loop expression (1.3) for $T_{\mu\nu}$ we can get

$$T_{\mu\nu} = T_{\mu\nu}^{QED}\Big[1 - \frac{e^2}{24\pi^2}\ln\frac{2e^2\mathcal{F}}{\mu^4}\Big] + g_{\mu\nu}\frac{e^2}{24\pi^2}\mathcal{F}, \qquad \mathcal{G} = 0. \tag{1.12}$$

The $T_{\mu\nu}$ becomes proportional to the space-time metric tensor $g_{\mu\nu}$ at the extreme magnetic field $H_0^2 = H_c^2 \exp(6\pi/\alpha)$ and therefore induces a positive effective cosmological constant (see Fig.1).

To calculate the energy momentum tensor $T_{\mu\nu}$ in pure $SU(N)$ YM theory one should use the expression (1.4). Considering the QCD in the limit of chiral fermions, one should also add the quark contribution (1.5) by using the substitution $11N \to b = 11N - 2N_f$:

$$T_{\mu\nu} = T_{\mu\nu}^{QCD}\Big[1 + \frac{b\,g^2}{96\pi^2}\ln\frac{2g^2\mathcal{F}}{\mu^4}\Big] - g_{\mu\nu}\frac{b\,g^2}{96\pi^2}\mathcal{F}, \qquad \mathcal{G} = 0. \tag{1.13}$$

It is worth to compare the expressions for the energy momentum tensors in QED (1.12) and QCD (1.13). They clearly stress the physical differences between the two gauge field theories, exposed in the form of opposite signs in front of the logarithmic terms and in the front of the anomalous contributions to the energy momentum tensor traces. The last terms are proportional to the metric tensor of the space time and naturally generate the effective contribution to the cosmological constant [18, 19, 20]. These contributions are well defined and are finite physical quantities in renormalisable gauge field theories. Both expressions for the energy momentum tensors allow directly analyse the structure and the physical properties of the vacuum in respective gauge field theories [27, 28, 29, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 45, 46, 47]

Let us consider first the vacuum energy density $T_{00} \equiv \epsilon(\mathcal{F})$ which has in QCD the following form [13]:

$$\epsilon(\mathcal{F}) = \mathcal{F} + \frac{b\,g^2}{96\pi^2}\mathcal{F}\Big(\ln\frac{2g^2\mathcal{F}}{\mu^4} - 1\Big). \tag{1.14}$$

The energy density has its new minimum outside the perturbative vacuum state $G_{\mu\nu}^2(\langle A\rangle) = 0$, at Lorentz and renormalisation group invariant field strength which receives a nonzero contribution through the chromomagnetic gauge field condensate [13]

$$\langle 2g^2\mathcal{F}\rangle_{vac} = \mu^4 \exp\Big(-\frac{96\pi^2}{b\,g^2(\mu)}\Big) = \Lambda_{QCD}^4, \tag{1.15}$$

where $b = 11N - 2N_f$ and $\langle ...\rangle_{vac}$ means the average over the vacuum gauge field orientations. The Lorentz invariant form of the effective action (1.4) suggests that there are many
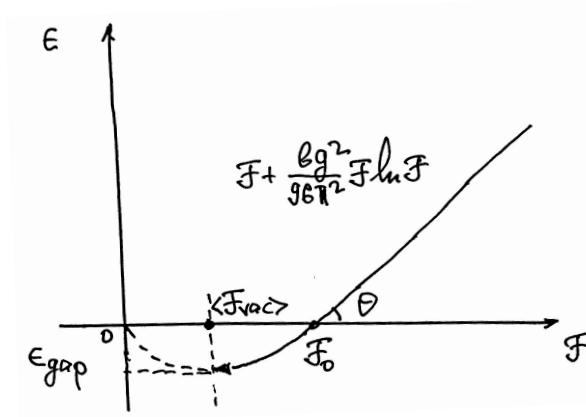
Figure 2: The graph shows the qualitative behaviour of the vacuum energy density $\epsilon(\mathcal{F})$ (1.14). At the intersection point $\mathcal{F}_0$ (2.11) the effective coupling constant is small (2.12) and the intersection angle θ is strictly positive (2.13). The energy density curve can be continuously extended from point $\mathcal{F}_0$ deep into the negative energy density region arbitrary close to the value of the vacuum condensate $\langle\mathcal{F}\rangle_{vac}$ by considering a larger values of N and keeping the t'Hooft coupling constant $g^2 N$ fixed (2.14), (2.15). This proves that there is a nonzero energy gap $\epsilon_{gap} > 0$ between perturbative and true vacuum states. The vacuum is characterised by the nonzero value of the chromomagnetic field strength tensor (1.15), (2.7) and the energy density gap ϵ_{gap} (1.17) [13, 27].

