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Introduction to the Intersection Theory for Twisted Homology and Cohomology Groups

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We give an introduction to the intersection theory for twisted homology and cohomology groups associated with Euler type integrals of hypergeometric functions. We introuduce twisted homology and cohomology groups motivated by Twisted Stokes' Theorem, and give their dimension formulas. We define an intersection form between twisted homology groups and that between twisted cohomology groups, and explain how to compute them. These intersection forms are compatible with the natural pairing between the twisted homology and cohomology groups. This compatibility yields a twisted period relation, which relates intersection numbers and period integrals regarded as some kinds of hypergeometric functions. In Appendix, we show that Elliott's identity can be obtained from the twisted period relation.

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1. Introduction

This article is an introduction to the intersection theory for twisted homology and cohomology groups based on results in [AK], [CM], [KY1], [M1] and [Y2].

The hypergeometric series $F(a, b, c; x)$ is defined by

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where x is a main variable in $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$, a, b, c are complex parameters with $c \neq 0, -1, -2, \dots,$ and

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

Under $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$, $F(a, b, c; x)$ admits an Euler type integral

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^a (1-t)^{c-a} (1-tx)^{-b} \frac{dt}{t(1-t)}. \tag{1}$$

Here we assign a branch of the multi-valued function $u(t) = t^a(1-t)^{c-a}(1-tx)^{-b}$ in (1) on the integration interval $(0,1)$ by $\arg(t) = \arg(1-t) = 0$, and $-\pi/2 < \arg(1-tx) < \pi/2$ for $x \in \mathbb{D}$. We separate $u(t) = t^a(1-t)^{c-a}(1-tx)^{-b}$ and $\varphi = \dfrac{dt}{t(1-t)}$ from the integrand in (1), and regard this integral as a pairing between the 1-form φ and a pair $I \otimes u_I(t)$ of $I = (0,1)$ and a branch of $u_I(t) = t^a(1-t)^{c-a}(1-tx)^{-b}$ on I. With respect to this, any branch $u_I(t)$ on I vanishes at the boundary of I under $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$, and the exterior derivative d acts on $u(t)\varphi$ as

$$d(u(t)\varphi) = u(t)\cdot d\varphi + du(t)\wedge\varphi = u(t)\cdot\left(d\varphi + \frac{du(t)}{u(t)}\wedge\varphi\right) = 0.$$

Motivated by the above, we define twisted homology and cohomology groups associated with an Euler type integral of Lauricella's hypergeometric series F_D in m-variables given in (8) as follows. We set a multi-valued function

$$u(t) = t^{\alpha_0}(t-x_1)^{\alpha_1}\cdots(t-x_m)^{\alpha_m}(t-1)^{\alpha_{m+1}}$$

on the space $T_x = \mathbb{P}^1 - \{x_0, x_1, \ldots, x_m, x_{m+1}, x_{m+2}\}$ for mutually different fixed complex variables $x_0 = 0, x_1, \ldots, x_m, x_{m+1} = 1, x_{m+2} = \infty$ with parameters

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2}) \in (\mathbb{C}-\mathbb{Z})^{m+3}, \quad \alpha_{m+2} = -\alpha_0 - \alpha_1 - \cdots - \alpha_m - \alpha_{m+1}.$$

Let C_k^u be the space of finite linear combinations $\sum_j a_j\Delta_j \otimes u_{\Delta_j}(t)$, where $a_j \in \mathbb{C}$, Δ_j is a k-simplex in T_x and $u_{\Delta_j}(t)$ is a branch of $u(t)$ on Δ_j. A twisted boundary operator is given by the linear extension of

$$\partial_\omega : C_k^u \ni \Delta\otimes u_\Delta(t) \mapsto \partial\Delta \otimes u_\Delta(t)|_{\partial\Delta} \in C_{k-1}^u \quad (k = 0,1,2),$$

where Δ is a k-simplex in T_x, ∂ is the topological boundary operator, and $u_\Delta(t)|_{\partial\Delta}$ is the restriction of $u_\Delta(t)$ to $\partial\Delta$. A twisted homology group is defined as the quotient

$$H_1(C_\bullet^u, \partial_\omega) = \ker(\partial_\omega : C_1^u \to C_0^u)/\partial_\omega(C_2^u).$$

On the other hand, we set a single-valued rational 1-form ω on T_x by the logarithmic exterior derivative of the multi-valued function $u(t)$:

$$\omega = d\log(u(t)) = \frac{du(t)}{u(t)} = \sum_{j=0}^{m+1}\frac{\alpha_j dt}{t-x_j}.$$

A twisted cohomology group is defined as the quotient

$$H^1(\mathcal{E}^\bullet, \nabla_\omega) = \ker(\nabla_\omega : \mathcal{E}^1 \to \mathcal{E}^2)/\nabla_\omega(\mathcal{E}^0)$$

by a twisted exterior derivative

$$\nabla_\omega : \mathcal{E}^k \ni \psi \mapsto (d+\omega\wedge)\psi \in \mathcal{E}^{k+1} \quad (k = 0,1,2),$$

where $\mathcal{E}^k$ is the space of smooth k-forms on T_x. These groups are dual to each other by the pairing

$$\langle\psi,\gamma\rangle = \sum_i a_i\int_{I_i} u_{I_i}(t)\psi,$$

where ψ is a ∇_ω-closed 1-form on T_x satisfying $\nabla_\omega\psi = 0$ and $\gamma = \sum_i a_i I_i \otimes u_{J_i}(t)$ is a twisted 1-cycle satisfying $\partial_\omega(\gamma) = 0$. We show that the dimensions of these groups are equal to $-\chi(T_x) = m + 1$, where $\chi(T_x)$ is the Euler number of T_x.

By using $\partial_{-\omega}$ and $\nabla_{-\omega}$ given by $u^{-1}(t) = 1/u(t)$ instead of $u(t)$, we have $H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$ and $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$. There are an intersection form between $H_1(C_\bullet^u, \partial_\omega)$ and $H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$, and that between $H^1(\mathcal{E}^\bullet, \nabla_\omega)$ and $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$. We explain how to evaluate intersection numbers for twisted 1-cycles in §6 and those for $\nabla_{\pm\omega}$-closed 1-forms in §8.

It is known that the intersection forms are compatible with the natural pairings between twisted homology and cohomology groups. This compatibility yields a twisted period relation, which relates intersection numbers and period integrals regarded as some kinds of hypergeometric functions. In case of $m = 0$, we have the simplest example, which is the inversion formula

$$B(p,q) \cdot B(-p,-q) = \frac{2\pi\sqrt{-1}(p+q)}{pq} \cdot \frac{1 - e^{2\pi\sqrt{-1}(p+q)}}{(1 - e^{2\pi\sqrt{-1}p})(1 - e^{2\pi\sqrt{-1}q})} \tag{2}$$

for the Beta function

$$B(p,q) = \int_0^1 t^p (1-t)^q \frac{dt}{t(1-t)} \quad (\mathrm{Re}(p), \mathrm{Re}(q) > 0), \tag{3}$$

extended to a meromorphic function on $\mathbb{C}^2$ by the functional equations

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p\Gamma(p) = \Gamma(p+1) = \int_0^\infty e^{-t} t^p \, dt \quad (\mathrm{Re}(p) > -1).$$

Here $B(p,q)$ and $B(-p,-q)$ are regarded as period integrals, and the right hand side of (2) is the product of the intersection number of $\nabla_{\pm\omega}$-closed 1-forms and that of twisted 1-cycles.

For case $m = 1$, we also show that the twisted period relation yields identities among several values of the hypergeometric series such as

$$F(a,b,c;x)F(1-a,1-b,2-c;x) = F(b-c+1,a-c+1,2-c;x)F(c-a,c-b,c;x).$$

For general m, we give identities among several values of the hypergeometric series F_D by the twisted period relation.

In Appendix, we show that Elliott's identity can be obtained from the twisted period relation.

2. Regularization of open intervals

For the integrals (1) or (3), we need the convergence condition $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$ or $\mathrm{Re}(p), \mathrm{Re}(q) > 0$. Let us change these conditions into $a, c-a \notin \mathbb{Z}$ or $p, q \notin \mathbb{Z}$.

Let $C_0^{+\varepsilon}$ be a circle with center $t = 0$, radius ε and terminal $t = \varepsilon$, and $C_1^{-\varepsilon}$ be that with center $t = 1$, radius ε and terminal $t = 1 - \varepsilon$, where ε is a sufficiently small positive number; see Figure 1.

Proposition 1. *If* $\mathrm{Re}(p), \mathrm{Re}(q) > 0$, $p, q \notin \mathbb{Z}$ *then*

$$B(p,q) = \frac{1}{e^{2\pi\sqrt{-1}p} - 1} \int_{C_0^{+\varepsilon}} u(t)\varphi + \int_\varepsilon^{1-\varepsilon} u(t)\varphi - \frac{1}{e^{2\pi\sqrt{-1}q} - 1} \int_{C_1^{-\varepsilon}} u(t)\varphi, \tag{4}$$

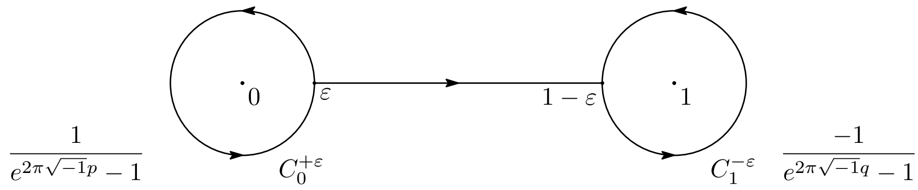


Figure 1: Regularization of the open interval $(0, 1)$

where $\varphi = \dfrac{dt}{t(1-t)}$*,* $u(t) = t^p(1-t)^q$*, its branch on* $[\varepsilon, 1-\varepsilon]$ *is assigned by* $\arg(t) = \arg(1-t) = 0$*, and those on* $C_0^{+\varepsilon}$ *and* $C_1^{-\varepsilon}$ *are assigned by the* $\arg(t) = \arg(1-t) = 0$ *at their start points.*

Proof. We show that the right hand side of (4) is independent of ε. This property yields the identity, since

$$\lim_{\varepsilon\to 0}\int_{C_0^{+\varepsilon}} u(t)\varphi = \lim_{\varepsilon\to 0}\int_{C_1^{-\varepsilon}} u(t)\varphi = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1-\varepsilon} u(t)\varphi = B(p,q)$$

under $\mathrm{Re}(p), \mathrm{Re}(q) > 0$. Take a circle $C_0^{+\delta}$ for $0 < \delta < \varepsilon$, and consider the difference

$$\frac{1}{e^{2\pi\sqrt{-1}p}-1}\left[\int_{C_0^{+\varepsilon}} u(t)\varphi - \int_{C_0^{+\delta}} u(t)\varphi\right] - \int_{\delta}^{\varepsilon} u(t)\varphi.$$

It is equal to

$$\frac{1}{e^{2\pi\sqrt{-1}p}-1}\int_C t^p(1-t)^q\frac{dt}{t(1-t)},$$

which vanishes by Cauchy's integral theorem, where the path C is in Figure 2. We can similarly

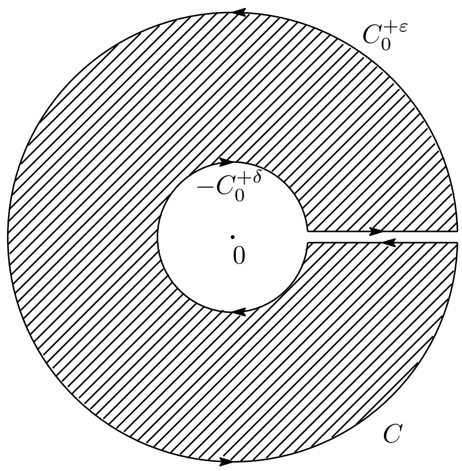


Figure 2: Path C

show that the right hand side of (4) is independent of ε for the circle $C_1^{-\varepsilon}$. □

Definition 1 ([AK, Example 2.1])**.** We define the regularization of the open interval $(0,1)$ with respect to $u(t) = t^p(1-t)^q$ by the formal sum

$$\frac{1}{e^{2\pi\sqrt{-1}p}-1}C_0^{+\varepsilon} + [\varepsilon, 1-\varepsilon] - \frac{1}{e^{2\pi\sqrt{-1}q}-1}C_1^{-\varepsilon}$$

with assignments of branches of $u(t)$ on $C_0^{+\varepsilon}$, $[\varepsilon, 1-\varepsilon]$, $C_1^{-\varepsilon}$ as in Proposition 1.

