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Stability and Causality of the relativistic third order hydrodynamics

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We present the analysis of linear stability and causality of the third order relativistic hydrodynamics derived in the PRC 88, 021903 [1]. Here a third order evolution equation for shear stress tensor is derived from relativistic Boltzmann equation using Chapman-Enskog expansion. We perturb the fluid system, which is initially in equilibrium and at rest, by slightly changing the energy density and fluid velocity to study its propagation. The dispersion relation for longitudinal and transverse modes of propagation is derived. It was found that there exists an acausal mode in theory around the static equilibrium. Our results match with the more detailed and elaborate analysis done in [3].

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1. Introduction

The hot and dense QCD medium created in high-energetic heavy-ion collisions behaves like a fluid system and is successfully studied by tools of relativistic hydrodynamics. A theory of relativistic hydrodynamics should be causal and stable so that the disturbance in the fluid medium propagates with finite velocity and decays in time. Causality is the restriction imposed by special relativity, which doesn't allow any information to travel faster than the speed of light. The earliest formulations of hydrodynamic equations for non-ideal fluids were covariant generalizations of the Navier-Stokes equations of Newtonian non-perfect fluids by Eckart [4] and Landau-Lifshitz [5]. These are first-order theories that involve parabolic differential equations and violate causality and face instability problems. There were many attempts to get rid of acausality and remove the instability of first-order hydrodynamics, and to obtain a hyperbolic second-order theory, which led to the derivation of Israel-Stewart equations. In this generalized theory, dissipative fluxes such as heat flux, shear, and bulk stresses are treated as independent variables, and their evolution equations are hyperbolic in nature. The second-order theories allow the existence of a relaxation time for dissipative processes, so the system doesn't return to the equilibrium states instantaneously, unlike Navier-Stokes theory, which restores causality. Hiscock and Lindblom later showed that the perturbations evolve causally in Israel-Stewart theory around equilibrium states. It was also shown that causality ensures the linear stability of homogeneous equilibrium states [6]. Despite the success of Israel-Stewart theory in explaining a wide range of collective phenomena observed in heavy-ion collisions, it has resulted in unphysical effects such as reheating of the expanding medium [7] and negative longitudinal pressure [8]. This motivates the improvisation of the relativistic second-order theory by incorporating higher-order corrections. In Ref. [1] a new relativistic third-order evolution equation for the shear stress tensor from kinetic theory is derived. Here we are analyzing the causality and stability properties of the third-order relativistic hydrodynamics by studying the evolution of perturbations in the fluid.

2. Analysis

Third order evolution equation for the shear stress tensor, $\pi^{\mu\nu}$ derived in Ref. [1, 2] from the relativistic Boltzmann equation using the Chapman-Enskog method is

$$\begin{aligned}
\frac{\pi^{\mu\nu}}{\tau_\pi} =& -\dot{\pi}^{\langle\mu\nu\rangle} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta \\
&- \frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma} - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) \\
&- \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) \\
&- \frac{2}{7}\tau_\pi\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{2}{7}\tau_\pi\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{10}{63}\tau_\pi\pi^{\mu\nu}\theta^2 + \frac{26}{21}\tau_\pi\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta, \qquad (1)
\end{aligned}$$

where we use the notation $\dot{A} \equiv u^\mu\partial_\mu A$ for co-moving derivative, $\theta \equiv \partial_\mu u^\mu$ for the expansion scalar, $\sigma^{\rho\gamma} \equiv \nabla^{(\rho}u^{\gamma)} - (\theta/3)\Delta^{\rho\gamma}$ for the velocity stress tensor, $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity tensor and τ_π is the shear relaxation time. The notation $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\beta}$ represents traceless symmetric projection orthogonal to velocity four-vector u^μ. The relativistic Navier-Stokes equation takes the form as follows,

$$\begin{aligned} \dot{\epsilon} + (\epsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} &= 0\,, \\ (\epsilon + p)\dot{u}^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} &= 0\,, \end{aligned} \tag{2}$$

where ϵ and p are energy density and pressure respectively. We slightly perturb the energy density and the fluid velocity of the system that is initially in equilibrium and at rest as,

$$\epsilon = \epsilon_0 + \delta\epsilon(t, x), \qquad u^\mu = (1, \vec{0}) + \delta u^\mu(t, x). \tag{3}$$

Where ϵ_0 is the equilibrium energy density. First, we study the longitudinal propagation of the perturbation. Keeping only perturbations to first order, eq. (2) becomes,

$$(\epsilon_0 + p_0)\partial_t \delta u^x + \partial_x p + \partial_\mu \delta\Pi^{\mu x} = 0. \tag{4}$$