states which have the same energy density as the covariantly constant chromomagnetic field. In a series of articles [21, 22, 23, 24] the authors found and explored spatially homogeneous solutions of the YM equations which are invariant with respect to the Lorentz transformations and conveniently represent the gauge field fluctuations in the vacuum. The average $\langle...\rangle_{vac}$ can be understood as average over these gauge field configurations. The condensate generates a dynamical symmetry breaking of the scaling invariance[1]:

$$\langle T_{\mu\mu}\rangle_{vac} = -\frac{b}{48\pi^2}\langle 2g^2\mathcal{F}\rangle_{vac}.$$

Substituting the vacuum field intensity (1.15) into the expression for the energy momentum tensor (1.13) one can get that in the vacuum the tensor $T_{\mu\nu}$ is proportional to the space-time metric $g_{\mu\nu}$:

$$\langle T_{\mu\nu}\rangle_{vac} = -g_{\mu\nu}\,\frac{b}{96\pi^2}\langle g^2\mathcal{F}\rangle_{vac}. \tag{1.16}$$

In this form the energy momentum tensor represents the relativistically invariant equation of the vacuum state $\epsilon_{vac} = -P_{vac}$ and uniquely characterises the vacuum [18, 19, 20] with its negative energy density ϵ_{vac}. The vacuum energy momentum tensor (1.16) generates the effective cosmological constant Λ_{eff}

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = g_{\mu\nu}\Lambda_{eff} + \frac{8\pi G}{c^4}T_{\mu\nu} = \frac{8\pi G}{c^4}(\langle T_{\mu\nu}\rangle_{vac} + T_{\mu\nu})$$

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[1]The Λ_{QCD} is defined here through the covariant subtraction scheme (1.2). The relation with other renormalisation schemes was derived in [27].

of the form:

$$\epsilon_{vac} = \frac{c^4 \Lambda_{eff}}{8\pi G} = -\frac{b}{96\pi^2}\langle g^2\mathcal{F}\rangle_{vac} = -\frac{b}{192\pi^2}\,\Lambda^4_{QCD}\,, \tag{1.17}$$

where the chromomagnetic condensate (1.15) is $< 2g^2\mathcal{F} >_{vac} = \Lambda^4_{QCD}$. The magnetic permeability (1.8) in the vacuum state (1.15) is equal to zero:

$$\mu^{QCD}_{vac} = 1 + \frac{b\,g^2}{96\pi^2}\log\frac{\langle 2g^2\mathcal{F}\rangle_{vac}}{\mu^4} = 0. \tag{1.18}$$

The chromomagnetic condensate (1.15) is of order Λ^4_{QCD}, and the vacuum energy density is negative and is about $\epsilon_{vac} \approx -b\;10^{-8}GeV^4$. The value of the cosmological constant measured in the observation of the high-z Type Ia supernovae [48, 49, 50, 51] and by the Plank Collaboration [52, 53] $\epsilon_\Lambda = c^4\Lambda_{obser}/8\pi G \approx 10^{-47}GeV^4$ is about 39 decimal places smaller and positive. The energy gap depends on a gauge group and a matter content, the b parameter in beta function, as well as of the temperature of the universe [44]. At high temperatures the curve of the effective potential moves upward, the value of the chromomagnetic gluon condensate tends to zero, as well as the Λ_{eff}, and the scaling invariance get restored. The phase transition is of the second-order [44]. In the article [54] the authors suggested a possible cancelation mechanism between chromomagnetic and its "mirror chromoelectric" condensates.

