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Symmetries of the curved momentum space compatible with κ-Minkowski

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We compute the isometry group for the curved momentum spaces compatible with $\kappa-$Minkowski following the classification introduced in [1]. Momentum spaces are associated to the non degenerated orbits of a $5D$ representation of the group manifold AN_3 generated by the $\kappa-$Minkowski algebra $\mathfrak{an}_3$. Each inequivalent momentum space belongs to one of three classes (space,light, or time-like), depending on the nature of the fiducial 5-dimensional vector used to construct the orbit. We compute the isometry group of each one of these momentum spaces as the Inönü-Wigner contraction of the global symmetry group of the embedding $5D$ space with respect to the stabilizer subgroup of the corresponding fiducial vector.

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Introduction

κ-Minkowski [2–4] is the noncommutative spacetime whose coordinates satisfy the following commutation relations

$$[x^0,x^i] = \frac{\mathrm{i}}{\kappa}x^i\,, \qquad [x^i,x^j] = 0\,, \qquad i,j = 1,\ldots,3, \tag{1}$$

where the parameter κ has the dimension of an energy (in $\hbar = 1$ units). The brackets in (1) close the Lie algebra $\mathfrak{an}_3$ [5–7] . More in general, any algebra of the form $[x^\mu,x^\nu] = \mathrm{i}(v^\mu x^\nu - v^\nu x^\mu)$, $\mu = 0,\ldots,3$, where v^μ is any set of four real numbers, is isomorphic to (1) by a linear redefinition of generators. The algebra (1) is unchanged under the left action

$$\Delta_L[x^\mu] = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes \mathbb{1}, \tag{2}$$

of the elements a^μ, Λ^μ_ν generating κ-Poincaré quantum group [2–4, 8–13]. In this sense, κ-Minkowski is the "quantum homogeneous space" of the κ-Poincaré quantum group, and the x^0 and the x^i are interpreted as the time and spatial coordinates respectively.

Interestingly, the κ-Minkowski spacetime is associated to a *curved momentum space*; as it was first discussed in [14], and then also in a variety of works [14–20]. The curvature of the momentum space can be understood in terms of the plane waves obtained by exponentianting $\mathfrak{an}_3$. Since the algebra is not abelian,the plane waves (elements of the AN_3 group) have a non linear composition rule [21, 22], which reduces to the familiar one when wave vectors are much smaller than κ [23, 24]. The Lie group theory ensures that the wave parameters can be regarded as curved coordinates over the group manifold AN_3. The above mentioned plane waves can be used to discuss field theories on κ-Minkowski [25–29]. Hence, it is legitimate to associate the momentum space of on $\kappa-$Minkowski with the group manifold AN_3. There are more then one momentum spaces compatible with the κ-Minkowski spacetime [19, 30]. In [1] it has been proposed a method to obtain and categorize these momentum spaces .

In this paper we will discuss the isometries of these momentum spaces.

1. Momentum Spaces of $\kappa-$Minkowski

It has been noticed in [14] that the algebra $\mathfrak{so}(4,1)$ has a subalgebra which is isomorphic to (1): $x^\mu \sim M_{0\mu} + M_{4\mu}$, where $M_{AB}\,(A,B = 0,\ldots,4)$ are the standard antisymmetric 5×5 matrix representation of Lorentz generators. This isomorphism induces a five dimensional representation of x^0 and x^i

$$\rho(x^0) = -\frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \hat{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}\,, \qquad \rho(x^i) = -\frac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_i & 0 \\ \mathbf{e}_i & \hat{0} & \mathbf{e}_i \\ 0 & -\mathbf{e}_i & 0 \end{pmatrix}\,, \tag{1.1}$$

