

Invariant Differential Operators for the Real Exceptional Lie Algebra F_4'

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In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact exceptional Lie algebra F_4' which is split real form of the exceptional Lie algebra F_4 . We consider induction from a maximal parabolic algebra. We classify the reducible Verma modules over F_4 which are compatible with this induction. Thus, we obtain the classification of the corresponding invariant differential operators.

*Corfu Summer Institute 2019 "School and Workshops on Elementary Particle Physics and Gravity"
(CORFU2019)
31 August - 25 September 2019
Corfu, Greece*

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1. Introduction

Invariant differential operators play very important role in the description of physical symmetries. The general scheme for constructing these operators was given some time ago [1]. In recent papers [2,3] we started the systematic explicit construction of the invariant differential operators.

The first task in the construction is to make the multiplet classification of the reducible Verma modules over the algebra in consideration following [4]. Such classification provides the weights of embeddings between the Verma modules via the singular vectors, and thus, by [1], the weights of the invariant differential operators.

We have done the multiplet classification for many real non-compact algebras, first from the class of algebras that have discrete series representations, see [5]. In the present paper we focus on the complex exceptional Lie algebra F_4 and on its split real form algebra F_4' . Our scheme requires that we use induction from parabolic subalgebras. In the present paper we choose a parabolic subalgebra containing the factor $\mathcal{M} \oplus \mathcal{A}$, where $\mathcal{M} = sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$, $\mathcal{A} = o(2)$. This choice is motivated by the fact that the complexification of $\mathcal{M} \oplus \mathcal{A}$ and the corresponding compact form $\mathcal{M}_c \oplus \mathcal{A}_c = su(3) \oplus su(2) \oplus u(1)$ have applications in physics being the Lie algebra symmetry of the standard model of elementary particles [6] (see also [7]).¹

We present the multiplet classification of the reducible Verma modules over F_4 which are compatible with the chosen parabolic of F_4' . We give also the weights of the singular vectors between these modules. By the scheme of [1] these singular vectors will produce the invariant differential operators.

2. Preliminaries

2.1 Lie algebra

We start with the complex exceptional Lie algebra $\mathcal{G}^{\mathbb{C}} = F_4$. We use the standard definition of $\mathcal{G}^{\mathbb{C}}$ given in terms of the Chevalley generators X_i^{\pm} , H_i , $i = 1, 2, 3, 4 (= \text{rank } F_4)$, by the relations :

$$\begin{aligned} [H_j, H_k] &= 0, [H_j, X_k^{\pm}] = \pm a_{jk} X_k^{\pm}, [X_j^+, X_k^-] = \delta_{jk} H_j, \\ \sum_{m=0}^n (-1)^m \binom{n}{m} (X_j^{\pm})^m X_k^{\pm} (X_j^{\pm})^{n-m} &= 0, j \neq k, n = 1 - a_{jk}, \end{aligned} \quad (2.1)$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}; \quad (2.2)$$

is the Cartan matrix of $\mathcal{G}^{\mathbb{C}}$, $\alpha_j^{\vee} \equiv \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$ is the co-root of α_j , (\cdot, \cdot) is the scalar product of the roots, so that the nonzero products between the simple roots are: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2(\alpha_3, \alpha_3) = 2(\alpha_4, \alpha_4) = 2$, $(\alpha_1, \alpha_2) = -1$, $(\alpha_2, \alpha_3) = -1$, $(\alpha_3, \alpha_4) = -1/2$. The elements H_i span the

¹More precisely the symmetry is presented on the group level as : $G = SU(3) \times SU(2) \times U(1)/Z$, where Z belongs to the center of G .

Cartan subalgebra \mathcal{H} of $\mathcal{G}^{\mathbb{C}}$, while the elements X_i^{\pm} generate the subalgebras \mathcal{G}^{\pm} . We shall use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_-, \quad \mathcal{G}_{\pm} \equiv \bigoplus_{\alpha \in \Delta^{\pm}} \mathcal{G}_{\alpha}, \quad (2.3)$$

where Δ^+ , Δ^- , are the sets of positive, negative, roots, resp. Explicitly we have that there are roots of two lengths with length ratio 2 : 1.