Note that each path integral of (4) in Proposition 1 reduces to an integral of a continuous function over a bounded closed interval. Since it is well defined under the condition $p, q \notin \mathbb{Z}$, we remove the convergence condition $\mathrm{Re}(p), \mathrm{Re}(q) > 0$ for implicit integrals for $B(p,q)$.

We can apply this cycle to the Euler type integral (1) for hypergeometric series. We change the convergence condition $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$ into $a, c-a \notin \mathbb{Z}$ by using this cycle. To prove (1) integrated along this cycle, we use the Taylor expansion

$$(1-tx)^{-b} = \sum_{n=0}^{\infty} \frac{(b)_n}{n!} t^n x^n$$

and change the order of the infinite series and the integral. This order change is permitted under the uniform convergence of the series for the fixed variable $x \in \mathbb{D}$. Note that we need much more strong conditions for changing the order of an infinite series and an implicit integral.

Moreover, we will see an advantage to be able to define the intersection form by introducing the regularization of open intervals.

3. Lauricella's hypergeometric system $\mathcal{F}_D$

In this section, we generalize the Euler type integral (1) of $F(a, b, c; x)$ or (3) for $B(p,q)$ by referring to [Y1, §6]. By the variable change $s = 1/t$, (1) and (3) are transformed into

$$\int_1^{\infty} s^{b-c}(s-x)^{-b}(s-1)^{c-a}\frac{ds}{s-1}, \qquad \int_1^{\infty} s^{-p-q}(s-1)^q \frac{ds}{s-1}.$$

Here we generalize $u(t)$ to

$$u(t) = u(t, x) = \prod_{j=0}^{m+2}(t-x_j)^{\alpha_j} = t^{\alpha_0}\Big[\prod_{j=1}^{m}(t-x_j)^{\alpha_j}\Big](t-1)^{\alpha_{m+1}}, \tag{5}$$

where $x_0 = 0, x_1, \ldots, x_m, x_{m+1} = 1, x_{m+2} = \infty$ are mutually different complex variables, and α_j are complex parameters satisfying

$$\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2} \notin \mathbb{Z}, \qquad \sum_{j=0}^{m+2}\alpha_j = 0. \tag{6}$$

We set

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2})$$

and assume (6) throughout this article unless otherwise specified. We consider the integral

$$\int_1^{\infty} u(t)\varphi, \qquad \varphi = \frac{dt}{t-1}, \tag{7}$$

which converges under our assumption (6) if we use the regularization of $(1, \infty)$ with respect to $u(t)$. Introduce parameters a, $b = (b_1, \dots, b_m)$, c from α as

$$a = \alpha_{m+2}, \quad b = (-\alpha_1, \dots, -\alpha_m), \quad c = \alpha_{m+1} + \alpha_{m+2}.$$

In case of $m = 0$, the integral (7) is equal to $B(\alpha_1, \alpha_2) = B(a, c - a)$; in case of $m = 1$, it is equal to

$$B(\alpha_2, \alpha_3)F(\alpha_3, -\alpha_1, \alpha_2 + \alpha_3; x_1) = B(a, c - a)F(a, b, c; x_1)$$

for $c = \alpha_2 + \alpha_3 \neq 0, -1, -2, \dots$ and $x_1 \in \mathbb{D}$. For general m, if $c = \alpha_{m+1} + \alpha_{m+2}$ is different from $0, -1. - 2, \dots$ and $x = (x_1, \dots, x_m)$ belongs to

$$\mathbb{D}_m = \{x \in \mathbb{C}^m \mid \max_{1 \le j \le m} |x_j| < 1\}$$

then the integral (7) can be expressed as

$$B(\alpha_{m+1}, \alpha_{m+2})F_D(\alpha_{m+2}, -\alpha_1, \dots, -\alpha_m, \alpha_{m+1} + \alpha_{m+2}; x) = B(c - a, a)F_D(a, b, c; x),$$

where $F_D(a, b, c; x)$ is Lauricella's hypergeometric series defined by

$$F_D(a, b, c; x) = \sum_{n \in \mathbb{N}_0^m} \frac{(a)_{n_1 + \cdots + n_m} \prod_{j=1}^m (b_j)_{n_j}}{(c)_{n_1 + \cdots + n_m} \prod_{j=1}^m n_j!} \prod_{j=1}^m x_j^{n_j}. \tag{8}$$

The differential operators

$$\begin{aligned} x_j(1-x_j)\partial_j^2 + (1-x_j) \sum_{1 \le k \le m}^{k \neq j} x_k \partial_j \partial_k + [c - (a+b_j+1)x_j]\partial_j - b_j \sum_{1 \le k \le m}^{k \neq j} x_k \partial_k - ab_j, \quad (1 \le j \le m) \\ (x_j - x_k)\partial_j \partial_k - b_k \partial_j + b_j \partial_k, \quad (1 \le j < k \le m) \end{aligned}$$

annihilate the series $F_D(a, b, c; x)$, where $\partial_j = \dfrac{\partial}{\partial x_j}$ $(1 \le j \le m)$. The partial differential equations given by their actions generate Lauricella's hypergeometric system $\mathcal{F}_D(a, b, c)$, which is a holonomic system of rank $m + 1$ with regular locus

$$X = \big\{x \in \mathbb{C}^m \big| \prod_{j=1}^m [x_j(1 - x_j)] \prod_{1 \le j < k \le m} (x_j - x_k) \neq 0\big\}.$$

This means that the vector space of solutions to $\mathcal{F}_D(a, b, c)$ on a small neighborhood of any $x \in X$ is $m + 1$ dimensional.

4. Twisted Stokes' Theorem

We consider the treatment of a multi-valued 1-form $u(t)\varphi(t)$ in (7) by referring to [AK, §2.1]. Let ψ be a smooth k-form on $T_x = \mathbb{P}^1 - \{x_0, x_1, \dots, x_{m+2}\} = \mathbb{C} - \{0, x_1, \dots, x_m, 1\}$, and let Δ be a k-simplex in T_x. To define an integral

$$\int_\Delta u(t)\psi, \tag{9}$$

we need to assign a branch $u_\Delta(t)$ of $u(t)$. Though $u(t)$ is multi valued on T_x, the restriction of any branch of $u(t)$ to Δ is single valued since Δ is simply connected. Thus we separate the multi-valued function $u(t)$ and ψ from $u(t)\psi$, and regard the integral (9) with assignment of a branch $u_\Delta(t)$ of $u(t)$ on Δ as a pairing $\langle\psi, \Delta\otimes u_\Delta(t)\rangle$ between ψ and the pair $\Delta\otimes u_\Delta(t)$ of Δ and $u_\Delta(t)$.

We rewrite Stokes' theorem

$$\int_D d(u_D(t)\psi) = \int_{\partial D} u_D(t)\psi \tag{10}$$

under this policy, where ψ is a smooth k-form, D is a $(k+1)$-simplex, $u_D(t)$ is a branch of $u(t)$ and ∂ is the boundary operator. The left hand side of (10) is

$$d(u_D(t)\psi) = u_D(t)d\psi + du_D(t)\wedge\psi = u_D(t)(d\psi+\omega\wedge\psi) = u_D(t)\nabla_\omega\psi,$$

where we set

$$\omega = d\log(u_D(t)) = \frac{du_D(t)}{u_D(t)} = \sum_{j=0}^{m+1}\frac{\alpha_j dt}{t-x_j},$$

and introduce a twisted exterior derivative

$$\nabla_\omega = d + \omega\wedge .$$

Here note that $\omega = d\log u_D(t) = d\log u(t)$ is a single-valued smooth 1-form on T_x though $u(t)$ is multi valued on T_x.

On the other hand, the right hand side of (10) is $\langle\psi, (\partial D)\otimes u_D(t)|_{\partial D}\rangle$ under this policy, where $u_D(t)|_{\partial D}$ is the restriction of $u_D(t)$ to ∂D. Thus we define a twisted boundary operator ∂_ω by

$$\partial_\omega(D\otimes u_D(t)) = (\partial D)\otimes u_D(t)|_{\partial D}.$$

We summarize these results as follows.

Theorem 1 (Twisted Stokes' Theorem ([AK, §2.1.2,2.1.3]))**.**

$$\langle\nabla_\omega\psi, D\otimes u_D(t)\rangle = \langle\psi, \partial_\omega(D\otimes u_D(t))\rangle.$$

Let ψ be a smooth k-form ψ on T_x. If $\nabla_\omega\psi = 0$ then ψ is said to be ∇_ω-closed, and if there exists a smooth $(k-1)$-form f on T_x such that $\nabla_\omega f = \psi$, then ψ is said to be ∇_ω-exact. Since $\nabla_\omega\circ\nabla_\omega = 0$, if ψ is ∇_ω-exact then ψ is ∇_ω-closed. Any holomorphic 1-form φ on T_x is ∇_ω-closed since $\nabla_\omega(\varphi) = d\varphi+\omega\wedge\varphi = 0$ by $dt\wedge dt = 0$. The 1-form ω is ∇_ω-exact since $\omega = \nabla_\omega 1$.

A finite linear combination

$$\gamma = \sum_j a_j\Delta_j\otimes u_{\Delta_j}(t)$$

is said to be a twisted k-chain, where $a_j\in\mathbb{C}$, each $\Delta_j\otimes u_{\Delta_j}(t)$ is a pair of a k-simplex Δ_j in T_x and a branch $u_{\Delta_j}(t)$ of $u(t)$ on Δ_j. If $\partial_\omega(\gamma) = 0$ then γ is said to be a twisted k-cycle, where the twisted boundary operator ∂_ω is extended linearly. Since $\partial_\omega\circ\partial_\omega = 0$, $\partial_\omega(\gamma)$ is a twisted $(k-1)$-cycle for any twisted k-chain γ.