2. Effective Lagrangian and Renormalisation Group

It is useful to derive the expression of the effective Lagrangian by using the renormalisation group equation [13, 14]. The solution of the renormalisation group equation in terms of effective coupling constant $\bar{g}(g,t)$, with the boundary condition $\bar{g}(g,0) = g$, has the following form [13, 14]:

$$\frac{\partial\mathcal{L}}{\partial\mathcal{F}} = -\frac{g^2}{\bar{g}^2(t)}, \qquad \frac{d\bar{g}}{dt} = \beta(\bar{g})\,, \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4). \tag{2.1}$$

The derivative (2.1) of the effective Lagrangian has transparent expression in terms of the effective coupling constant and allows to obtain the effective Lagrangian by integration over $\mathcal{F}$ in all order of the perturbative expansion:

$$\mathcal{L}(\mathcal{F}) = -\mu^4 \int \frac{e^{2t}}{\bar{g}^2(t)}dt\,, \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4), \tag{2.2}$$

and find out the expressions for the physical quantities beyond the one-loop approximation. One can calculate different observables of physical interest that will include the effective energy momentum tensor, vacuum energy density, the magnetic permeability, the effective coupling constants and their behaviour as a function of the external fields. In particular, the vacuum magnetic permeability introduced in (1.6) will take the following form [11]:

$$\mu_{vac} = \frac{g^2}{\bar{g}^2(t)}, \qquad \mathcal{G} = 0. \tag{2.3}$$

and the energy momentum tensor (1.9) can expressed in the form:

$$T_{\mu\nu} = -\Big(G_{\mu\lambda}G_{\nu\lambda} - g_{\mu\nu}\frac{1}{4}G_{\lambda\rho}^2\Big)\frac{g^2}{\bar{g}^2(t)} + g_{\mu\nu}\Big(\int \frac{e^{2t}}{\bar{g}^2(t)}dt - \frac{1}{2}\frac{e^{2t}}{\bar{g}^2(t)}\Big)\mu^4. \qquad (2.4)$$

The vacuum energy density in terms of the trace $T_{\mu\mu}$ is:

$$\epsilon = T_{00} = \frac{\vec{\mathcal{H}}_a^{\ 2}}{2}\frac{g^2}{\bar{g}^2(t)} + \frac{1}{4}T_{\mu\mu}, \qquad \mathcal{G} = 0, \qquad (2.5)$$

where the trace of the energy momentum tensor $T_{\mu\mu}$ is given by the following expression:

$$T_{\mu\mu} = 4\mu^4 \int \frac{e^{2t}\beta(\bar{g}(t))}{\bar{g}(t)^3}dt\ , \qquad t = \frac{1}{2}\ln(2g^2\mathcal{F}/\mu^4). \qquad (2.6)$$

The last formula provides all-loop expression for the conformal anomaly in gauge field theories[2]. As far as the beta function $\bar{\beta}(g)$ has no zeros, is negative analytical function of the coupling constant and $\int_g^\infty \frac{dg}{\beta(g)} < \infty$, the minimum of the energy density curve is defined by the extremum, where the derivative (2.1) vanishes. It follows that the value of the chromomagnetic condensate is [13]

$$\langle 2g^2\mathcal{F}\rangle_{vac} = \mu^4 \exp\Big(2\int_{g(\mu)}^\infty \frac{dg}{\beta(g)}\Big). \qquad (2.7)$$

Let us analyse the behaviour of the effective coupling constant. The Callan-Symanzik beta function can be calculated by using (2.1) and (1.4):

$$\beta = \frac{1}{2}g\frac{\partial\mathcal{M}}{\partial t}|_{t=0} = -\frac{11N}{96\pi^2}g^3 \qquad (2.8)$$

and the effective coupling constant as a function of the field takes the form

$$\bar{g}^2(\mathcal{F}) = \frac{g^2}{1 + \frac{11g^2N}{96\pi^2}\ln\frac{2g^2\mathcal{F}}{\mu^4}}, \qquad (2.9)$$

where we introduced the Casimir operator $C_2(G) = N$ for the gauge group $G = SU(N)$. Let us consider the value of the field strength tensor $\mathcal{F}_0$ at which the vacuum energy density (1.14) vanishes $\epsilon(\mathcal{F}_0) = 0$, as it is shown on Fig.2:

$$2g^2\mathcal{F}_0 = \mu^4 \exp\big(-\frac{96\pi^2}{11g^2N} + 1\big) = e\langle 2g^2\mathcal{F}\rangle_{vac}. \qquad (2.10)$$

The effective coupling constant (2.9) at this field strength has the value

$$\bar{g}^2(\mathcal{F}_0) = \frac{96\pi^2}{11N}\ . \qquad (2.11)$$

It follows that the effective coupling constant at the intersection point $\mathcal{F}_0$ is small:

$$\bar{g}^2(\mathcal{F}_0) = \frac{96\pi^2}{11N} \ll 1 \quad \text{if} \quad N \gg \frac{96\pi^2}{11}\ . \qquad (2.12)$$

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[2]If one considers the approximation in which $\bar{g}(t)$ is field independent $\bar{g}(t) \equiv g$ then (2.6) will reduce to the one given in literature [55, 56, 57, 58].