The long roots are: $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. With the chosen normalization they have length 2.

The short roots are: $\alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$, and they have length 1.

(Note that the short roots are exactly those which contain α_3 and/or α_4 with odd coefficient, while the long roots contain α_3 and α_4 with even coefficients.)

Thus, F_4 is 52-dimensional ($52 = |\Delta| + \text{rank } F_4$).

In terms of the normalized basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ we have:

$$\Delta^+ = \{ \varepsilon_i, 1 \leq i \leq 4; \varepsilon_j \pm \varepsilon_k, 1 \leq j < k \leq 4; \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4), \text{ all signs} \}. \quad (2.4)$$

The simple roots are:

$$\pi = \{ \alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \}. \quad (2.5)$$

The maximal compact subalgebra is $\mathcal{K} = sp(3) \oplus su(2)$. Its complexification $\mathcal{K}^{\mathbb{C}}$ may be embedded most easily in F_4 as the Lie algebra generated by the subalgebras with simple roots $\{\alpha_2, \alpha_3, \alpha_4\}$ and $\{\alpha_1\}$. The long roots of $sp(3, \mathbb{C})$ in this embedding are: $\alpha_2, \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4$. The short roots are: $\alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4$.

Note that the 16 roots on the first line of (2.4) form the positive root system of B_4 with simple roots $\varepsilon_i - \varepsilon_{i+1}, i = 1, 2, 3, \varepsilon_4$.

The Weyl group of F_4 is the semidirect product of S_3 with a group which itself is the semidirect product of S_4 with $(\mathbb{Z}/2\mathbb{Z})^3$, thus, $|W| = 2^7 3^2 = 1152$.

2.2 Verma modules

Let us recall that a *Verma module* V^{Λ} is defined as the HWM over $\mathcal{G}^{\mathbb{C}}$ with highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^{\Lambda}$, induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(\mathcal{B})$, where $\mathcal{B} = \mathcal{H} \oplus \mathcal{G}_+$ is a Borel subalgebra of $\mathcal{G}^{\mathbb{C}}$, such that:

$$\begin{aligned} X v_0 &= 0, \quad \forall X \in \mathcal{G}_+ \\ H v_0 &= \Lambda(H) v_0, \quad \forall H \in \mathcal{H} \end{aligned} \quad (2.6)$$

Verma modules are generically irreducible. A Verma module V^{Λ} is reducible [8] iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$(\Lambda + \rho, \beta^{\vee}) = m \quad (2.7)$$

holds, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, ($\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4$).

If (2.7) holds then the reducible Verma module V^Λ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda - m\beta$. This statement is equivalent to the fact that V^Λ contains a *singular vector* $v_s \in V^\Lambda$, such that $v_s \neq \xi v_0$, ($0 \neq \xi \in \mathbb{C}$), and :

$$\begin{aligned} X v_s &= 0, \quad \forall X \in \mathcal{G}_+ \\ H v_s &= \Lambda'(H) v_s, \quad \Lambda' = \Lambda - m\beta, \quad \forall H \in \mathcal{H} \end{aligned} \quad (2.8)$$

More explicitly, [1],

$$v_{m,\beta}^s = \mathcal{P}_{m,\beta} v_0. \quad (2.9)$$

The general reducibility conditions (2.7) for V^Λ spelled out for the simple roots in our situation are:

$$\begin{aligned} m_1 \equiv m_{\alpha_1} &= (\Lambda + \rho, \alpha_1), & m_2 \equiv m_{\alpha_2} &= (\Lambda + \rho, \alpha_2), \\ m_3 \equiv m_{\alpha_3} &= (\Lambda + \rho, 2\alpha_3), & m_4 \equiv m_{\alpha_4} &= (\Lambda + \rho, 2\alpha_4) \end{aligned} \quad (2.10)$$