In case of $m = 0$, we define a twisted 1-chain as

$$\gamma_0 = \frac{1}{e^{2\pi\sqrt{-1}p}-1}C_0^{+\varepsilon}\otimes u_0(t) + [\varepsilon, 1-\varepsilon]\otimes u_I(t) - \frac{1}{e^{2\pi\sqrt{-1}q}-1}C_1^{-\varepsilon}\otimes u_1(t) \tag{11}$$

by $I = [\varepsilon, 1-\varepsilon]$, $C_0^{+\varepsilon}$, $C_1^{-\varepsilon}$ and the branches $u_I(t)$, $u_0(t)$, $u_1(t)$ of $u(t) = t^p(1-t)^q$ on them as in §2. (Strictly speaking, we should divide $C_0^{+\varepsilon}$ and $C_1^{-\varepsilon}$ into upper and lower semi-circles, since $C_0^{+\varepsilon}$ and $C_1^{-\varepsilon}$ are not isomorphic to a 1-simplex in T_x.) It is a twisted 1-cycle since

$$\begin{aligned}\partial_\omega(\gamma_0) =& \frac{[t]_{t=\varepsilon} \otimes e^{2\pi\sqrt{-1}p} u_0(\varepsilon) - [t]_{t=\varepsilon} \otimes u_0(\varepsilon)}{e^{2\pi\sqrt{-1}p} - 1} + \left([t]_{t=1-\varepsilon} \otimes u_I(1-\varepsilon) - [t]_{t=\varepsilon} \otimes u_I(\varepsilon)\right)\\ &- \frac{[t]_{t=1-\varepsilon} \otimes e^{2\pi\sqrt{-1}q} u_1(1-\varepsilon) - [t]_{t=1-\varepsilon} \otimes u_1(1-\varepsilon)}{e^{2\pi\sqrt{-1}q} - 1}\\ =&0,\end{aligned}$$

where $u_0(\varepsilon)$ and $u_1(1-\varepsilon)$ denote the values of $u_0(t)$ and $u_1(t)$ at the start points of $C_0^{+\varepsilon}$ and $C_1^{-\varepsilon}$, respectively. Here note that the values of $u_0(t)$ and $u_1(t)$ at the end points of $C_0^{+\varepsilon}$ and $C_1^{-\varepsilon}$ are equal to $e^{2\pi\sqrt{-1}p} u_0(\varepsilon)$ and $e^{2\pi\sqrt{-1}q} u_1(1-\varepsilon)$, respectively, and that

$$u_0(\varepsilon) = u_I(\varepsilon), \quad u_1(1-\varepsilon) = u_I(1-\varepsilon).$$

We can regard the integral (1) or (3) as a pairing between a ∇_ω-closed 1-form and a twisted 1-cycle.

Remark 1. Since the open interval $(0,1)$ in $T_x = \mathbb{C} - \{0,1\}$ cannot be expressed by a finite sum of 1-simplexes in T_x, we cannot regard the pair $(0,1) \otimes u_{(0,1)}(t)$ of $(0,1)$ and a branch $u_{(0,1)}(t)$ of $u(t)$ as a twisted 1-chain for this definition. To regard $(0,1) \otimes u_{(0,1)}(t)$ as a twisted 1-chain, we need to extend "finite sums" to "infinite sums with local finiteness", which are not treated in this article.

5. Twisted homology groups

The vector space of twisted k-chains in T_x is denoted by C_k^u. Let φ be a ∇_ω-closed 1-form on T_x, and γ be a twisted 1-cycle in T_x. Since

$$\langle \varphi, \partial_\omega(G) \rangle = \langle \nabla_\omega \varphi, G \rangle = \langle 0, G \rangle = 0,$$

for any $G \in C_k^u$ by Twisted Stocks' theorem, we have

$$\langle \varphi, \gamma + \partial_\omega(G) \rangle = \langle \varphi, \gamma \rangle.$$

This means that $\partial_\omega(C_2^u)$ has no effect on the pairing $\langle \varphi, \gamma \rangle$ between a ∇_ω-closed 1-form φ and a twisted 1-cycle γ in T_x.

Definition 2 (Twisted homology group)**.** The k-th twisted homology group $H_k(C_\bullet^u, \partial_\omega)$ is defined by

$$H_k(C_\bullet^u, \partial_\omega) = \ker(\partial_\omega : C_k^u \to C_{k-1}^u)/\partial_\omega(C_{k+1}^u) \quad (k = 0, 1, 2).$$

Proposition 2 (Dimension formula [AK, Lemma 2.14])**.** *Under our assumption (6) on α, we have*

$$\dim H_1(C_\bullet^u, \partial_\omega) = m + 1.$$

Proof. Since C^u_{-1} and C^u_3 are isomorphic to the zero vector space, we have

$$H_0(C^u_\bullet, \partial_\omega) = C^u_0/\partial_\omega(C^u_1), \quad H_2(C^u_\bullet, \partial_\omega) = \ker(\partial_\omega : C^u_2 \to C^u_1).$$

Let p be any point in T_x and C be a loop in T_x with terminal p turning around x_0 once positively. Then

$$\partial_w(C \otimes u_C(t)) = (e^{2\pi\sqrt{-1}\alpha_0} - 1) \cdot [t]_{t=p} \otimes u_C(p),$$

where $u_C(p)$ is the value of a branch $u_C(t)$ of $u(t)$ at the start point. Thus we have $C^u_0 = \partial_\omega(C^u_1)$ which means $H_0(C^u_\bullet, \partial_\omega) = 0$ under the condition $\alpha_0 \notin \mathbb{Z}$. Since any twisted 2-chain consists of finite sum of $c_j\Delta_j \otimes u(t)$ for $c_j \in \mathbb{C}$ and 2-simplexes Δ_j, we cannot eliminate their boundaries. Thus we have $\ker(\partial_\omega : C^u_2 \to C^u_1) = 0$, which means $H_2(C^u_\bullet, \partial_\omega) = 0$.

We use the fact

$$\chi(T_x) = \dim H_0(C^u_\bullet, \partial_\omega) - \dim H_1(C^u_\bullet, \partial_\omega) + \dim H_2(C^u_\bullet, \partial_\omega),$$

where $\chi(T_x)$ is the Euler number of T_x. Since

$$\chi(T_x) = \chi(\mathbb{P}^1) - \#\{x_0, \dots, x_{m+2}\} = 2 - (m+3) = -m - 1,$$

we have $\dim H_1(C^u_\bullet, \partial_\omega) = m + 1$. □

Remark 2. This dimension formula is valid under the condition $\alpha \notin \mathbb{Z}^{m+3}$. If $\alpha_j \in \mathbb{Z}$ then we cannot use the regularization of cycles for paths with terminal x_j. For example, we give some cases $(p, q, -p-q) \notin \mathbb{Z}^3$ for $B(p, q)$: $T_x = \mathbb{P}^1 - \{0, 1, \infty\}$, $u(t) = t^p(1-t)^q$ and $\dim H_1(C^u_\bullet, \partial_\omega) = 1$.

If $p \in \mathbb{Z}$ and $q, p+q \notin \mathbb{Z}$, then the regularization of $(0, 1)$ with respect to $u(t)$ cannot be defined. In this case, $C^\varepsilon_0 \otimes u_0(t)$ becomes a twisted 1-cycle, and it spans $H_1(C^u_\bullet, \partial_\omega)$. Note that this twisted 1-cycle can be regarded as $\partial_\omega(\Delta_0 \otimes u_{\Delta_0}(t))$ if we permit infinite sums with locally finiteness, where $\Delta_0 = \{t \in \mathbb{C} \mid 0 < |t| \le \varepsilon\}$ and $u_{\Delta_0}(t)$ is the continuation of $u_0(t)$ to Δ_0.

If $p, q \notin \mathbb{Z}$ and $p+q \in \mathbb{Z}$ then we have the regularization of $(0, 1)$ with respect to $u(t)$, and it gives a non-zero element of $H_1(C^u_\bullet, \partial_\omega)$. In this case, if we permit infinite sums with locally finiteness then $(0, 1) \otimes u_{(0,1)}(t)$ can be regarded as a twisted 1-cycle and it coincides with $\partial_\omega(\Delta_I \otimes u_{\Delta_I}(t))$ up to a non-zero constant, where $\Delta_I = T_x - (0, 1)$ and a branch $u_{\Delta_I}(t)$ of $u(t)$ is assigned by $\arg(t), \arg(1-t) \in (0, 2\pi)$ on Δ_I.

6. Intersection form between twisted homology groups

By using $1/u(t) = u^{-1}(t)$ instead of u(t), we define the spaces $C^{u^{-1}}_k$ of twisted k-chains, the twisted boundary operator $\partial_{-\omega}$ and the twisted homology groups $H_k(C^{u^{-1}}_\bullet, \partial_{-\omega})$ $(k = 0, 1, 2)$.

Definition 3 (Intersection number of twisted 1-cycles)**.** Let $\gamma = \sum_i a_i I_i \otimes u_{I_i}(t)$ and $\delta = \sum_j b_j J_j \otimes u^{-1}_{J_j}(t)$ be twisted cycles, where $a_i, b_j \in \mathbb{C}$, I_i and J_j are 1-simplexes in T_x. Suppose that any I_i and J_j intersect transversally at every intersection point p of them. The intersection number of γ and δ is defined by

$$\langle \gamma, \delta \rangle = \sum_{i,j} a_i b_j \sum_{p \in I_i \cap J_j} \langle I_i, J_j \rangle_p \cdot [u_{I_i}(p)] \cdot [u^{-1}_{J_j}(p)], \tag{12}$$

where $\langle I_i, J_j \rangle_p$ denotes the topological intersection number of I_i and J_i at p.

As in [KY1, §1], we have the following theorems.

Theorem 2. *The intersection number* (12) *induces a bi-linear form between the spaces of twisted cycles* $\ker(\partial_\omega : C_1^u \to C_0^u)$ *and* $\ker(\partial_{-\omega} : C_1^{u^{-1}} \to C_0^{u^{-1}})$*, which descends to that between* $H_1(C_\bullet^u, \partial_\omega)$ *and* $H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$*. It is non-degenerate.*

Theorem 3 ([AK, §2.3.3], [KY1, Theorem 2.1] and [Y2, §4.7])**.** *Suppose that* $x_1, \dots, x_m \in \mathbb{R}$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$, $x_{m+2} = \infty$*. Let* γ_j *be the twisted cycle given by the regularization of* $I_j = (x_j, x_{j+1})$ *with respect to* $u(t)$*, and let* δ_k *be that of* $I_k = (x_k, x_{k+1})$ *with respect to* $u^{-1}(t)$*. Then we have*

$$\langle \gamma_j, \delta_k \rangle = \begin{cases} \dfrac{-\theta_j}{1-\theta_j} & \text{if } k = j-1, \\ \dfrac{1-\theta_j\theta_{j+1}}{(1-\theta_j)(1-\theta_{j+1})} & \text{if } k = j, \\ \dfrac{-1}{1-\theta_{j+1}} & \text{if } k = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_j = \exp(2\pi\sqrt{-1}\alpha_j)$.

