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# Appell-Lauricella's hypergeometric functions and intersection theory

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We study hypergeometric functions by applying the intersection theory for twisted homology and cohomology groups, which arise from their Euler-type integral representations.
In this article, we consider Appell-Lauricella's hypergeometric functions which are classical generalizations of Gauss' hypergeometric function. Especially, we focus on Lauricella's $F_A$ and $F_C$, and study them by using intersection theory. By evaluating some intersection numbers, we obtain quadratic relations between $F_A$ or $F_C$ as a consequence of the twisted period relations.

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## 1. Introduction

Gauss' hypergeometric function is defined by

$$
{}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n,
$$

where $x \in \mathbb{C}$ is a variable, $a, b, c \in \mathbb{C}$ are parameters ($c \notin \mathbb{Z}_{\leq 0}$), and we set $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This series converges on $\{x \in \mathbb{C} \mid |x| < 1\}$. It is well-known that ${}_2F_1(a,b,c;x)$ satisfies the hypergeometric differential equation

$$
\left( x(1-x)\frac{d^2}{dx^2} + (c-(a+b+1)x)\frac{d}{dx} - ab \right) f(x) = 0,
$$

and that if $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$, then ${}_2F_1(a,b,c;x)$ admits an Euler-type integral representation:

$$
{}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^a (1-t)^{c-a} (1-xt)^{-b} \frac{dt}{t(1-t)}. \tag{1}
$$

As classical generalizations of ${}_2F_1$, Lauricella's multi-variable hypergeometric functions $F_A$, $F_B$, $F_C$ and $F_D$ are well-known. These are also called Appell's hypergeometric functions when they are in two variables.

The hypergeometric function ${}_2F_1$ and its generalizations are studied from various view points. In this article, we study them by applying the intersection theory for twisted homology and cohomology groups. Twisted (co)homology groups are associated with local systems defined by multi-valued functions that are integrands of Euler-type integral representations. For example, when we consider ${}_2F_1$, we use the local system defined by the multi-valued function $t^a(1-t)^{c-a}(1-xt)^{-b}$ in $t$ which appears in (1). Aomoto applied such homology and cohomology groups to study of hypergeometric functions.

Intersection pairings of (co)homology groups with coefficients in local systems were defined in [16]. However, since these definitions are written in terms of homological algebra, it seems not easy to evaluate the intersection numbers directly. By [9], intersection numbers of twisted homology groups can be evaluated in terms of topological intersection numbers and branches of the multi-valued function. By [2] and [12], intersection numbers of twisted cohomology groups are expressed by residues of logarithmic forms.

The intersection theory for twisted homology groups are applied to study of the monodromy representations, connection problems, and so on. That for twisted cohomology groups are applied to study of Pfaffian equations, contiguity relations, and so on. By the compatibility of these intersection pairings and the pairings between twisted homology and cohomology groups given by integrations, we can obtain the twisted period relations which imply quadratic relations between hypergeometric functions.

In [15] which is in the same proceedings, Matsumoto introduces the intersection theory for twisted (co)homology groups associated with multi-valued functions in one variable, which arise from ${}_2F_1$ and Lauricella's $F_D$. In this article, we treat those associated with multi-valued functions in multi-variables, focusing on two examples arising from Lauricella's $F_A$ and $F_C$.



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This article is arranged as follows. In Section 2, we give the definitions of Lauricella's hypergeometric functions. We also introduce their Euler-type integral representations and the systems $E_A$, $E_B$, $E_C$ and $E_D$ of differential equations satisfied by $F_A$, $F_B$, $F_C$ and $F_D$, respectively. In Section 3, we give settings of twisted (co)homology groups and intersection pairings to study $F_A$ and $F_C$. In Section 4, we explain how to construct twisted cycles that corresponds to series solutions to $E_A$ or $E_C$. By evaluating intersection numbers, we obtain quadratic relations between $F_A$ or $F_C$ from the twisted period relations.

## 2. Appell-Lauricella's hypergeometric functions

In this section, we give the definitions of Appell-Lauricella's multi-variable hypergeometric functions, and collect some basic facts referring to [14].