The numbers m_i from (2.10) corresponding to the simple roots are called Dynkin labels, while the more general Harish-Chandra parameters are:

$$m_\beta = (\Lambda + \rho, \beta^\vee), \quad \beta \in \Delta^+ \quad (2.11)$$

2.3 Structure theory of the real form

The split real form of F_4 is denoted as F_4' , sometimes as $F_{4(4)}$. This real form has discrete series representations since $\text{rank} F_4' = \text{rank} \mathcal{K}$. We can use the same basis (but over \mathbb{R}) and the same root system.

The Iwasawa decomposition of the real split form $\mathcal{G} \equiv F_4'$, is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_0 \oplus \mathcal{N}_0, \quad (2.12)$$

the Cartan decomposition is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}, \quad (2.13)$$

where we use: the maximal compact subgroup $\mathcal{K} \cong sp(3) \oplus su(2)$, $\dim_{\mathbb{R}} \mathcal{Q} = 28$, $\dim_{\mathbb{R}} \mathcal{A}_0 = 4$, $\mathcal{N}_0 = \mathcal{N}_0^+$, or $\mathcal{N}_0 = \mathcal{N}_0^- \cong \mathcal{N}_0^+$, $\dim_{\mathbb{R}} \mathcal{N}_0^\pm = 24$.

Since \mathcal{G} is maximally split, then the centralizer \mathcal{M}_0 of \mathcal{A}_0 in \mathcal{K} is zero, thus, the minimal parabolic \mathcal{P}_0 and the corresponding Bruhat decomposition are:

$$\mathcal{P}_0 = \mathcal{A}_0 \oplus \mathcal{N}_0, \quad \mathcal{G} = \mathcal{A}_0 \oplus \mathcal{N}_0^+ \oplus \mathcal{N}_0^- \quad (2.14)$$

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of the group under consideration [9–11].

We recall that in general a parabolic subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ is any subalgebra of \mathcal{G} which contains a minimal parabolic subalgebra \mathcal{M}_0 . In general, \mathcal{M} contains the subalgebra \mathcal{M}_0 , while \mathcal{A} is contained in \mathcal{A}_0 , \mathcal{N} is contained in \mathcal{N}_0 .

On the other extreme are the maximal parabolic subalgebras for which $\dim \mathcal{A} = 1$.

2.4 Elementary representations

Further, let G, K, P, M, A, N be Lie groups with Lie algebras $\mathcal{G}_0, \mathcal{H}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$.

Let ν be a (non-unitary) character of A , $\nu \in \mathcal{A}^*$. Let μ fix a finite-dimensional (non-unitary) representation D^μ of M on the space V_μ . In the case when M is cuspidal then we may use also the discrete series representation of M with the same Casimirs as D^μ . (We ignore a possible discrete center of M since its representations are not relevant for the construction of invariant differential operators [5].)

We call the induced representation $\chi = \text{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [12]. (These are called *generalized principal series representations* (or *limits thereof*) in [13].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \} \quad (2.15)$$

where $a = \exp(H) \in A$, $H \in \mathcal{A}$, $m \in M$, $n \in N$. The representation action is the *left regular action*:

$$(\mathcal{I}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \quad (2.16)$$

An important ingredient in our considerations are the highest/lowest weight representations of \mathcal{G} . These can be realized as (factor-modules of) Verma modules V^Λ over \mathcal{G} , where $\Lambda \in (\mathcal{H})^*$, the weight $\Lambda = \Lambda(\chi)$ being determined uniquely from χ [1].

As we have seen when a Verma module is reducible and (2.7) holds then there is a singular vector (2.9). Relatedly, then there exists [1] an *invariant differential operator*

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \quad (2.17)$$

given explicitly by:

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{N}^-}) \quad (2.18)$$

where $\widehat{\mathcal{N}^-}$ denotes the *right action* on the functions \mathcal{F} .