Proof. Let us compute $\langle \gamma_j, \delta_j \rangle$. We can regard the branches of $u(t)$ and $u^{-1}(t)$ on any components of γ_j and δ_j as the analytic continuations of branches $u_{\mathbb{H}}(t)$ and $u_{\mathbb{H}}^{-1}(t)$ defined on the upper half space of T_x satisfying $u_{\mathbb{H}}(t) \cdot u_{\mathbb{H}}^{-1}(t) = 1$. These continuations are denoted by $u_{\gamma_j}(t)$ and $u_{\delta_j}^{-1}(t)$. We can see by Figure 3 that the topological intersection number of the 1-simplexes of γ_j and δ_j at p_1 is -1, that at p_2 is $+1$, and

$$u_{\gamma_j}(p_1) \cdot u_{\delta_j}^{-1}(p_1) = u_{\mathbb{H}}(p_1) \cdot u_{\mathbb{H}}^{-1}(p_1) = 1, \quad u_{\gamma_j}(p_2) \cdot u_{\delta_j}^{-1}(p_2) = \theta_{j+1} u_{\mathbb{H}}(p_2) \cdot u_{\mathbb{H}}^{-1}(p_2) = \theta_{j+1}.$$

Here note that the value of $u_{\gamma_j}(t)$ at p_2 is multiplied θ_{j+1} to $u_{\mathbb{H}}(t)$ since the variable t turns once around x_{j+1} positively along the circle. Thus the intersection number $\langle \gamma_j, \delta_j \rangle$ is

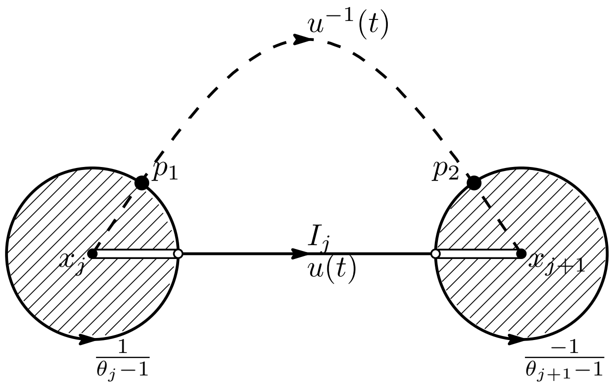


Figure 3: Intersection of twisted cycles

$$(-1)\frac{1}{\theta_j - 1} + \theta_{j+1}\frac{-1}{\theta_{j+1} - 1} = \frac{1-\theta_j\theta_{j+1}}{(1-\theta_j)(1-\theta_{j+1})}.$$

We can similarly compute other $\langle \gamma_j, \delta_k \rangle$.

We can show that the intersection form between $H_1(C^u_\bullet, \partial_\omega)$ and $H_1(C^{u^{-1}}_\bullet, \partial_{-\omega})$ is non-degenerate by the regularity of the intersection matrix

$$H_h = (\langle \gamma_j, \delta_k \rangle)_{0 \le j,k \le m} = \begin{pmatrix} \frac{1-\theta_0\theta_1}{(1-\theta_0)(1-\theta_1)} & \frac{-1}{1-\theta_1} & 0 & \cdots & 0 \\ \frac{-\theta_1}{1-\theta_1} & \frac{1-\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)} & \frac{-1}{1-\theta_2} & \ddots & \\ 0 & \frac{-\theta_2}{1-\theta_2} & \frac{1-\theta_2\theta_3}{(1-\theta_2)(1-\theta_3)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & \cdots & & \frac{1-\theta_m\theta_{m+1}}{(1-\theta_m)(1-\theta_{m+1})} \end{pmatrix}.$$

Under our assumption (6), $\det(H_h) = \dfrac{1-\theta_{m+2}^{-1}}{(1-\theta_0)(1-\theta_1)\cdots(1-\theta_{m+1})}$ does not vanish. □

By Theorem 3, the intersection number between twisted cycles defining $B(p,q)$ and $B(-p,-q)$ is

$$\frac{1-e^{2\pi\sqrt{-1}(p+q)}}{(1-e^{2\pi\sqrt{-1}p})(1-e^{2\pi\sqrt{-1}q})}.$$

7. Twisted cohomology groups

Recall that Twisted Stokes' Theorem

$$\langle \nabla_\omega \psi, D \otimes u(t) \rangle = \langle \psi, \partial_\omega (D \otimes u(t)) \rangle,$$

where $\omega = d\log u(t)$, $\nabla_\omega = d + \omega\wedge$, ψ is a smooth k-form, D is a $(k+1)$-chain in T_x. Let φ be a ∇_ω-closed 1-form on T_x, and γ be a twisted 1-cycle in T_x. Since

$$\langle \nabla_\omega f, \gamma \rangle = \langle f, \partial_\omega \gamma \rangle = \langle f, 0 \rangle = 0$$

for any smooth function f on T_x by Twisted Stocks' theorem, we have

$$\langle \varphi + \nabla_\omega f, \gamma \rangle = \langle \varphi, \gamma \rangle.$$

It means that $\nabla_\omega(\mathcal{E}^0)$ has no effect on the pairing $\langle \varphi, \gamma \rangle$ between a ∇_ω-closed 1-form φ and a twisted 1-cycle γ in T_x, where $\mathcal{E}^0$ is the space of smooth functions on T_x.

Let $\mathcal{E}^k$ and $\mathcal{E}^k_c$ be the space of smooth k-forms on T_x and that with compact support. Note that

$$\mathcal{E}^k_c \subset \mathcal{E}^k, \quad \frac{dt}{t(1-t)}, \frac{dt}{t-1} \in \mathcal{E}^1 - \mathcal{E}^1_c,$$

and that $\psi \in \mathcal{E}^k$ belongs to $\mathcal{E}^k_c$ if and only if $\psi \equiv 0$ on a small neighborhood U_j of x_j ($j = 0, 1, \dots, m+2$).

Definition 4 (Twisted cohomology groups)**.** A twisted cohomology group and that with compact support are defined as

$$\begin{aligned} H^k(\mathcal{E}^\bullet, \nabla_\omega) &= \ker(\nabla_\omega : \mathcal{E}^k \to \mathcal{E}^{k+1})/\nabla_\omega(\mathcal{E}^{k-1}), \\ H^k(\mathcal{E}^\bullet_c, \nabla_\omega) &= \ker(\nabla_\omega : \mathcal{E}^k_c \to \mathcal{E}^{k+1}_c)/\nabla_\omega(\mathcal{E}^{k-1}_c), \end{aligned}$$

for $k = 0, 1, 2$.

By regarding $\psi \in \mathcal{E}_c^1$ as an element of $\mathcal{E}^1$, we have a natural map

$$H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega). \tag{13}$$

Proposition 3 (Dimension formula [KN, Main Theorem, Remark 2])**.** *Under our assumption* (6) *on* α*, we have*

$$\dim H^1(\mathcal{E}^\bullet, \nabla_\omega) = \dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega) = m+1,$$

and the natural map (13) *is an isomorphism.*

Proof. To obtain the claim for the dimension, we show that the spaces

$$\begin{aligned} H^0(\mathcal{E}^\bullet, \nabla_\omega) &= \ker(\nabla_\omega : \mathcal{E}^0 \to \mathcal{E}^1), \quad H^2(\mathcal{E}^\bullet, \nabla_\omega) = \mathcal{E}^2/\nabla_\omega(\mathcal{E}^1), \\ H_c^0(\mathcal{E}^\bullet, \nabla_\omega) &= \ker(\nabla_\omega : \mathcal{E}_c^0 \to \mathcal{E}_c^1), \quad H_c^2(\mathcal{E}^\bullet, \nabla_\omega) = \mathcal{E}_c^2/\nabla_\omega(\mathcal{E}_c^1), \end{aligned}$$

vanish, and use the fact

$$\begin{aligned} \chi(T_x) &= \dim H^0(\mathcal{E}^\bullet, \nabla_\omega) - \dim H^1(\mathcal{E}^\bullet, \nabla_\omega) + \dim H^2(\mathcal{E}^\bullet, \nabla_\omega) \\ &= \dim H^0(\mathcal{E}_c^\bullet, \nabla_\omega) - \dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega) + \dim H^2(\mathcal{E}_c^\bullet, \nabla_\omega). \end{aligned}$$

Let us show $H^0(\mathcal{E}^\bullet, \nabla_\omega) = H_c^0(\mathcal{E}^\bullet, \nabla_\omega) = 0$. Regard $\nabla_\omega f = 0$ as a first order linear differential equation for unknown function f. Its solution is $c/u(t)$ ($c \in \mathbb{C}$) in a neighborhood of any $t \in T_x$. In fact,

$$\nabla_\omega \frac{1}{u(t)} = d\frac{1}{u(t)} + \omega \wedge \frac{1}{u(t)} = -\frac{du(t)}{u(t)^2} + \frac{du(t)}{u(t)^2} = 0.$$

Since $1/u(t)$ is not single valued on T_x under $\alpha \in (\mathbb{C} - \mathbb{Z})^{m+3}$, its global solution is only 0. Thus the spaces vanish.

Let us show $H^2(\mathcal{E}^\bullet, \nabla_\omega) = 0$. For any $\eta \in \mathcal{E}^2$, we find $\psi \in \mathcal{E}^1$ such that $\nabla_\omega \psi = \eta$. Note that η is $(1,1)$-form and $\overline{\partial}\eta = 0$. By the $\overline{\partial}$-Poincaré lemma, there exists a $(1,0)$-form ψ such that $\overline{\partial}\psi = \eta$. It takes the form $\psi = g(t)dt$ and satisfies

$$\nabla_\omega \psi = (\partial + \overline{\partial}) \cdot (g(t)dt) + \omega \wedge (g(t)dt) = \overline{\partial}(g(t)dt) = \eta.$$

Thus any element of $\mathcal{E}^2$ is a ∇_ω-image of $\mathcal{E}^1$, and $H^2(\mathcal{E}^\bullet, \nabla_\omega) = 0$. Hence we have

$$\dim H^1(\mathcal{E}^\bullet, \nabla_\omega) = -\chi(T_x) = m+1.$$

To obtain $H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$, we show the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$ is surjective. For any $\psi \in \mathcal{E}^1$ satisfying $\nabla_\omega \psi = 0$, we find $\psi' \in \mathcal{E}_c^1$ such that $\psi - \psi' \in \nabla_\omega(\mathcal{E}^0)$. Set

$$f_j(t) = \frac{1}{(e^{2\pi\sqrt{-1}\alpha_j} - 1)u(t)} \int_{C(t)} u(t)\psi, \tag{14}$$

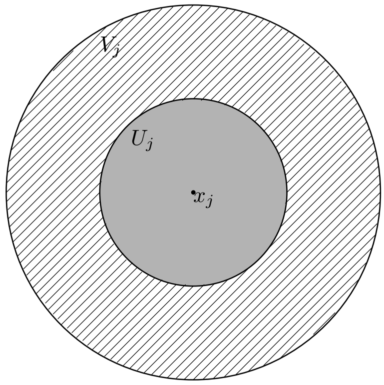
where $C(t)$ is a loop with terminal t turning once around x_j positively. It is independent of the choice of loops by $\nabla_\omega \psi = 0$ and Twisted Stocks' theorem. Thus f_j is single valued around x_j and satisfies

$$\begin{aligned} \nabla_\omega(f_j) &= \frac{-du(t)}{(e^{2\pi\sqrt{-1}\alpha_j} - 1)u(t)^2} \int_{C(t)} u(t)\psi + \frac{e^{2\pi\sqrt{-1}\alpha_j} u(t)\psi - u(t)\psi}{(e^{2\pi\sqrt{-1}\alpha_j} - 1)u(t)} \\ &\quad + \frac{du(t)}{u(t)} \wedge \frac{1}{(e^{2\pi\sqrt{-1}\alpha_j} - 1)u(t)} \int_{C(t)} u(t)\psi \\ &= \psi. \end{aligned}$$

Let h_j be a smooth function on $\mathbb{P}^1$ satisfying

$$h_j(t) = \begin{cases} 1 & \text{if} \quad t \in U_j, \\ 0 & \text{if} \quad t \in V_j^c, \end{cases} \tag{15}$$

where U_j and V_j are neighborhoods of x_j satisfying $U_j \subset V_j$. Though f_j are defined only around x_j, the function $\sum_{j=0}^{m+2} h_j(t)f_j(t)$ can be regarded as a function on T_x.