The energy density curve $\epsilon(\mathcal{F})$ (1.14) intersects the horizontal zero energy line at the nonzero angle θ (see Fig.2):

$$\tan\theta = \frac{11g^2 N}{96\pi^2} > 0. \tag{2.13}$$

This means that i) the true vacuum state is below the perturbative vacuum and that ii) there is a nonzero chromomagnetic field in the vacuum. Now the question is, how far into the infrared region one can continue the energy density curve by using the one-loop result ? Let us consider the fields which are approaching the infrared pole. This can be done, in particular, by using the following parametrisation:

$$\mathcal{F}_n = e^{1-n}\langle\mathcal{F}\rangle_{vac}, \tag{2.14}$$

where the parameter n is less than one, and we have $\mathcal{F}_n \to \langle\mathcal{F}\rangle_{vac}$ when n tends to unity from below. At these fields values the effective coupling constant (2.9) tends to zero:

$$\bar{g}^2(\mathcal{F}_n) = \frac{96\pi^2}{11N(1-n)} \to 0 \tag{2.15}$$

if the product $N(1-n) \to \infty$ is large and the t'Hooft coupling constant $g^2 N = \lambda$ is fixed and small. It follows then that the effective coupling constant can be made small in order to justify the use of the one-loop result and the behaviour of energy density curve which is continuously extended infinitesimally close to the value of the vacuum field $\langle\mathcal{F}\rangle_{vac}$, as it is shown on Fig.2. Let us analyse how the field at the intersection point (2.10) and the effective coupling constant (2.11) are changing when we include the two-loop contribution. The two-loop[3] effective Lagrangian has the form [11]

$$\mathcal{L} = -\mathcal{F} - \left(\frac{11}{6(4\pi)^2}g^2 N + \frac{34}{6(4\pi)^4}(g^2 N)^2\right)\mathcal{F}\left(\ln\frac{2g^2\mathcal{F}}{\mu^4} - 1\right). \tag{2.16}$$

The field at the intersection point (2.10) is shifted by an exponentially small correction

$$2g^2\mathcal{F}_0' = \exp\left(-\frac{96\pi^2}{11\lambda} \cdot \frac{1}{1 + \frac{17}{88\pi^2}\lambda} + 1\right). \tag{2.17}$$

At this field the effective coupling constant is smaller by the factor $1/1 + \frac{17}{88\pi^2}\lambda$

$$\bar{g}^2(\mathcal{F}_0') = \frac{96\pi^2}{11N} \cdot \frac{1}{1 + \frac{17}{88\pi^2}\lambda} \ll 1, \tag{2.18}$$

and the inequality (2.18) is fulfilled at smaller values of N than in the first approximation (2.12). The chromomagnetic condensate in the two-loop approximation will take the following form:

$$\langle 2g^2\mathcal{F}\rangle_{vac} = \mu^4 \exp\left(-\frac{1}{\beta_1 g^2}\left[1 - \frac{\beta_2 g^2}{\beta_1}\ln(1 + \frac{\beta_1}{\beta_2 g^2})\right]\right). \tag{2.19}$$

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[3]The beta function coefficients $\bar{\beta} = -\beta_1 g^3 - \beta_2 g^5 + ..$ are given by $\beta_1 = \frac{11N}{6(4\pi)^2}$ and $\beta_2 = \frac{34N^2}{6(4\pi)^4}$ [60, 61].

It is interesting to know if the energy density curve is a continuous function of the field strength $\mathcal{F}$ in the infrared region $[0, \langle \mathcal{F} \rangle_{vac}]$, which is outside of the validity of the perturbative calculations, and if the energy density curve is a convex function for all values of fields: from ultraviolet region down to infrared region. A non-perturbative functional method developed by Zwanziger in [42] is answering to these questions affirmatively. It seems that further development of his approach can shed even more light to the behaviour of the effective Lagrangian in the infrared region $[0, \langle \mathcal{F} \rangle_{vac}]$.