Actually, since our ERs are induced from finite-dimensional representations of \mathcal{M} the corresponding Verma modules are always reducible. Thus, it is more convenient to use *generalised Verma modules* \tilde{V}^Λ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu v_0$.

Algebraically, the above is governed by the notion of \mathcal{M} -compact roots of $\mathcal{G}^\mathbb{C}$. These are the roots of $\mathcal{G}^\mathbb{C}$ which can be identified as roots of $\mathcal{M}^\mathbb{C}$ as the latter is a subalgebra of $\mathcal{G}^\mathbb{C}$. The consequence of this is that (2.7) is always fulfilled for the \mathcal{M} -compact roots of $\mathcal{G}^\mathbb{C}$. That is why we consider generalised Verma modules. Relatedly, the invariant differential operators corresponding to \mathcal{M} -compact roots are trivial.

3. Invariant differential operators for F_4'

The real form F_4' has several parabolic subalgebras [2]. We shall consider the maximal parabolic subalgebra [2]:

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \\ \mathcal{M} &= sl(3, \mathbb{R}) \oplus (2, \mathbb{R}), \\ \dim \mathcal{A} &= 1, \quad \dim \mathcal{N} = 20 \end{aligned} \quad (3.1)$$

such that the embedding of \mathcal{M} and $\mathcal{M}^{\mathbb{C}}$ in $\mathcal{G}^{\mathbb{C}}$ is given by:

$$sl(3, \mathbb{R})^{\mathbb{C}} : \{\alpha_1, \alpha_2, \alpha_{12} = \alpha_1 + \alpha_2\}, \quad sl(2, \mathbb{R})^{\mathbb{C}} : \{\alpha_4\} \quad (3.2)$$

Remark: Note that F_4' has a another maximal parabolic subalgebra that is also written as (3.1) but the embedding of \mathcal{M} and $\mathcal{M}^{\mathbb{C}}$ flips the short and long roots [2]:

$$sl(3, \mathbb{R})^{\mathbb{C}} : \{\alpha_3, \alpha_4, \alpha_{34} = \alpha_3 + \alpha_4\}, \quad sl(2, \mathbb{R})^{\mathbb{C}} : \{\alpha_1\} \quad (3.3)$$

That case is also very interesting and will be considered next [14]. \diamond

Further we classify the generalized Verma modules (GVM) relative to the maximal parabolic subalgebra (3.1). This also provides the classification of the P -induced ERs with the same Casimirs. The classification is done as follows. We group the reducible Verma modules (also the corresponding ERs) related by nontrivial embeddings in sets called *multiplets* [1, 4]. These multiplets may be depicted as a connected graph, the vertices of which correspond to the GVMs and the lines between the vertices correspond to the GVM embeddings (and also the invariant differential operators between the ERs). The explicit parametrization of the multiplets and of their Verma modules (and ERs) is important for understanding of the situation.

The result of our classification is as follows. The multiplets of GVMs (and ERs) induced from (3.1) are parametrized by four positive integers $\chi = [m_1, m_2, m_3, m_4]$. Each multiplet contains 96 GVMs (ERs). They are presented in Fig. 1.

On the figure each arrow represents an embedding between two Verma modules, V^{Λ} and $V^{\Lambda'}$, the arrow pointing to the embedded module $V^{\Lambda'}$. Each arrow carries a number n , $n = 1, 2, 3, 4$, which indicates the level of the embedding, $\Lambda' = \Lambda - m_n \beta$ [5]. Another feature is indicated by the enumeration of the GVMs (ERs). Namely, if Λ corresponds to signature $\chi_{k, \ell}^-$, $k < 9$, then Λ' corresponds to signature $\chi_{k+1, \ell'}^-$ (where ℓ, ℓ' are secondary enumerations that are absent in some cases). Analogously: if Λ corresponds to signature $\chi_{k, \ell}^+$, $k > 10$, then Λ' corresponds to signature $\chi_{k-1, \ell'}^+$; if Λ corresponds to signature $\chi_{9, \ell}^-$, then Λ' corresponds to signature $\chi_{10, \ell'}^*$, (where $*$ may happen to be $' + '$ or $' - '$); if Λ corresponds to signature $\chi_{10, \ell}^*$, then Λ' corresponds to signature $\chi_{9, \ell'}^+$.