Since f_j satisfies $\nabla_\omega f_j = \psi$ around x_j, and $h_j f_j$ identically vanishes around x_k $(j \neq k)$, we have

$$\psi' = \psi - \nabla_\omega\Big(\sum_{j=0}^{m+2} h_j(t)f_j(t)\Big) \in \mathcal{E}_c^1,$$

which shows that the natural map (13) is surjective.

Let us show $H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$. For any $\eta \in \mathcal{E}_c^2$, there exists a $(1,0)$-form ψ such that $\nabla_\omega \psi = \eta$. We have seen the existence of ψ as an element of $\mathcal{E}_1$, ψ may not belong to $\mathcal{E}_c^1$. However $\nabla_\omega\psi (= \eta)$ vanishes identically around x_j $(j = 0, \ldots, m+2)$, there exists a smooth function f_j around x_j such that $\nabla_\omega f_j = \psi$ as in (14). Though f_j is defined only around x_j, $h_j(t)f_j(t)$ can be regarded as defined on T_x. Thus we have

$$\psi - \nabla_\omega\big(\sum_{j=0}^{m+2} h_j(t)f_j(t)\big) \in \mathcal{E}_c^1, \quad \nabla_\omega\Big(\psi - \nabla_\omega\big(\sum_{j=0}^{m+2} h_j(t)f_j(t)\big)\Big) = \eta,$$

which mean that $H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$. Hence we have

$$\dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega) = -\chi(T_x) = m + 1.$$

We have seen that the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$ is surjective. Because of $\dim H^1(\mathcal{E}^\bullet, \nabla_\omega) = \dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$, this map is isomorphic. □

Definition 5. We define

$$\imath_\omega : H^1(\mathcal{E}^\bullet, \nabla_\omega) \to H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$$

by the inverse of the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$.

Here we mention about the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$ in case of $\alpha \notin \mathbb{Z}^{m+3}$ and $\alpha_j \in \mathbb{Z}$. We may assume $\alpha_{m+2} \notin \mathbb{Z}$ otherwise use a projective transformation sending x_k with $\alpha_k \notin \mathbb{Z}$ to ∞. In this case, f_j in (14) cannot be defined. Instead of the circle $C(t)$ in (14), we take a path $C_\infty(t)$ starting from t, approaching to $x_{m+2} = \infty$ turning this point once around positively, and tracing back to t. Then the function

$$g_j(t) = \frac{1}{(e^{2\pi\sqrt{-1}\alpha_{m+2}} - 1)u(t)} \int_{C_\infty(t)} u(t)\psi$$

is single valued on a small neighborhood of t, and satisfies $\nabla_\omega g_j(t) = \psi$. However, if $u(t)\psi$ includes the term $\dfrac{cdt}{t - x_j}$ then $c\log(t - x_j)$ appears and $g_j(t)$ cannot be single valued on U_j. Thus the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$ is not surjective in this case.

In the proof of $H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$, we can kill this term $\dfrac{cdt}{t - x_j}$ by adding a 1-form $\dfrac{-cdt}{(t - x_j)^{\alpha_j+1}}$ to ψ. Since $\nabla_\omega(\dfrac{-dt}{(t - x_j)^{\alpha_j+1}}) = 0$, the property $\nabla_\omega\psi = \eta$ is kept by this addition. Hence we obtain $H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$ in this case. Thus we have the following proposition.

Proposition 4. *If $\alpha \notin \mathbb{Z}^{m+3}$ then*

$$H^0(\mathcal{E}^\bullet, \nabla_\omega) = 0, \quad H^2(\mathcal{E}^\bullet, \nabla_\omega) = 0, \quad H^0(\mathcal{E}_c^\bullet, \nabla_\omega) = 0, \quad H^2(\mathcal{E}_c^\bullet, \nabla_\omega) = 0,$$

$$\dim H^1(\mathcal{E}^\bullet, \nabla_\omega) = \dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega) = m + 1.$$

In the above observation, we see that if $\alpha \notin \mathbb{Z}^{m+3}$ and $\alpha_j \in \mathbb{Z}$ then the natural map $H^1(\mathcal{E}_c^\bullet, \nabla_\omega) \to H^1(\mathcal{E}^\bullet, \nabla_\omega)$ is not surjective though $\dim H^1(\mathcal{E}^\bullet, \nabla_\omega) = \dim H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$. We given an element of the kernel of the natural map in this case. Note that

$$\frac{dh_j(t)}{u(t)} \in \mathcal{E}_c^1, \quad \frac{h_j(t)}{u(t)} \in \mathcal{E}^0 - \mathcal{E}_c^0,$$

where $h_j(t)$ is given in (15). Since

$$\begin{aligned}
\nabla_\omega\left(\frac{dh_j(t)}{u(t)}\right) &= \frac{d \circ d(h_j(t))}{u(t)} - dh_j(t) \wedge d\left(\frac{1}{u(t)}\right) + \omega \wedge \frac{dh_j(t)}{u(t)} \\
&= -dh_j(t) \wedge \nabla_\omega\left(\frac{1}{u(t)}\right) = 0, \\
\nabla_\omega\left(\frac{h_j(t)}{u(t)}\right) &= \frac{d(h_j(t))}{u(t)} + h_j(t) \cdot d\left(\frac{1}{u(t)}\right) + \omega \wedge \frac{h_j(t)}{u(t)} \\
&= \frac{dh_j(t)}{u(t)} + h_j(t) \cdot \nabla_\omega\left(\frac{1}{u(t)}\right) = \frac{dh_j(t)}{u(t)},
\end{aligned}$$

$\dfrac{dh_j(t)}{u(t)}$ is ∇_ω-closed and belongs to $\nabla_\omega(\mathcal{E}^0)$. Thus $\dfrac{dh_j(t)}{u(t)}$ is zero as elements of $H^1(\mathcal{E}^\bullet, \nabla_\omega)$. Recall that $\ker(\nabla_\omega : \mathcal{E}^0 \to \mathcal{E}^1)$ is spanned by $1/u(t)$, which is not single valued on T_x under $\alpha \notin \mathbb{Z}^{m+3}$. Thus the global solution to $\nabla_\omega(f) = \dfrac{dh_j(t)}{u(t)}$ on T_x is unique. Since $\dfrac{h_j(t)}{u(t)} \notin \mathcal{E}_c^0$, $\dfrac{dh_j(t)}{u(t)}$ is different from 0 as elements of $H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$. Hence this is a non-zero element in the kernel of the natural map.

Remark 3. Since there exists $\psi \in \mathcal{E}^1$ such that $\nabla_\omega \psi = \eta$ for any $\eta \in \mathcal{E}^2$, and $1/u(t)$ does not have a compact support,

$$H^2(\mathcal{E}^\bullet, \nabla_\omega) = 0, \quad H^0(\mathcal{E}_c^\bullet, \nabla_\omega) = 0$$

without any conditions on α.

8. Intersection form between twisted cohomology groups

By using $\nabla_{-\omega} = d - \omega\wedge$ instead of ∇_ω, we have twisted cohomology groups $H^k(\mathcal{E}^\bullet, \nabla_{-\omega})$ ($k = 0, 1, 2$), which satisfy

$$H^0(\mathcal{E}^\bullet, \nabla_{-\omega}) = 0, \quad H^2(\mathcal{E}^\bullet, \nabla_{-\omega}) = 0, \quad \dim H^1(\mathcal{E}^\bullet, \nabla_{-\omega}) = m + 1.$$

Definition 6 (Intersection number of $\nabla_{\pm\omega}$-closed 1-forms)**.** The intersection number of $\varphi \in \ker(\nabla_\omega : \mathcal{E}_c^1 \to \mathcal{E}_c^2)$ and $\psi \in \ker(\nabla_{-\omega} : \mathcal{E}^1 \to \mathcal{E}^2)$ is defined by

$$\langle \varphi, \psi \rangle = \iint_{T_x} \varphi \wedge \psi. \tag{16}$$

It induces a bi-linear form between the space of ∇_ω-closed 1-forms and that of $\nabla_{-\omega}$-closed 1-forms.

Since $\varphi \in \mathcal{E}_c^1$ has a compact support, the support of $\varphi \wedge \psi$ is also compact, and the integral in (16) converges.

Theorem 4. *The intersection form between* $\ker(\nabla_\omega : \mathcal{E}_c^1 \to \mathcal{E}_c^2)$ *and* $\ker(\nabla_{-\omega} : \mathcal{E}^1 \to \mathcal{E}^2)$ *descends to that between* $H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$ *and* $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$*. The isomorphism* $\imath_\omega : H^1(\mathcal{E}^\bullet, \nabla_\omega) \to H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$ *given in Definition 5 induces the intersection form between* $H^1(\mathcal{E}^\bullet, \nabla_\omega)$ *and* $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$ *by*

$$\langle \varphi, \psi \rangle = \iint_{T_x} \imath_\omega(\varphi) \wedge \psi.$$

Proof. For a $\nabla_{-\omega}$-closed 1-form $\psi \in \mathcal{E}^1$, $f \in \mathcal{E}_c^0$, we have

$$\begin{aligned}
&\iint_{T_x} (\nabla_\omega f) \wedge \psi = \iint_{T_x} (df + \omega \wedge f) \wedge \psi = \iint_{T_x} (df) \wedge \psi + f \cdot \omega \wedge \psi \\
= &\iint_{T_x} \big(d(f \cdot \psi) - f \cdot d\psi\big) + f \cdot \omega \wedge \psi = \iint_{T_x} d(f \cdot \psi) - f \cdot \nabla_{-\omega}\psi \\
= &\int_{\partial T_x} f \cdot \psi - \iint_{T_x} f \cdot 0 = 0,
\end{aligned}$$

since $f \cdot \psi \in \mathcal{E}_c^1$. Similarly, we can show $\displaystyle\iint_{T_x} \varphi \wedge (\nabla_{-\omega} g) = 0$ for a ∇_ω-closed 1-form $\varphi \in \mathcal{E}_c^1$, $g \in \mathcal{E}^0$. Thus $\langle\ ,\ \rangle$ descends to the intersection form between $H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$ and $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$.