where $\mathbf{e}_i^a = \delta_i^a$, three-dimensional vector quantities are in boldface, $\hat{0}$ is the zero 3×3 matrix. The (1.1) is a $*$-representation under the involution compatible with the Lorentz group $(\rho^\alpha{}_\beta)^* = \eta^{\alpha\lambda}\eta_{\gamma\beta}\overline{\rho^\gamma{}_\lambda}$. The plane waves/group elements are represented as $G^*(p_\mu) = e^{\mathrm{i}p_i\rho(x^i)}e^{\mathrm{i}p_0\rho(x^0)}$. In [1] we noticed that the non degenerate orbits of this representation are diffeomorphic to the group manifold. Given a fiducial five-dimensional vector u^A, an orbit is obtained by acting upon it obtain an five-dimensional vector with $G^*(p_\mu)$ for all choices of p^μ:

$$X^A = X^A(p_\mu) = G^*(p_\mu)^A{}_B\, u^B\,. \tag{1.2}$$

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The $X^A(p_\mu)$ are the parametric representation of a four-dimensional submanifold embedded in a five-dimensional Minkowski space, which is diffeomorphic to the momentum space (group manifold of $AN(3)$). Furthermore, $X^A X_A = X^A(p)X^B(p)\eta_{AB} = u^A u^B \eta_{AB}$ with $\eta_{AB} = \mathrm{diag}(+,-,-,-,-)$ for all $p^\mu \in \mathbb{R}^4$. The $X^A(p)$ induce on the orbit a metric

$$\mathrm{d}s^2 = -\frac{\partial X^A}{\partial p_\mu}\frac{\partial X^B}{\partial p_\nu}\eta_{AB}\mathrm{d}p_\mu \mathrm{d}p_\nu, \tag{1.3}$$

which riproduce the results in [31] for $u^A = \delta^A_4$. In this case the relation $X^0 + X^4 > 0$ is verified for all choices of p_μ, and therefore we are actually dealing with half of de Sitter spacetime. A different choice of the fiducial vector leads to a different momentum spaces associated to κ-Minkowski. In particular one has three families of inequivalent momentum spaces depending on whether u^A is space,time or light-like [1]. Furthermore, the algebra $\mathfrak{an}_3$ is also isomorphic to a sub algebra of $\mathfrak{so}(3,2)$, which induce the following representation of the x^0, x^i

$$\begin{aligned}
\rho'(x^0) &= -\tfrac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \hat{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \quad \rho'(x^1) = \tfrac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & -\mathbf{e}_1 & 0 \\ \mathbf{e}_1 & \hat{0} & \mathbf{e}_1 \\ 0 & \mathbf{e}_1 & 0 \end{pmatrix}, \\
\rho'(x^2) &= \tfrac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_2 & 0 \\ \mathbf{e}_2 & \hat{0} & \mathbf{e}_2 \\ 0 & -\mathbf{e}_2 & 0 \end{pmatrix}, \quad \rho'(x^3) = \tfrac{\mathrm{i}}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_3 & 0 \\ \mathbf{e}_3 & \hat{0} & \mathbf{e}_3 \\ 0 & -\mathbf{e}_3 & 0 \end{pmatrix}
\end{aligned} . \tag{1.4}$$

Hence the orbit-based construction of the momentum space can be recast in terms of $SO(3,2)$ group producing three more families [1,30]. In some sense, the momentum space has become fuzzy [32]. In this paper we will derive the symmetries of momentum spaces belonging to the families listed in [1].

2. Inönü-Wigner contractions and momentum symmetries

We want to study the isometry group of the momentum spaces obtained in [1]. There are three classes of momentum space each corresponding to the family of nondegenrated orbits $(AN_3)\,u$ built using a a space-like, light-like or time-like fiducial vector. The symmetries of a momentum space coincide with the symmetries of the corresponding orbit. Our idea is to obtain the symmetry group as the Inönü Wigner group contraction [33] of the global symmetry group of the embedding space with respect to the stabilizing subgroup of the fiducial vector u (little group). We consider the Lie algebra of generators L_{AB} with the following commutation relation