Let us see what can be obtained in this respect analysing the renormalisation group results. We already obtained the first derivative of the energy density curve (2.1),(2.4) and can calculate its second derivative as well:

$$\frac{\partial \epsilon}{\partial \mathcal{F}} = \frac{g^2}{\bar{g}^2}, \qquad \mathcal{F}\frac{\partial^2 \epsilon}{\partial \mathcal{F}^2} = -g^2 \; \frac{\beta(\bar{g})}{\bar{g}^3}. \tag{2.20}$$

Thus the convexity of the energy density curve is defined by its second derivative which depends on the sign of the beta function. In QCD, in the perturbative regime the $\beta(\bar{g})$ is negative and the second derivative in (2.20) is positive:

$$\mathcal{F}\frac{\partial^2 \epsilon}{\partial \mathcal{F}^2} = \frac{11}{6(4\pi)^2}g^2 N + \frac{34}{6(4\pi)^4}(g^2 N)^2 + ... \tag{2.21}$$

and the energy density curve is convex (see Fig.2). In QED the overall sign is negative and the energy density curve is concave (see Fig.1).

In summary one can conclude that any non-perturbative information about the ratio $\beta(\bar{g})/\bar{g}^3$ can be translated into the information about property of the energy density curve $\epsilon(\mathcal{F})$. As far as the beta function $\beta(g)$ has no zeros, is a negative analytical function of the coupling constant and $\int_g^\infty \frac{dg}{\beta(g)} < \infty$, then it follows from (2.20) that the energy density curve is convex and that the minimum of the energy density curve $\epsilon(\mathcal{F})$ is defined by the extremum where its first derivative vanishes.

In this presentation we discussed the phenomena of the YM condensation investigating the effective action $\Gamma[\langle A \rangle]$, a functional which depends on the vacuum expectation value of the gauge field operators $A_\mu^a(x) \equiv \langle 0|\mathcal{A}_\mu^a(x)|0\rangle$, as it was defined in (1.1). The condensate is of the chromomagnetic type and has a physical value proportional to the Λ_{QCD}^4 which is of the order of few hundred MeV^4

$$\langle 2g^2\mathcal{F}\rangle_{vac} = \langle \frac{g^2}{2}G_{\mu\nu}^2\rangle_{vac} = \langle g^2(\vec{\mathcal{H}}_a^2 - \vec{\mathcal{E}}_a^2)\rangle_{vac} = \mu^4 \exp\left(2\int_g^\infty \frac{dg}{\beta(g)}\right) = \Lambda_{QCD}^4 \; > \; 0 \tag{2.22}$$

or in terms of the strong coupling constant

$$\langle \frac{\alpha_s}{\pi}G_{\mu\nu}^2\rangle_{vac} = \langle \frac{g^2}{4\pi^2}G_{\mu\nu}^2\rangle_{vac} = \frac{\Lambda_{QCD}^4}{2\pi^2} \; . \tag{2.23}$$

It should be stressed that in our investigation we were studying the effective action $\Gamma[A]$ (1.1), which depends on the VEV of the gauge field operator $\langle 0|\mathcal{A}_\mu^a(x)|0\rangle \equiv A_\mu^a(x)$. The effective action is well defined functional, renormalisable in all order of the loop expansion and fulfils the renormalisation group equation. Its first variational derivative provides a complete quantum mechanical equations of motion $\delta\Gamma[A]/\delta A_\mu^a(x) = 0$.

3. Conclusion

The short overview of the publications devoted to the chromomagnetic gluon condensation and QCD vacuum are given below. The confinement problem from the point of view of the QCD vacuum and chromomagnetic gluon condensate was considered in the articles of Mandelshatam [69, 70, 71], Nambu [72], Adler and Piran [74] and Nielsen and Olesen [34]. The thermodynamics of the Yang-Mills gas by Linde [73]. The publication on generation of galactic magnetic field due to the condensation of vector field was considered in [67]. The induced gravity was considered by Adler [59]. The magnetostatics was considered in [68]. The phenomenology of hadrons and the properties of the QCD vacuum by Shuryak [66]. The mechanism of dynamical supersymmetry breaking and string compactification to four dimension due to the properties of the non-Abelian effective action was suggested by Veneziano and Taylor [62]. The dynamical mass generation in QCD and glueballs spectrum by Cornwall [75, 76]. The string-like solution in YM theory were considered in [63, 64, 65]. Knot-like string solutions in YM theory stabilised in the presence of the chromomagnetic condensate by Faddeev and Niemi [77, 78] and the monopole condensation by Cho [79]. A complementary approach for the construction of effective actions at different scales, the Wilsonian effective actions, was developed by Reuter and Wetterich in [80, 81, 82, 83]. The effective action in generalised YM theory was considered in [97].