We used $p = p_0 + \delta p$ and $\Pi^{\mu\nu} = \Pi^{\mu\nu}_{(0)} + \delta\Pi^{\mu\nu}$. We have to calculate the last term in the L.H.S of the above equation using eq. (1). There are total 19 terms in the R.H.S of eq. (1). We evaluate the contribution of each term to $\delta\Pi^{\mu x}$ individually considering only perturbations to the first order. The linear order terms of perturbations are $-\dot{\pi}^{\langle\mu\nu\rangle}$, $2\beta_\pi\sigma^{\mu\nu}$, $\frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right)$, $-\frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right)$, $-\frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right)$. Therefore we obtain,

$$\partial_\mu\delta\pi^{\mu x} = -\frac{4}{3}\eta_0\partial_x^2\delta u^x + (-\tau_\pi\partial_t + \frac{27}{105}\tau_\pi^2\partial_x^2)\partial_\mu\delta\pi^{\mu x}, \tag{5}$$

where η_0 is the unperturbed shear viscosity coefficient. Using eq. (4), the above equation can be rewritten as,

$$\begin{aligned} \partial_\mu\delta\pi^{\mu x} =& -\frac{4}{3}\eta_0\partial_x^2\delta u^x + \tau_\pi\partial_t\left((\epsilon_0 + p_0)\partial_t\delta u^x + \partial_x p\right) \\ & - \frac{27}{105}\tau_\pi^2\partial_x^2\left((\epsilon_0 + p_0)\partial_t\delta u^x + \partial_x p\right). \end{aligned}$$

Substituting this back in eq. (4) and using the relation $\partial_x p = \frac{dp}{d\epsilon}\partial_x\epsilon = c_s^2\partial_x\delta\epsilon$, one obtains

$$\begin{aligned} &(\epsilon_0 + p_0)\partial_t\delta u^x + c_s^2\partial_x\delta\epsilon - \frac{4}{3}\eta_0\partial_x^2\delta u^x + \tau_\pi(\epsilon_0 + p_0)\partial_t^2\delta u^x + \\ &c_s^2\tau_\pi\partial_t\partial_x\delta\epsilon - \frac{27}{105}\tau_\pi^2(\epsilon_0 + p_0)\partial_t\partial_x^2\delta u^x - \frac{27}{105}\tau_\pi^2 c_s^2\partial_x^3\delta\epsilon = 0, \end{aligned}$$

where c_s is the speed of sound. Applying Fourier ansatz

$$\delta\epsilon = e^{i\omega t - ikx}\delta\epsilon_{\omega,k} \quad \text{and} \quad \delta u^i = e^{i\omega t - ikx}\delta u^i_{\omega,k}\,,$$

one obtains,

$$\begin{aligned} &i\omega(\epsilon_0 + p_0)\delta u^x - \frac{ik^2}{\omega}c_s^2(\epsilon_0 + p_0)\delta u^x + \frac{4}{3}\eta_0 k^2\delta u^x - \tau_\pi\omega^2(\epsilon_0 + p_0)\delta u^x + \\ &c_s^2\tau_\pi k^2(\epsilon_0 + p_0)\delta u^x + i\frac{27}{105}\tau_\pi^2\omega k^2(\epsilon_0 + p_0)\delta u^x - i\frac{27}{105}\tau_\pi^2 c_s^2\frac{k^4}{\omega}(\epsilon_0 + p_0)\delta u^x = 0, \end{aligned} \tag{6}$$

where ω and k are frequency and wave number respectively. We find the dispersion relation for the longitudinal perturbation as shown in the figure.

Now we check the causality and the stability of transverse modes of propagation. So we consider the y-component of the eq. (2),

$$(\epsilon_0 + p_0)\partial_t \delta u^y + \partial_\mu \delta\Pi^{\mu y} = 0. \tag{7}$$

Following the same method we obtain the dispersion relation for transverse modes,

$$i\omega(\epsilon_0 + p_0)\left(1 + i\omega\tau_\pi + \frac{8}{35}\tau_\pi^2 k^2\right) + \eta_0 k^2 = 0.$$

Solving this quadratic equation, we get dispersion relation as shown in the figure.

3. Results and Discussions

From the dispersion relations plotted in Fig. 1, we can see that there exists a nonhydrodynamic mode in both transverse and longitudinal perturbation which increase as the wave number increase and doesn't saturate, thus, the theory displays acausal behavior. The transverse modes of third order theory are purely imaginary, and the imaginary part of longitudinal modes is always positive, so the theory is stable.

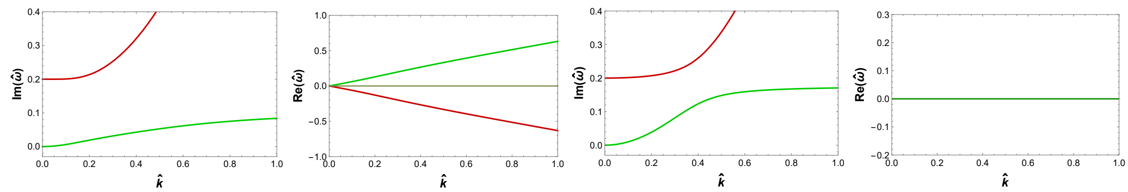

Figure 1: The imaginary and real parts of the frequency of the longitudinal and the transverse modes for perturbation around a static fluid. The red and green curves are propagating modes.

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