$$[L_{AB}, L_{CD}] = g_{AD}L_{BC} - g_{AC}L_{BD} + g_{BC}L_{AD} - g_{BD}L_{AC}\,, \tag{2.1}$$

where $g_{AB} = \mathrm{diag}(+,-,-,-,\lambda)$, and $\lambda = \pm 1$ distinguishes between the de Sitter $\mathfrak{so}(4,1)$ the anti-de Sitter $\mathfrak{so}(3,2)$ Lie algebras [34]. We split the generators L_{AB} as $L_{ij} = \varepsilon_{ijk}J_k$, $L_{0j} = K_j$, $L_{4j} = M_j$, $L_{04} = B$, hence the algebra (2.1) reads

$$\begin{aligned}
&[J_i, J_j] = \varepsilon_{ijk}J_k, \quad [J_i, M_j] = \varepsilon_{ijk}M_k, \quad [J_i, K_j] = \varepsilon_{ijk}K_k, \quad [K_i, K_j] = -\varepsilon_{ijk}J_k, \quad [M_i, M_j] = \lambda\varepsilon_{ijk}J_k, \\
&[K_i, M_j] = \delta_{ij}B, \qquad [K_i, B] = M_i, \qquad [M_i, B] = \lambda K_i, \qquad [J_i, B] = 0.
\end{aligned} \tag{2.2}$$

The group contraction will be cast as follows. Given a fiducial vector u^A, we first isolate the combination of generators L which changes u^A from those that leaves it invariant (little group). Then, we rescale them using a u^A-dependent dimensionful parameter in such a way that the algebra will change in a singular way. We finalize the contraction blowing up the parameter. All that remains, will be the Lie algebra generating the symmetry group of the momentum space associated with u^A.

$SO(4,1)$ **Space-like fiducial vector:** we consider the space-like fiducial vector $v_1^A = (0,0,0,0,\alpha)$. The little group that stabilizes v is generated by the J_i's and the K_i's. On the other hand v is changed by the M_i's and B. Then, following the contraction mechanism, we rescale these last two generators as $P_0 = B/\alpha$, and $P_i = M_i/\alpha$, where α is the only non-vanishing component of the fiducial vector v. Sending $\alpha \to \infty$ the algebra (2.2) becomes

$$\begin{aligned}
&[J_i,J_j] = \varepsilon_{ijk}J_k, \quad [J_i,P_j] = \varepsilon_{ijk}P_k, \quad [J_i,K_j] = \varepsilon_{ijk}K_k, \quad [K_i,K_j] = -\varepsilon_{ijk}J_k, \quad [P_i,P_j] = 0,\\
&[K_i,P_j] = \delta_{ij}P_0, \quad [K_i,P_0] = P_i, \quad [M_i,P_0] = 0, \quad [J_i,P_0] = 0.
\end{aligned} \tag{2.3}$$

Not surprisingly we obtain the Poincaré algebra $\mathfrak{iso}(3,1)$. Indeed, since the orbit of the dS group acting on the fiducial vector $(0,0,0,0,\alpha)$ gives an half de Sitter hyperboloid oriented along the temporal axis, it looks like Minkowski space-time, whose isometry group is $ISO(3,1)$, in a neighbourhood of the fiducial vector.

$SO(4,1)$ **Light-like fiducial vector:** this time we consider the light-like fiducial vector $v_2^A = (\beta,0,0,0,\beta)$ is lightlike whose stabilizing subgroup is generated by the L_{ij} (*i.e.* J_i) and the $N_i^+ := K_i + M_i$. It is changed by the action of B and $N_i^- := K_i - M_i$, thus we rescale those elements of $\mathfrak{so}(4,1)$ as $Q_0 = B/\beta$, and $Q_i = N_i^-/\beta = (K_i - M_i)/\beta$. Then, by sending $\beta \to \infty$ in (2.2) we get

$$\begin{aligned}
&[J_i,J_j] = \varepsilon_{ijk}J_k, \quad [J_i,N_j^+] = \varepsilon_{ijk}N_k^+, \quad [J_i,Q_j] = \varepsilon_{ijk}Q_k, \quad [N_i^+,N_j^+] = 0, \quad [N_i^+,Q_j] = -2\delta_{ij}Q_0,\\
&[Q_i,Q_j] = 0, \quad [Q_i,Q_0] = 0, \quad [N_i^+,Q_0] = 0, \quad [J_i,Q_0] = 0.
\end{aligned} \tag{2.4}$$