Alternative approach for the investigation of the Yang-Mills condensates was suggested in [84, 85, 86, 87, 88] by performing the Monte Carlo lattice simulations. The lattice formulation is offering a non-perturbative regularisation of the YM theory and in principle allows to measure different condensates. The aims was to extract the VEV of the composite operator:

$$\langle 0| \frac{\alpha_s}{\pi} \mathcal{G}_{\mu\nu} \mathcal{G}_{\mu\nu} |0\rangle. \tag{3.1}$$

In perturbation theory this VEV is represented by the vacuum diagrams without external particles, these diagrams are diverging as the fourth power of the cutoff and in a standard renormalisation scheme are set equal to zero. As it was stressed in references [84, 85, 86, 87, 88], the main difficulty in measuring the condensates of the type (3.1) lies in the necessity to subtract the dominant and diverging perturbative contribution and then extract the exponentially falling non-perturbative term which is the only one of interest from the point of view of continuum theory. The evaluation of the VEV requires the calculation of the expression: $\lim_{a\to 0} \frac{C}{a^4} \langle 0|(1-P)|0\rangle_{meas} - \langle 0|(1-P)|0\rangle_{pert} = \langle 0|\frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}|0\rangle + ...,$ where P is the plaquette operator and dots denote the operators of higher dimension. The perturbative VEV is represented by a diverging series [89, 90, 91]: $\langle 0|(1-P)|0\rangle_{pert} = \sum_{n=0}^{\infty} c_n \alpha_s^n$. It was demonstrated in [84, 85, 86, 87, 88], that the separation of perturbative and non-perturbative contributions has arbitrariness which makes the determination of the composite operator condensate (3.1) ambiguous. In our investigation we were studying a different observable, the effective action $\Gamma[A]$, which depends on the VEV of the gauge field theory operator $A \equiv \langle 0|\mathcal{A}|0\rangle$. The effective action (1.1) is well defined functional renormalisable in all order of the perturbative loop expansion, fulfils the renormalisation group equations and providing the exact quantum mechanical field equations in terms of its first functional derivative.

The phenomena of chromomagnetic gluon condensation [13] initiated series of publications by the ITEP group where they used the gluon condensate to improve their perturbative sum rule equations [92, 93] [4]. The determination of the gluon condensate numerical value from hadronic τ decay data and from the charmonium sum rules is reviewed in [94]. The values of the condensate, extracted from QCD sum rules from experimental data, are given in Table 1 [94]. The separation of perturbative and non-perturbative contributions has arbitrariness [93] as it was pointed out by Ioffe [94, 95, 96]. The sum rule equations are defined in terms of the operator product expansion of currents that includes the VEVs of composite operators of the type $\langle 0|\mathcal{G}_{\mu\nu}^2|0\rangle$ [94]. In the actual calculations of these VEVs terms, the quark and gluon propagators $\Delta(x, y, A)$ in the background field [12] of the gluon condensate $A \equiv \langle 0|\mathcal{A}|0\rangle$ [13] are used [93, 94], that yields into the quantity $G_{\mu\nu}^2(A)$ instead of $\langle 0|\mathcal{G}_{\mu\nu}^2|0\rangle$. This has resulted in an apparent theoretical inconsistency of the sum rule equations and echoes the remarks made above concerning the VEV of composite operator $\langle 0|\mathcal{G}_{\mu\nu}^2|0\rangle$ and arbitrariness in separation of perturbative and non-perturbative contributions.

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[4]In 1977 the author gave a theoretical seminar on the chromomagnetic gluon condensation [13] in ITEP. At end of the seminar one of the participants, Victor Novikov, on our way back to the metro station by tram, remarked to the author that the theoretical prediction of the chromomagnetic condensate presented at the seminar [13] can be crucial in improving the naive sum rule equations published earlier in [92] by introducing the chromomagnetic condensate in the form of power corrections. A year later, the proposal was realised in [93].

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PoS(CORFU2019)162

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