Though $\displaystyle\iint_{T_x} \varphi \wedge \psi$ is not well defined for a ∇_ω-closed 1-form $\varphi \in \mathcal{E}^1$ and a $\nabla_{-\omega}$-closed 1-form $\psi \in \mathcal{E}^1$, $\langle\ ,\ \rangle$ is a well-defined form between $H^1(\mathcal{E}_c^\bullet, \nabla_\omega)$ and $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$ and $\imath_\omega$ is isomorphic. Thus $\langle\ ,\ \rangle$ can be extended to the intersection form between $H^1(\mathcal{E}^\bullet, \nabla_\omega)$ and $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$. □

Theorem 5 ([CM, Theorem 1],[M1, §4]). *The intersection form is non-degenerate. For*

$$\varphi_j = d\log\frac{t-x_j}{t-x_{j+1}}, \quad \psi_k = d\log\frac{t-x_k}{t-x_{k+1}},$$

we have

$$\langle\varphi_j,\psi_k\rangle = 2\pi\sqrt{-1}\begin{cases} \dfrac{-1}{\alpha_j} & \text{if } k=j-1, \\ \dfrac{\alpha_j+\alpha_{j+1}}{\alpha_j\alpha_{j+1}} & \text{if } k=j, \\ \dfrac{-1}{\alpha_{j+1}} & \text{if } k=j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We consider $\iota_\omega : H^1(\mathcal{E}^\bullet,\nabla_\omega) \to H^1(\mathcal{E}_c^\bullet,\nabla_\omega)$. Note that φ_j is a rational 1-form admitting poles only on x_j,x_{j+1} with residues $1,-1$ on $\mathbb{P}^1$. Recall that we have expressed a solution to $\nabla_\omega(f)=\psi$ by the integral in (14). In this time, we express a solution f_ℓ to $\nabla_\omega f_\ell = \varphi_j$ in V_ℓ as a power series.

Put $f_\ell(t) = \sum_{n=0}^\infty a_n(t-x_\ell)^n$ with unknowns a_n's in V_ℓ. Then we can determine uniquely a_n by comparing coefficients of the both sides of $\nabla_\omega f_\ell = \varphi_j$. Since the Laurent expansion of ω around $t=x_\ell$ is

$$\omega = \left[\frac{\alpha_\ell}{t-x_\ell} + b_0 + b_1(t-x_\ell)+\cdots\right]dt,$$

the starting term a_0 of $f_\ell(t)$ is

$$a_0 = \begin{cases} 0 & \text{if} \quad \ell \neq j, j+1, \\ \dfrac{1}{\alpha_\ell} & \text{if} \quad \ell = j, \\ \dfrac{-1}{\alpha_{\ell+1}} & \text{if} \quad \ell = j+1. \end{cases} \tag{17}$$

By the existence of a solution, each series has a positive radius of convergence.

We can regard

$$f = \sum_{\ell=0}^{m+2} h_\ell f_\ell$$

as a function on T_x, where h_ℓ is given in (15). Note that $\varphi_j - \nabla_\omega f$ vanishes identically on U_ℓ, and $\varphi_j - \nabla_\omega f \in \mathcal{E}_c^1$. Hence $\iota_\omega(\varphi_j)$ is represented by $\varphi_j - \nabla_\omega f$, and $\varphi_j - \nabla_\omega f = \varphi_j$ on the complement of $\cup_{\ell=0}^{m+2} V_\ell$. Thus we have

$$\begin{aligned}\langle \varphi_j, \psi_k \rangle =& \iint_{T_x} (\varphi_j - \nabla_\omega f) \wedge \psi_k = \sum_{\ell=0}^{m+2} \iint_{V_\ell - U_\ell} (\varphi_j - \nabla_\omega h_\ell f_\ell) \wedge \psi_k \\ =& -\sum_{\ell=0}^{m+2} \iint_{V_\ell - U_\ell} (\nabla_\omega h_\ell f_\ell) \wedge \psi_k = -\sum_{\ell=0}^{m+2} \iint_{V_\ell - U_\ell} (f_\ell dh_\ell + h_\ell (\nabla_\omega f_\ell)) \wedge \psi_k \\ =& -\sum_{\ell=0}^{m+2} \iint_{V_\ell - U_\ell} f_\ell dh_\ell \wedge \psi_k = -\sum_{\ell=0}^{m+2} \int_{\partial(V_\ell - U_\ell)} h_\ell f_\ell \psi_k = \sum_{\ell=0}^{m+2} \int_{\partial U_\ell} f_\ell \psi_k \\ =& 2\pi\sqrt{-1} \sum_{\ell=0}^{m+2} \mathrm{Res}_{t=x_\ell}(f_\ell \psi_k).\end{aligned}$$

Here we use Stokes' Theorem and Residue Theorem. Note that h_ℓ vanishes identically on ∂V_ℓ, and becomes identically 1 on ∂U_ℓ. Residue calculuses together with (17) yield the expression of $\langle \varphi_j, \psi_k \rangle$ in this proposition.

We have the intersection matrix

$$H_c = \left(\langle \varphi_j, \psi_k \rangle\right)_{0 \le j,k \le m} = 2\pi\sqrt{-1} \begin{pmatrix} \frac{\alpha_0+\alpha_1}{\alpha_0\alpha_1} & \frac{-1}{\alpha_1} & 0 & \cdots & 0 \\ \frac{-1}{\alpha_1} & \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2} & \frac{-1}{\alpha_2} & \ddots & \\ 0 & \frac{-1}{\alpha_2} & \frac{\alpha_2+\alpha_3}{\alpha_2\alpha_3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & \cdots & & \frac{\alpha_m+\alpha_{m+1}}{\alpha_m\alpha_{m+1}} \end{pmatrix}$$

with $\det(H_c) = \dfrac{-(2\pi\sqrt{-1})^{m+1}\alpha_{m+2}}{\alpha_0\alpha_1\cdots\alpha_{m+1}} \neq 0$. Therefore the intersection form is non-degenerate. □

9. Twisted period relation

There is a natural pairing between the twisted cohomology group $H^1(\mathcal{E}^\bullet, \nabla_\omega)$ and the twisted homology group $H_1(C_\bullet^u, \partial_\omega)$ defined by

$$\langle \varphi, \gamma \rangle = \sum_i a_i \int_{I_i} u_{I_i}(t)\varphi, \tag{18}$$

where $\varphi \in H^1(\mathcal{E}^\bullet, \nabla_\omega)$ and $\gamma = \sum_i a_i I_i \otimes u_{I_i}(t) \in H_1(C_\bullet^u, \partial_\omega)$. It is known that this pairing is perfect, and yields solutions to Lauricella's hypergeometric system $\mathcal{F}_D(a,b,c)$. Similarly, we have a natural perfect pairing

$$\langle \psi, \delta \rangle = \sum_j b_j \int_{J_j} u_{J_j}^{-1}(t)\psi, \tag{19}$$

between $\psi \in H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$ and $\delta = \sum_j b_j J_j \otimes u_{J_j}^{-1}(t) \in H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$.

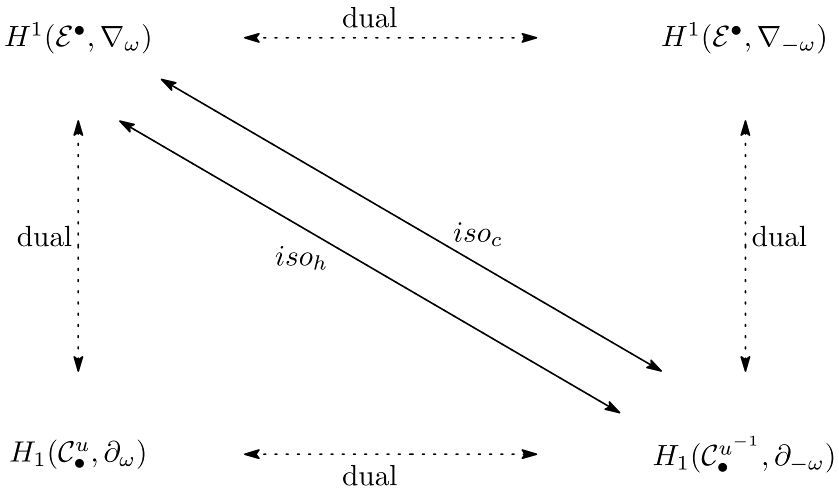
We can regard $H^1(\mathcal{E}^\bullet, \nabla_\omega)$ and $H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$ as dual spaces of $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$ with respect to the intersection form between twisted cohomology groups, and to (19), respectively. Thus we have an isomorphism $iso_c : H^1(\mathcal{E}^\bullet, \nabla_\omega) \to H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$ such that

$$\langle \varphi, \psi \rangle = \langle \psi, iso_c(\varphi) \rangle$$

for any $\psi \in H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$. On the other hand, it is also possible to regard them as dual spaces of $H_1(C^u_\bullet, \partial_\omega)$ with respect to (18), and to the intersection form between twisted homology groups, respectively. There is an isomorphism $iso_h : H^1(\mathcal{E}^\bullet, \nabla_\omega) \to H_1(C^{u^{-1}}_\bullet, \partial_{-\omega})$ such that

$$\langle \varphi, \gamma \rangle = \langle \gamma, iso_h(\varphi) \rangle$$

for any $\gamma \in H_1(C^u_\bullet, \partial_\omega)$. See the diagram below.



As shown in [KY1, §1] and [CM, §3], the following theorem holds.

Theorem 6 (Compatibility of intersection forms)**.** *The isomorphisms iso_c and iso_h coincide.*

Remark 4. Definition 3 does not directly yield a bilinear form between the spaces of twisted 1-cycles. Note that there is a case that simplexes I_i and J_j do not intersect transversally, where I_i and J_j are 1-simplexes of twisted 1-cycles $\gamma = \sum_i a_i I_i \otimes u(t)$ and $\delta = \sum_j b_j J_j \otimes u^{-1}(t)$, respectively. By using iso_c, we can define the intersection form between twisted homology groups $H_1(C^u_\bullet, \partial_\omega)$ and $H_1(C^{u^{-1}}_\bullet, \partial_{-\omega})$ by

$$\langle \gamma, \delta \rangle = \langle iso_c^{-1}(\delta), \gamma \rangle = \sum_i a_i \int_{I_i} u(t) iso_c^{-1}(\gamma). \tag{20}$$

Though there is no ambiguity in this definition, we cannot directly evaluate $\langle \gamma, \delta \rangle$ by this definition. We can show that (20) is equal to the value given in Definition 3, refer to [M3, §7].