The brackets in (2.4) define the Lie algebra $\mathfrak{carr}(3,1)$ of the Carroll group [35–39], in which J_i and Q_i are interpreted as spatial rotation and translation generators respectively, N_i^+ plays the role of Carrollian boost and Q_0 is the time translation generator. Indeed, the orbit of a light-like fiducial vector is just the future-oriented fold of the light cone, and the induced metric has one zero eigenvalue (and the other eigenvalues have all the same sign) [1]. The Carroll group Carr$(3,1)$ can be defined as the inhomogeneous group associated to those boost which independently preserve the two metrics $\eta_{\mu\nu} = \mathrm{diag}(0,1,1,1)$ and $\eta^{\mu\nu} = \mathrm{diag}(1,0,0,0)$[1].

$SO(4,1)$ **Time-like fiducial vector:** the only case left is that of a time-like vector, say $v_3^A = (\gamma,0,0,0,0)$. Such a vector is stabilized by the little group of generators L_{ij} (the spatial rotations J_i) and $L_{4i} = M_i$ while it is changed by the action of $L_{0i} = K_i$ and $L_{04} = B$. Repeating the now familiar procedure, we introduce $T_i = K_i/\gamma$, $T_0 = B/\gamma$, and the algebra $\mathfrak{so}(4,1)$ and sending $\gamma \to \infty$ we get

$$\begin{aligned}
&[J_i,J_j] = \varepsilon_{ijk}J_k, \quad [J_i,M_j] = \varepsilon_{ijk}M_k, \quad [J_i,T_j] = \varepsilon_{ijk}T_k, \quad [T_i,T_j] = 0, \quad [M_i,M_j] = \varepsilon_{ijk}J_k,\\
&[T_i,M_j] = \delta_{ij}T_0, \quad [T_i,T_0] = 0, \quad [M_i,T_0] = T_i, \quad [J_i,T_0] = 0
\end{aligned} \tag{2.5}$$

The Lie algebra above algebra generates the Euclidean group in four dimensions $\mathfrak{iso}(4)$ [19], with T_μ as translation generators and J_i, M_j as $SO(4)$ generators. Indeed, the orbit of the dS group generated by v_3^A is one of the sheets of the two-sheeted hyperboloid aligned along the X^0 axis [1]. In fact the hyperboloid looks like the Euclidean plane $\mathbb{R}^4$ near it axis.

[1]The name Carroll is a reference to the author of the famous novel *Trough the Looking-glass* [37] because the Carolliann time somehow fits the description of time given to Alice by the Red Queen.

$SO(3,2)$ **Space-like fiducial vector:** now, we switch to the AdS group $SO(3,1)$ (generated by algebra (2.2) with $\lambda=-1$). When discussing the contraction of AdS we will adopt the following nomenclature

$$J_3 = I_{12}, \quad B = I_{34}, \quad M_1 = I_{41},\, M_2 = I_{42}, \quad K_1 = I_{31}, \quad K_2 = I_{32}. \tag{2.6}$$