Further, we can see there is an additional symmetry. It is relevant for the ERs and indicates the integral intertwining Knapp-Stein (KS) operators acting between the ERs. Due to this symmetry in the actual parametrization we shall use the conformal weight $d = 7/2 + c$, more precisely, the parameter c , instead of the non-compact Dynkin label m_3 . The parameter c is more convenient since the KS operators flip its sign. The KS operators also involve an $sl(3)$ flip of the Dynkin labels m_1, m_2 (see below). Thus, the entries are:

$$\chi = \{n_1, n_2, c, n_4\} \quad (3.4)$$

so that for the top ER (GVM) on the figure Λ_0^- we have:

$$\chi_0^- = \{n_1 = m_1, n_2 = m_2, c = -(m_1 + m_2 + m_3 + m_4/2), n_4 = m_4\} \quad (3.5)$$

Furthermore the $sl(3)$ flip $(n_1, n_2)^{\pm}$ will be given below by:

$$(n_1, n_2)^+ = (n_1, n_2), \quad (n_1, n_2)^- = (n_2, n_1) \quad (3.6)$$

Altogether, the explicit parametrization of the multiplets is given by:

$$\begin{aligned}
\chi_0^\mp &= \{ (m_1, m_2)^\pm, \mp(m_1 + m_2 + m_3 + m_4/2), m_4 \}, \\
\chi_{1,1}^\mp &= \{ (m_1, m_{23})^\pm, \mp(m_{12} + m_{34}/2), m_{34} \}, \\
\chi_{2,1}^\mp &= \{ (m_{12}, m_{23})^\pm, \mp(m_{12} + m_{34}/2), m_{24} + m_2 \}, \\
\chi_{2,2}^\mp &= \{ (m_1, m_{24})^\pm, \mp(m_{12} + m_3/2), m_3 \}, \\
\chi_3^\mp &= \{ (m_{12}, m_{24})^\pm, \mp(m_{12} + m_3/2), m_{23} + m_2 \}, \\
\chi_{3,1}^\mp &= \{ (m_2, m_{13})^\pm, \mp(m_2 + m_{34}/2), m_{14} + m_{12} \}, \\
\chi_{3,2}^\mp &= \{ (m_{13}, m_2)^\pm, \mp(m_{13} + m_4/2), m_{24} + m_{23} \}, \\
\chi_{4,1}^\mp &= \{ (m_{14}, m_2)^\pm, \mp(m_2 + m_3/2), m_{13} + m_{12} \}, \\
\chi_{4,2}^\mp &= \{ (m_{13}, m_{24})^\pm, \mp(m_{12} + m_3/2), m_2 + m_{23} \}, \\
\chi_{4,3}^\mp &= \{ (m_{23}, m_{12}^\pm, \mp(m_{23} + m_4/2), m_{14} + m_{13} \}, \\
\chi_{4,4}^\mp &= \{ (m_{14}, m_2)^\pm, \mp(m_{13} + m_4/2), m_{24} + m_{23} \}, \\
\chi_5^\mp &= \{ (m_{23}, m_{14})^\pm, \mp(m_2 + m_3/2), m_{13} + m_{12} \}, \\
\chi_{5,1}^\mp &= \{ (m_{13} + m_2, m_{24})^\pm, \mp(m_{12} + m_3/2), m_3 \}, \\
\chi_{5,2}^\mp &= \{ (m_{14}, m_{23})^\pm, \mp(m_{12} + m_{34}/2), m_{24} + m_2 \}, \\
\chi_{5,3}^\mp &= \{ (m_{24}, m_{12})^\pm, \mp(m_{23} + m_4/2), m_{14} + m_{13} \}, \\
\chi_{5,4}^\mp &= \{ (m_{23}, m_1)^\pm, \mp(m_{23} + m_4/2), m_{14} + m_{13} + 2m_2 \}, \\
\chi_{6,1}^\mp &= \{ (m_{23}, m_{14} + m_2)^\pm, \mp(m_3/2), m_{13} + m_{12} \}, \\
\chi_{6,2}^\mp &= \{ (m_{24}, m_{13})^\pm, \mp(m_2 + m_{34}/2), m_{14} + m_{12} \}, \\
\chi_{6,3}^\mp &= \{ (m_{14} + m_2, m_{23})^\pm, \mp(m_{12} + m_{34}/2), m_{34} \}, \\