Corollary 1 ([CM, Theorem 2])**.** *For bases ${}^t(\varphi_0, \dots, \varphi_m)$, $(\psi_0, \dots, \psi_m)$, $(\gamma_0, \dots, \gamma_m)$, ${}^t(\delta_0, \dots, \delta_m)$ of $H^1(\mathcal{E}^\bullet, \nabla_\omega)$, $H^1(\mathcal{E}^\bullet, \nabla_{-\omega})$, $H_1(C^u_\bullet, \partial_\omega)$, $H_1(C^{u^{-1}}_\bullet, \partial_{-\omega})$, construct four $(m+1) \times (m+1)$ matrices*

$$\Pi_\omega = \big(\langle \varphi_j, \gamma_k \rangle\big), \quad \Pi_{-\omega} = \big(\langle \psi_j, \delta_k \rangle\big), \quad H_c = \big(\langle \varphi_j, \psi_k \rangle\big), \quad H_h = \big(\langle \gamma_j, \delta_k \rangle\big).$$

Then they satisfy a twisted period relation

$$\Pi_\omega \, {}^t H_h^{-1} \, {}^t \Pi_{-\omega} = H_c, \quad i.e. \quad {}^t \Pi_{-\omega} H_c^{-1} \, \Pi_\omega = {}^t H_h. \tag{21}$$

Proof. By Theorem 6, we show this corollary. Let M be the representation matrix of $iso_c (= iso_h)$ with respect to bases ${}^t(\varphi_0, \dots, \varphi_m)$ and ${}^t(\delta_0, \dots, \delta_m)$, i.e.,

$${}^t(iso_h(\varphi_0), \dots, iso_h(\varphi_m)) = M \, {}^t(\delta_0, \dots, \delta_m).$$

We have

$$\begin{aligned}H_c =&(\langle\varphi_j,\psi_k\rangle)_{j,k} = {}^t(\varphi_0,\dots,\varphi_m)\cdot(\psi_0,\dots,\psi_m) = {}^t(iso_c(\varphi_0),\dots,iso_c(\varphi_m))\cdot(\psi_0,\dots,\psi_m)\\ =&M\,{}^t(\delta_0,\dots,\delta_m)\cdot(\psi_0,\dots,\psi_m) = M\,{}^t\Pi_{-\omega},\end{aligned}$$

where $\cdot$ denotes the pairing with matrix arrangement. Similarly, we have

$$\begin{aligned}\Pi_\omega =&(\langle\varphi_j,\gamma_k\rangle)_{j,k} = {}^t(\varphi_0,\dots,\varphi_m)\cdot(\gamma_0,\dots,\gamma_m) = {}^t(iso_h(\varphi_0),\dots,iso_h(\varphi_m))\cdot(\gamma_0,\dots,\gamma_m)\\ =&M\,{}^t(\delta_0,\dots,\delta_m)\cdot(\gamma_0,\dots,\gamma_m) = M\,{}^tH_h.\end{aligned}$$

By eliminating M from two identities $H_c = M\,{}^t\Pi_{-\omega}$ and $\Pi_\omega = M\,{}^tH_h$, we have $H_c = \Pi_\omega\,{}^tH_h^{-1}\,{}^t\Pi_{-\omega}$. □

We explain several examples in [CM, §4]. In case of $m = 0$, the twisted (co)homology groups in Corollary 1 are one dimensional, where $T_x = \mathbb{P}^1 - \{0,1,\infty\}$ and $u(t) = t^p(1-t)^q$, $1/u(t) = t^{-p}(1-t)^{-q}$. Take bases of $H^1(\mathcal{E}^\bullet,\nabla_\omega)$, $H^1(\mathcal{E}^\bullet,\nabla_{-\omega})$, $H_1(C_\bullet^u,\partial_\omega)$, $H_1(C_\bullet^{u^{-1}},\partial_{-\omega})$ as $\varphi_0 = \dfrac{dt}{t(1-t)}$, $\psi_0 = \dfrac{dt}{t(1-t)}$, γ_0 given in (11), δ_0 replaced p,q into $-p,-q$ for γ_0, respectively. Note that

$$\langle\varphi_0,\gamma_0\rangle = B(p,q),\qquad \langle\psi_0,\delta_0\rangle = B(-p,-q)$$

by (18), (19) and Proposition 1, and that they are well defined under $p,q\notin\mathbb{Z}$. By Theorems 3 and 5, we have

$$\langle\varphi_0,\psi_0\rangle = \frac{2\pi\sqrt{-1}(p+q)}{pq},\qquad \langle\gamma_0,\delta_0\rangle = \frac{1-e^{2\pi\sqrt{-1}(p+q)}}{(1-e^{2\pi\sqrt{-1}p})(1-e^{2\pi\sqrt{-1}q})}.$$

In this case, the twisted period relation (21) is equivalent to $\langle\varphi_0,\gamma_0\rangle\langle\psi_0,\delta_0\rangle = \langle\varphi_0,\psi_0\rangle\langle\gamma_0,\delta_0\rangle$, which yields the inversion formula (2) for the Beta function.

In case of $m = 1$, we derive identities for hypergeometric functions from the twisted period relation. Recall our setting

$$x_0 = 0,\quad x = x_1\ (0 < x_1 < 1),\quad x_2 = 1,\quad x_3 = \infty,$$

$$\alpha_0 = b-c,\quad \alpha_1 = -b,\quad \alpha_2 = c-a,\quad \alpha_3 = a,$$
$$u(t) = t^{b-c}(t-x)^{-b}(t-1)^{c-a},\quad \omega = d\log u(t).$$

To simplify the twisted period relation, we assume $\alpha_2+\alpha_3 = c\notin\mathbb{Z}$. We select ∇_ω-closed forms

$$\varphi_2 = d\log\frac{t-x_2}{t-x_3} = \frac{dt}{t-1},\qquad \varphi_0 = d\log\frac{t-x_0}{t-x_1} = \frac{-xdt}{t(t-x)},$$

and they are $\nabla_{-\omega}$-closed. Let γ_2,γ_0 and δ_2,δ_0 be bases of $H_1(C_\bullet^u,\partial_\omega)$ and $H_1(C_\bullet^{u^{-1}},\partial_{-\omega})$. Here recall that the twisted cycles γ_j and δ_j are given by the path I_j from x_j to x_{j+1} and branches of $u(t)$ and $1/u(t)$ on them, respectively. We have four matrices:

$$\Pi_\omega = \begin{pmatrix}\langle\varphi_2,\gamma_2\rangle & \langle\varphi_2,\gamma_0\rangle\\ \langle\varphi_0,\gamma_2\rangle & \langle\varphi_0,\gamma_0\rangle\end{pmatrix},\qquad \Pi_{-\omega} = \begin{pmatrix}\langle\varphi_2,\delta_2\rangle & \langle\varphi_2,\delta_0\rangle\\ \langle\varphi_0,\delta_2\rangle & \langle\varphi_0,\delta_0\rangle\end{pmatrix},$$

$$H_c = 2\pi\sqrt{-1}\begin{pmatrix} \frac{\alpha_2+\alpha_3}{\alpha_2\alpha_3} & 0 \\ 0 & \frac{\alpha_0+\alpha_1}{\alpha_0\alpha_1} \end{pmatrix}, \quad H_h = \begin{pmatrix} \frac{1-\theta_2\theta_3}{(1-\theta_2)(1-\theta_3)} & 0 \\ 0 & \frac{1-\theta_0\theta_1}{(1-\theta_0)(1-\theta_1)} \end{pmatrix}.$$

Any entry of $\Pi_{\pm\omega}$ can be expressed by the hypergeometric function. For example,

$$\begin{aligned} \langle\varphi_2, \gamma_2\rangle &= B(a, c-a)F(a, b, c; x) \\ \langle\varphi_2, \gamma_0\rangle &= -e^{\pi\sqrt{-1}(c-a-b)}x^{1-c}B(b-c+1, -b+1)F(b-c+1, a-c+1, 2-c; x). \end{aligned}$$

Here note that the second is given by the variable change $t = x/s$. By the $(1, 2)$-entry of the first identity of (21) in Corollary 1, we have

$$F(a, b, c; x)F(1-a, 1-b, 2-c; x) = F(b-c+1, a-c+1, 2-c; x)F(c-a, c-b, c; x).$$

By its $(1, 1)$-entry, we have

$$\begin{aligned} &F(a, b, c; x)F(-a, -b, -c; x) - 1 \\ =&\frac{ab(c-a)(c-b)}{c^2(c+1)(c-1)}x^2F(b-c+1, a-c+1, 2-c; x)F(c-b+1, c-a+1, 2+c; x). \end{aligned}$$

In case of general m, the twisted period relation yields identities among some values of Lauricella's hypergeometric series F_D with several parameters. Recall our setting

$$x_0 = 0, \quad x = (x_1, \dots, x_m) \in \mathbb{D}_m, \quad x_{m+1} = 1, \quad x_{m+2} = \infty,$$

$$\alpha_0 = -c + \sum_{j=1}^{m} b_j, \quad \alpha_j = -b_j \ (1 \le j \le m), \quad \alpha_{m+1} = c-a, \quad \alpha_{m+2} = a,$$

$$u = u(t) = t^{b_1+\dots+b_m-c}(t-x_1)^{-b_1}\cdots(t-x_m)^{-b_m}(t-1)^{c-a}.$$

$$\begin{aligned} \varphi_0 &= \frac{dt}{t-1}, \quad \varphi_j = \frac{dt}{t-x_j}, \\ \psi_0 &= \frac{dt}{t(t-1)}, \quad \psi_j = \frac{1}{x_j}\left(\frac{dt}{t-x_j} - \frac{dt}{t-x_0}\right) = \frac{dt}{t(t-x_j)}, \end{aligned}$$

where $1 \le j \le m$ and the index start from 0 for bases of twisted (co)homology groups. We assume that $\alpha_{m+1} + \alpha_{m+2} = c \notin \mathbb{Z}$.