Consider the little group of the space-like fiducial vector $w_1^A = (0,0,0,\alpha,0)$. Its stabilizer is generated by $L_{12} = J_3$, $L_{04} = B$, $L_{41} = M_1$, $L_{42} = M_2$, $L_{01} = K_1$ and $L_{02} = K_2$. Then, we have to rescale $U_a = J_a/\alpha$, $U_3 = K_3/\alpha$, $U_4 = M_3/\alpha$. Taking the limit $\alpha \to \infty$ in (2.2) we get:

$$[I_{\alpha,\beta}, I_{\gamma\delta}] = \gamma_{\alpha\delta} I_{\beta\gamma} - \gamma_{\alpha\gamma} I_{\beta\delta} + \gamma_{\beta\gamma} I_{\alpha\delta} - \gamma_{\beta\delta} I_{\alpha\gamma}, \quad [I_{\alpha,\beta}, U_\gamma] = \gamma_{\alpha\gamma} U_\beta - \gamma_{\beta\gamma} U_\alpha, \quad [U_\alpha, U_\beta] = 0, \tag{2.7}$$

where $\gamma_{\alpha\beta} = \mathrm{diag}(-,-,+,+)$, and the greek indices range from 1 to 4. The contracted algebra (2.7) is $\mathfrak{iso}(2,2)$, describing the isometries of a flat space of signature $(2,2)$. This is a hyperplane tangent in the fiducial vector to the orbit of w_1^A (a two-sheeted hyperboloid around the axis 3) [1].

$SO(3,2)$ **Light-like fiducial vector:** we choose $w_2^A = (0,0,0,\beta,\beta)$ as a fiducial light-like vector this time . The isotropy subgroup is generated by $L_{01} =$, L_{02}, L_{12}, (which close a $\mathfrak{so}(2,1)$ subalgebra), and $L_{03} + L_{04} = K_3 + B = N_0$, $L_{13} + L_{14} = -J_2 - M_1 = N_1$ and $L_{23} + L_{24} = J_1 - M_2 = N_2$. The generators that change w_2^A are $L_{03} - L_{04} = K_3 - B$, $L_{13} - L_{14} = -J_2 + M_1$, $L_{23} - L_{24} = J_1 + M_2$ and $L_{34} = B$. Hence, we rescale $V_0 = (K_3 - B)/\beta, V_1 = (M_1 - J_2)/\beta, V_2 = (J_1 + M_2)/\beta$ and $V_3 = B/\beta$, and the (2.2) become

$$\begin{aligned} &[V_\mu, V_\rho] = 0,\ [V_\mu, N_\rho] = 0,\ [N_\rho, N_\sigma] = 0,\ [L_{\rho\sigma}, V_3] = 0,\ [L_{\rho\sigma}, V_\tau] = h_{\rho\tau} V_\sigma - h_{\sigma\tau} V_\rho, \\ &[L_{\rho\sigma}, N_\tau] = h_{\rho\tau} N_\sigma - h_{\sigma\tau} N_\rho,\ [L_{\rho\sigma}, L_{\tau\lambda}] = h_{\rho\lambda} L_{\sigma\tau} - h_{\rho\tau} L_{\sigma\lambda} + h_{\sigma\tau} L_{\rho\lambda} - h_{\sigma\lambda} L_{\rho\tau}, \end{aligned} \tag{2.8}$$

where $\rho, \sigma, \tau, \lambda, ... = 0,1,2$ and $h_{\rho\sigma} = \mathrm{diag}(-1,1,1)$. This is the algebra $\mathfrak{carr}(2,2)$ which it generates the Carroll group in which one of the space-like axes has changed signature. These are the isometries of a light-like hyperplane in a flat spacetime of signature $(2,2)$, which the tangent space at w_2^A of the orbit of the AdS group generated by w_2^A [1].