\chi_{6,4}^\mp &= \{ (m_{13} + m_2, m_{14})^\pm, \mp(m_2 + m_3/2), m_3 \}, \\
\chi_{6,5}^\mp &= \{ (m_{24}, m_{13})^\pm, \mp(m_{23} + m_4/2), m_{14} + m_{13} + 2m_2 \}, \\
\chi_{6,6}^\mp &= \{ (m_{24}, m_{13})^\pm, \mp(m_2 + m_{34}/2), m_{14} + m_{13} + m_{23} + m_2 \}, \\
\chi_7^\mp &= \{ (m_{24}, m_{13} + m_2)^\pm, \mp(m_{34}/2), m_{14} + m_{12} \}, \\
\chi_{7,1}^\mp &= \{ (m_{13}, m_{14} + m_2)^\pm, \mp(m_3/2), m_{23} + m_2 \}, \\
\chi_{7,2}^\mp &= \{ (m_2, m_{14} + m_{23})^\pm, \pm(m_3/2), m_{13} + m_{12} \}, \\
\chi_{7,3}^\mp &= \{ (m_{24}, m_1)^\pm, \mp(m_2 + m_{34}/2), m_{14} + m_{13} + m_{23} + m_2 \}, \\
\chi_{7,4}^\mp &= \{ (m_{14} + m_{23}, m_2)^\pm, \mp(m_{13} + m_4/2), m_4 \}, \\
\chi_{7,5}^\mp &= \{ (m_{14} + m_2, m_{13})^\pm, \mp(m_2 + m_{34}/2), m_{34} \}, \\
\chi_{7,6}^\mp &= \{ (m_2, m_1)^\pm, \mp(m_2 + m_{34}/2), 2m_{14} + m_{23} + m_2 \}, \\
\chi_{8,1}^\mp &= \{ (m_{24}, m_{13} + m_2)^\pm, \mp(m_4/2), m_{13} + m_{14} \}, \\
\chi_{8,2}^\mp &= \{ (m_{14}, m_{13} + m_2)^\pm, \mp(m_{34}/2), m_{24} + m_2 \}, \\
\chi_{8,3}^\mp &= \{ (m_{24}, m_{12})^\pm, \mp(m_{34}/2), m_{14} + m_{13} + m_{23} + m_2 \}, \\
\chi_{8,4}^\mp &= \{ (m_{23}, m_1)^\pm, \mp(m_2 + m_3/2), 2m_{14} + m_{23} + m_2 \}, \\
\chi_{8,5}^\mp &= \{ (m_{14} + m_{23}, m_{12})^\pm, \mp(m_{23} + m_4/2), m_4 \}, \\
\chi_{8,6}^\mp &= \{ (m_2, m_{14} + m_{23})^\pm, \pm m_{34}/2, m_{14} + m_{12} \}, \\
\chi_{8,7}^\mp &= \{ (m_{12}, m_{14} + m_{23})^\pm, \pm m_3/2, m_{23} + m_2 \},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\chi_9^\mp &= \{ (m_{14}, m_{13} + m_2)^\pm, \mp(m_4/2), m_{24} + m_{23} \}, \\
\chi_{9,1}^\mp &= \{ (m_{24}, m_{13})^\pm, \mp(m_4/2), 2m_{14} + m_{23} + m_2 \}, \\
\chi_{9,2}^\mp &= \{ (m_{14} + m_{23} + m_2, m_1)^\pm, \mp(m_{23} + m_4/2), m_4 \}, \\
\chi_{9,3}^\mp &= \{ (m_{23}, m_{14} + m_2)^\pm, \pm m_4/2, m_{14} + m_{13} \}, \\
\chi_{9,4}^\mp &= \{ (m_{14}, m_2)^\pm, \mp(m_{34}/2), m_{14} + m_{13} + m_{23} + m_2 \}, \\
\chi_{9,5}^\mp &= \{ (m_1, m_{14} + m_{23} + m_2)^\pm, \pm(m_{12} + m_3/2), m_3 \}, \\
\chi_{9,6}^\mp &= \{ (m_{12}, m_{14} + m_{23})^\pm, \pm m_{34}/2, m_{24} + m_2 \}, \\
\chi_{9,7}^\mp &= \{ (m_{23}, m_{12})^\pm, \mp(m_3/2), 2m_{14} + m_{23} + m_2 \}, \\
\chi_{10,1}^\mp &= \{ (m_{14} + m_2, m_{13})^\pm, \mp m_4/2, m_{23} + m_{24} \}, \\
\chi_{10,2}^\mp &= \{ (m_{13}, m_2)^\pm, \mp m_3/2, 2m_{14} + m_{23} + m_2 \}, \\
\chi_{10,3}^\mp &= \{ (m_{14}, m_{23})^\pm, \mp m_4/2, m_{14} + m_{13} + 2m_2 \}, \\
\chi_{10,4}^\mp &= \{ (m_1, m_{14} + m_{23} + m_2)^\pm, \pm(m_2 + m_{34}/2), m_{34} \}
\end{aligned}$$