We set twisted cycles γ_0 and δ_0 by the regularization of the open interval $(1, \infty)$ with respect to $u(t)$ and that to $1/u(t)$, respectively. For any bases of $H_1(C_\bullet^u, \partial_\omega)$ and $H_1(C_\bullet^{u^{-1}}, \partial_{-\omega})$ including γ_0 and δ_0, the identity for the $(0, 0)$-entry of ${}^t\Pi_{-\omega}H_c^{-1}\Pi_\omega = {}^tH_h$ is

$$\begin{aligned} &(\int_1^\infty u^{-1}\psi_0, \dots, \int_1^\infty u^{-1}\psi_m)H_c^{-1}\,{}^t(\int_1^\infty u\varphi_0, \dots, \int_1^\infty u\varphi_m) \\ =\ &(0, 0)\text{-entry of } H_h = \frac{1-e^{2\pi\sqrt{-1}c}}{(1-e^{2\pi\sqrt{-1}(c-a)})(1-e^{2\pi\sqrt{-1}a})}. \end{aligned}$$

The intersection matrix H_c is diagonal and

$$H_c^{-1} = \frac{-1}{2\pi\sqrt{-1}}\mathrm{diag}(a-c, b_1x_1, \dots, b_mx_m).$$

The integrals are expressed as

$$\begin{pmatrix} \int_1^\infty u\varphi_0 \\ \int_1^\infty u\varphi_1 \\ \vdots \\ \int_1^\infty u\varphi_m \end{pmatrix} = \begin{pmatrix} B(a,c-a)F_D(a,b,c;x) \\ B(a,c+1-a)F_D(a,b+e_1,c+1;x) \\ \vdots \\ B(a,c+1-a)F_D(a,b+e_1,c+1;x) \end{pmatrix},$$

$$\begin{pmatrix} \int_1^\infty u^{-1}\psi_0 \\ \int_1^\infty u^{-1}\psi_1 \\ \vdots \\ \int_1^\infty u^{-1}\psi_m \end{pmatrix} = \begin{pmatrix} B(1-a,a-c)F_D(1-a,-b,1-c;x) \\ B(1-a,a-c+1)F_D(1-a,e_1-b,2-c;x) \\ \vdots \\ B(1-a,a-c+1)F_D(1-a,e_m-b,2-c;x) \end{pmatrix},$$

where e_j is the j-th unit vector of size m. By using the inversion formula for the Beta function

$$\begin{aligned} B(a,c-a)B(1-a,a-c) =& B(a,c-a)\cdot\Big(\frac{-a}{-c}\Big)B(-a,a-c) \\ =& 2\pi\sqrt{-1}\frac{1}{c-a}\cdot\frac{1-e^{2\pi\sqrt{-1}c}}{(1-e^{2\pi\sqrt{-1}(c-a)})(1-e^{2\pi\sqrt{-1}a})}, \\ B(a,c-a+1)B(1-a,a-c+1) =& \Big(\frac{c-a}{c}\cdot B(a,c-a)\Big)\cdot\Big(\frac{-a(a-c)}{-c(1-c)}B(-a,a-c)\Big) \\ =& 2\pi\sqrt{-1}\frac{a-c}{c(1-c)}\cdot\frac{1-e^{2\pi\sqrt{-1}c}}{(1-e^{2\pi\sqrt{-1}(c-a)})(1-e^{2\pi\sqrt{-1}a})}, \end{aligned}$$

we have an identity

$$\begin{aligned} & F_D(a,b,c;x)F_D(1-a,-b,1-c;x)-1 \\ = & \frac{c-a}{c(c-1)}\sum_{j=1}^{m} b_j x_j F_D(a,e_j+b,c+1;x)F_D(1-a,e_j-b,2-c;x). \end{aligned}$$

A. Elliott's identity as the twisted period relation

In Appendix, we show that Elliott's identity among hypergeometric series can be obtained from the twisted period relation (21) in Corollary 1 with detailed calculations.

A.1 Integral representations

Elliott's identity is given in [BNPV] as

$$\begin{aligned} & F(\frac{1}{2}+\lambda,-\frac{1}{2}-\nu,1+\lambda+\mu;r)F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r) \\ +& F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r) \\ -& F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r) \\ =& \frac{\Gamma(1+\lambda+\mu)\Gamma(1+\mu+\nu)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}, \end{aligned} \tag{22}$$

where λ, μ, ν are complex parameters with

$$1+\lambda+\mu, 1+\mu+\nu \neq 0, -1, -2, \dots, \tag{23}$$

and the main variable r satisfies inequalities $|r| < 1$ and $|1-r| < 1$. We have

$$F(\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r) = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^1 t^{\lambda-1/2}(1-t)^{\mu-1/2}(1-rt)^{\nu+1/2}dt,$$

$$F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r) = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^1 t^{\lambda-1/2}(1-t)^{\mu-1/2}(1-rt)^{\nu-1/2}dt,$$

$$F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r)$$
$$=\frac{\Gamma(\mu+\nu+1)}{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{1}{2})}\int_0^1 s^{-\lambda-1/2}(1-s)^{\lambda+\mu+\nu-1/2}(1-(1-r)s)^{-\nu-1/2}ds$$
$$=\frac{\Gamma(\mu+\nu+1)}{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{1}{2})}\int_{-\infty}^0 (-t)^{-\lambda-1/2}(1-t)^{-\mu-1/2}(1-rt)^{-\nu-1/2}dt,$$

$$F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r)$$
$$=\frac{\Gamma(\mu+\nu+1)}{\Gamma(-\lambda-\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}\int_0^1 s^{-\lambda-3/2}(1-s)^{\lambda+\mu+\nu+1/2}(1-(1-r)s)^{-\nu-1/2}ds$$
$$=\frac{\Gamma(\mu+\nu+1)}{\Gamma(-\lambda-\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}\int_{-\infty}^0 (-t)^{-\lambda-3/2}(1-t)^{-\mu-1/2}(1-rt)^{-\nu-1/2}dt,$$

under the condition

$$\lambda, \mu, \nu, \lambda+\mu+\nu \notin \{\frac{1}{2}+n \mid n \in \mathbb{Z}\}, \tag{24}$$

where we use the regularization of $(0,1)$ or $(-\infty, 0)$ with respect to the integrand if necessary. Here note that a variable change $s = t/(t-1)$ is used.

A.2 Setting of a local system

Hereafter we assume the conditions (23) and (24), and we set

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$
$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \Big(\frac{1}{t} - \frac{1}{t-1/r}\Big)dt,$$
$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \Big(\frac{1}{t} - \frac{1}{t-1}\Big)dt.$$

Then we have

$$1/u(t) = t^{-1/2-\lambda}(1-t)^{1/2-\mu}(1-rt)^{-1/2-\nu},$$

and

$$F(\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r) = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^1 u(t)\varphi_1,$$
$$F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r) = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^1 u(t)\varphi_2;$$

$$F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r) = \frac{\Gamma(\mu+\nu+1)e^{\pi\sqrt{-1}(\lambda+1/2)}}{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{1}{2})}\int_{-\infty}^{0}u(t)^{-1}\psi_1,$$
$$= \frac{\sqrt{-1}\Gamma(\mu+\nu+1)e^{\pi\sqrt{-1}\lambda}}{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}(\lambda+\mu+\nu+\frac{1}{2})\int_{-\infty}^{0}u(t)^{-1}\psi_1,$$
$$F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r) = \frac{\Gamma(\mu+\nu+1)e^{\pi\sqrt{-1}(\lambda+3/2)}}{\Gamma(-\lambda-\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}\int_{-\infty}^{0}u(t)^{-1}\psi_2$$
$$= \frac{-\sqrt{-1}\Gamma(\mu+\nu+1)e^{\pi\sqrt{-1}\lambda}}{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}(-\lambda-\frac{1}{2})\int_{-\infty}^{0}u(t)^{-1}\psi_2.$$

Thus integrals are expressed as

$$\Big(\int_0^1 u(t)\varphi_1,\int_0^1 u(t)\varphi_2\Big)$$
$$=\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}\Big(F(\frac{1}{2}+\lambda,-\frac{1}{2}-\nu,1+\lambda+\mu;r),F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)\Big); \tag{25}$$

$$\begin{pmatrix}\int_{-\infty}^{0}u(t)^{-1}\psi_1\\ \int_{-\infty}^{0}u(t)^{-1}\psi_2\end{pmatrix}$$
$$=\frac{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}{\sqrt{-1}e^{\pi\sqrt{-1}\lambda}\Gamma(\mu+\nu+1)}\begin{pmatrix}\frac{1}{\lambda+\mu+\nu+1/2} & 0\\ 0 & \frac{1}{\lambda+1/2}\end{pmatrix}\begin{pmatrix}F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\\ F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\end{pmatrix}. \tag{26}$$

A.3 Transform of a twisted period relation into Elliott's identity

The intersection matrix H_c of φ_1,φ_2 and ψ_1,ψ_2 is

$$H_c = 2\pi\sqrt{-1}\begin{pmatrix}\dfrac{1}{1/2+\lambda+\mu+\nu} & \dfrac{1}{1/2+\lambda}\\ 0 & \dfrac{1}{1/2+\lambda}\end{pmatrix}$$

by Theorem 5. We take a basis of the twisted homology group for $u(t)$ and that for $1/u(t)$ by extending γ and δ, respectively, where γ and δ are the twisted cycles given by the regularization of $(0,1)$ with respect to $u(t)$ and that of $(-\infty,0)$ with respect to $u^{-1}(t)$. The intersection number of these twisted cycles is $\dfrac{1}{-e^{2\pi\sqrt{-1}\lambda}-1}$ by Theorem 3. Consider the (1, 1)-entry of

$${}^t\Pi_\omega\ {}^tH_c^{-1}\Pi_{-\omega} = H_h$$

which is the transpose of the second identity of (21) in Corollary 1. Then it yields

$$\Big(\int_0^1 u(t)\varphi_1,\int_0^1 u(t)\varphi_2\Big)\ {}^tH_c^{-1}\begin{pmatrix}\int_{-\infty}^{0}\frac{1}{u(t)}\psi_1\\ \int_{-\infty}^{0}\frac{1}{u(t)}\psi_2\end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}. \tag{27}$$

Rewrite the integrals in the equality (27) in terms of hypergeometric series by (25) and (26). Then its exp and Gamma factors reduce to

$$
\begin{aligned}
&\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}\cdot\frac{\Gamma(-\lambda+\frac{1}{2})\Gamma(\lambda+\mu+\nu+\frac{3}{2})}{\sqrt{-1}e^{\pi\sqrt{-1}\lambda}\Gamma(\mu+\nu+1)}\\
=&\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(1-(\lambda+\frac{1}{2}))}{\sqrt{-1}e^{\pi\sqrt{-1}\lambda}}\cdot\frac{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}\\
=&\frac{\pi}{\sin(\pi(\lambda+\frac{1}{2}))}\cdot\frac{1}{\sqrt{-1}e^{\pi\sqrt{-1}\lambda}}\cdot\frac{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}\\
=&\frac{-2\pi\sqrt{-1}}{e^{2\pi\sqrt{-1}\lambda}+1}\cdot\frac{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)},
\end{aligned}
$$

and the product of ${}^tH_c^{-1}$ and the 2×2-matrix in (26) reduces to

$$
\frac{1}{2\pi\sqrt{-1}}\begin{pmatrix}\frac{1}{2}+\lambda+\mu+\nu & 0\\ -(\frac{1}{2}+\lambda+\mu+\nu) & \frac{1}{2}+\lambda\end{pmatrix}\begin{pmatrix}\frac{1}{1/2+\lambda+\mu+\nu} & 0\\ 0 & \frac{1}{1/2+\lambda}\end{pmatrix}=\frac{1}{2\pi\sqrt{-1}}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.
$$

Hence the equality (27) is transformed into

$$
\begin{aligned}
&\left(F(\frac{1}{2}+\lambda,-\frac{1}{2}-\nu,1+\lambda+\mu;r),F(\frac{1}{2}+\lambda,\frac{1}{2}-\nu,1+\lambda+\mu;r)\right)\\
&\cdot\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\cdot\begin{pmatrix}F(\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\\ F(-\frac{1}{2}-\lambda,\frac{1}{2}+\nu,1+\mu+\nu;1-r)\end{pmatrix}\\
=&\frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})},
\end{aligned}
$$

which is equivalent to Elliott's identity (22).

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"MathemAmplitudes 2019: Intersection Theory & Feynman Integrals"

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