$SO(3,2)$ **Time-like fiducial vector:** we consider $w_3^A = (\gamma,0,0,0,0)$ which is a time-like fiducial vector for the $\lambda = -1$ metric. Its stabilizer is generated by L_{ij} (the spatial rotations J_i) and $L_{4i} = M_i$. The fiducial vector is change by $L_{0i} = K_i$ and $L_{04} = B$ instead. Introducing $S_i = K_i/\gamma$ and $S_0 = B/\gamma$,and sending $\gamma \to \infty$ we get

$$\begin{aligned} &[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, M_j] = \varepsilon_{ijk} M_k, \quad [J_i, S_j] = \varepsilon_{ijk} S_k, \quad [S_i, S_j] = 0, \quad [M_i, M_j] = -\varepsilon_{ijk} J_k, \\ &[M_j, S_i] = -\delta_{ij} S_0, \quad [S_i, S_0] = 0, \quad [M_i, S_0] = -S_i, \quad [J_i, S_0] = 0 \quad . \end{aligned} \tag{2.9}$$

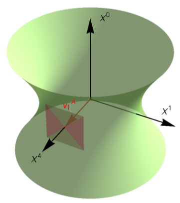
This is the Poincaré algebra $\mathfrak{iso}(3,1)$. Indeed, the orbit associated with w_3^A is a one-sheeted hyperboloid, with rotational symmetry in the $0-4$ plane . Near the 0 axis, this looks like Minkowski space-time.

3. Conclusions

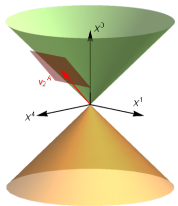
We developed a method to compute the isometry group of the momentum spaces compatible with κ-Minkowski spacetime. Inspired by the orbit-based classification introduced in [1], the isometry group has been computed as the contraction of $SO(4,1)$ (or $SO(3,2)$) with respect to the little group of the fiducial vector whose orbit correspond to the desired momentum space. The case with $\mathfrak{iso}(3,1)$ symmetry and the one with $\mathfrak{iso}(2,2)$ (see Table 1) symmetry are compatible with results in literature [19, 30]. On the other hand, the 4D-Euclidean case $\mathfrak{iso}(4)$ as well as the $\mathfrak{carr}(3,1)$, and $\mathfrak{carr}(2,2)$ are genuinely new. Since duality between the Caroll group and the Galilei group has both physical and mathematical implication [38, 40], the presence momentum spaces with Carollian symmetry group trills the authors curiosity. In conclusion, we have some new (momentum) spaces to be explored, whose physical interpretation may give new and unexpected application of $\kappa-$Minkowski space-time.

Fiducial Vector	Group	Contracted Algera	Group	Contracted Algera
space-like		$\mathfrak{iso}(3,1)$		$\mathfrak{iso}(2,2)$
light-like	$SO(4,1)$	$\mathfrak{carr}(3,1)$	$SO(3,2)$	$\mathfrak{carr}(2,2)$
time-like		$\mathfrak{iso}(4)$		$\mathfrak{iso}(3,1)$

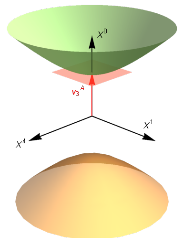
Table 1: *In the first column we report the nature of the fiducial vector. In the next two couples of column we report the isometry group associated to the momentum spaces in the class selected by the fiducial vector in the case of an embedding space symmetric under $SO(4,1)$ or $SO(3,2)$.*



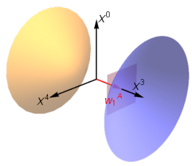
(a) *Orbit of the dS group generated by a spacelike fiducial vector.*



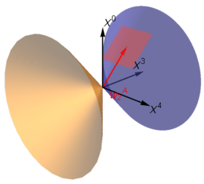
(b) *Orbit of the dS group generated by a lightlike fiducial vector.*



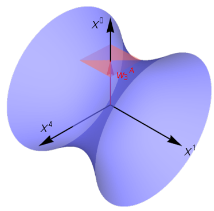
(c) *Orbit of the dS group generated by a timelike fiducial vector.*



(a) *Orbit of the AdS group generated by a spacelike fiducial vector.*



(b) *Orbit of the AdS group generated by a lightlike fiducial vector.*



(c) *Orbit of the AdS group generated by a timelike fiducial vector.*

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