The pairs $\Lambda^\pm(\chi^\pm)$ are symmetrically placed w.r.t. to the bullet in the middle of the figure.

4. Concluding remarks

Remark 1:

The integral intertwining KS operators act between the spaces \mathcal{C}_{χ^\mp} in opposite directions:

$$G_{KS}^+ : \mathcal{C}_{\chi^-} \longrightarrow \mathcal{C}_{\chi^+}, \quad G_{KS}^- : \mathcal{C}_{\chi^+} \longrightarrow \mathcal{C}_{\chi^-} \quad (4.1)$$

Remark 2:

The positive integers $\{m_1, m_2, m_3, m_4\}$ parametrize the finite-dimensional nonunitary irreps of F_4' (also the unitary finite-dimensional irreps of the compact Lie algebra f_4).

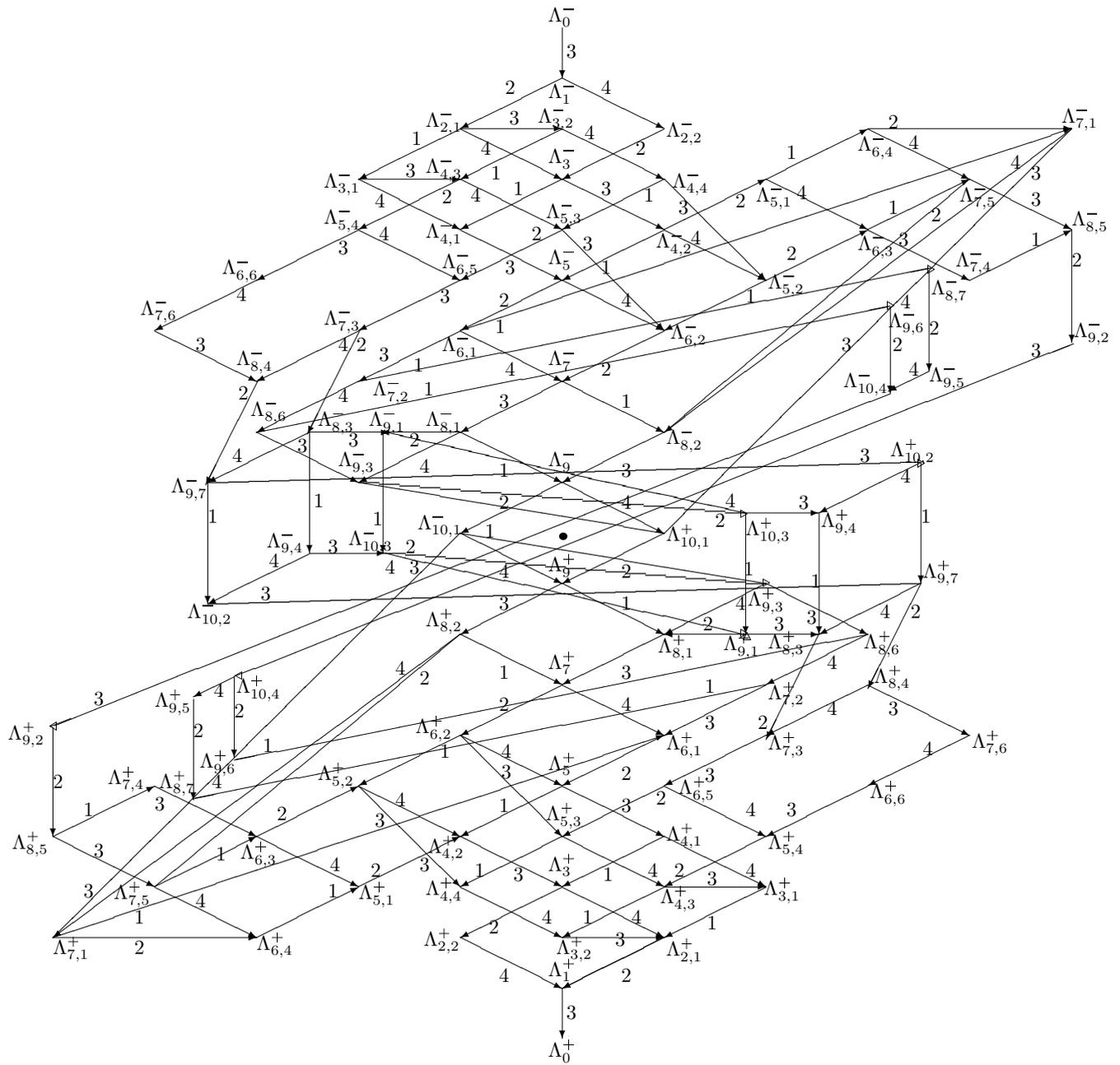
Remark 3:

We expect that the discrete series are contained in the representation χ_0^+ since it is dual to χ_0^- where are contained the finite-dimensional (non-unitary) irreps. Following the Harish-Chandra criterion we must check which \mathcal{M} -non-compact entries are negative. We recall that the \mathcal{M} -compact entries are $m'_1, m'_2, m'_{12}, m'_4$, all other are non-compact. It is easy to see that all the \mathcal{M} -non-compact entries are negative. The discrete series irrep with lowest possible conformal weight $d = 7$ happens naturally when $m_1 = m_2 = m_3 = m_4 = 1$. It corresponds to the one-dimensional irrep contained in χ_0^- .

Acknowledgments. The author has received partial support from Bulgarian NSF Grant DN-18/1.

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Fig. 1. Multiplets for the real split form F'_4 using maximal parabolic with $\mathcal{M} = sl(3, \mathbb{R})_{\text{long roots}} \oplus sl(2, \mathbb{R})_{\text{short roots}}$