### 2.1 Lauricella's hypergeometric series

Lauricella's hypergeometric functions of $m$ variables $x = (x_1, \dots, x_m)$ are defined as follows:

$$\begin{aligned}
&F_A(a, b_1, \dots, b_m, c_1, \dots, c_m; x)\\
&= \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a)_{n_1+\cdots+n_m}(b_1)_{n_1}\cdots(b_m)_{n_m}}{(c_1)_{n_1}\cdots(c_m)_{n_m} n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m} \qquad (|x_1| + \cdots + |x_m| < 1),\\
&F_B(a_1, \dots, a_m, b_1, \dots, b_m, c; x)\\
&= \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a_1)_{n_1}\cdots(a_m)_{n_m}(b_1)_{n_1}\cdots(b_m)_{n_m}}{(c)_{n_1+\cdots+n_m} n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m} \qquad (|x_1|, \dots, |x_m| < 1),\\
&F_C(a, b, c_1, \dots, c_m; x)\\
&= \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a)_{n_1+\cdots+n_m}(b)_{n_1+\cdots+n_m}}{(c_1)_{n_1}\cdots(c_m)_{n_m} n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m} \qquad (\sqrt{|x_1|} + \cdots + \sqrt{|x_m|} < 1),\\
&F_D(a, b_1, \dots, b_m, c; x)\\
&= \sum_{n_1,\dots,n_m=0}^{\infty} \frac{(a)_{n_1+\cdots+n_m}(b_1)_{n_1}\cdots(b_m)_{n_m}}{(c)_{n_1+\cdots+n_m} n_1!\cdots n_m!} x_1^{n_1}\cdots x_m^{n_m} \qquad (|x_1|, \dots, |x_m| < 1).
\end{aligned}$$

When $m = 2$, Lauricella's $F_A$ (resp. $F_B$, $F_C$, $F_D$) is also called Appell's hypergeometric function $F_2$ (resp. $F_3$, $F_4$, $F_1$).

| Lauricella | $F_A$ | $F_B$ | $F_C$ | $F_D$ |
|---|---|---|---|---|
| Appell | $F_2$ | $F_3$ | $F_4$ | $F_1$ |

**Table 1**

### 2.2 Euler-type integral representations

To obtain Euler-type integral representations of Lauricella's hypergeometric functions, we use

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt = B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

and its generalization

$$\int_{\Delta^m} \prod_{k=1}^{m} t_k^{p_k-1} \cdot \left(1-\sum_{k=1}^{m} t_k\right)^{p_{m+1}-1} dt_1 \wedge \cdots \wedge dt_m = \frac{\prod_{k=1}^{m+1} \Gamma(p_k)}{\Gamma(\sum_{k=1}^{m+1} p_k)},$$

where $\Delta^m = \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_1, \ldots, t_m, 1-\sum_{k=1}^m t_k > 0\}$.

#### 2.2.1 Euler-type integral representation of $F_A$

For simplicity, we denote $\sum = \sum_{k=1}^m$ and $\prod = \prod_{k=1}^m$. If $\mathrm{Re}(b_k), \mathrm{Re}(c_k - b_k) > 0$, then

$$\begin{aligned} &F_A(a, b_1, \ldots, b_m, c_1, \ldots, c_m; x) \\ &= \prod \frac{\Gamma(c_k)}{\Gamma(b_k)\Gamma(c_k-b_k)} \cdot \int_{(0,1)^m} \prod \left(t_k^{b_k-1} \cdot (1-t_k)^{c_k-b_k-1}\right) \cdot \left(1-\sum x_k t_k\right)^{-a} dt, \end{aligned} \tag{2}$$

where $dt = dt_1 \wedge \cdots \wedge dt_m$. To show this expression, it is sufficient to consider the power series expansion of $(1 - \sum x_k t_k)^{-a}$.

#### 2.2.2 Euler-type integral representation of $F_B$

If $\mathrm{Re}(b_k), \mathrm{Re}(c - \sum b_k) > 0$, then

$$\begin{aligned} &F_B(a_1, \ldots, a_m, b_1, \ldots, b_m, c; x) \\ &= \frac{\Gamma(c)}{\Gamma(c-\sum b_k)\prod \Gamma(b_k)} \cdot \int_{\Delta^m} \prod \left(t_k^{b_k-1} \cdot (1-x_k t_k)^{-a_k}\right) \cdot \left(1-\sum t_k\right)^{c-\sum b_k-1} dt. \end{aligned}$$

If we put $t_k = \frac{s_k}{x_k}$, the differential form changes into

$$\prod x_k^{-b_k} \cdot \prod \left(s_k^{b_k-1} \cdot (1-s_k)^{-a_k}\right) \cdot \left(1-\sum \frac{1}{x_k} s_k\right)^{c-\sum b_k-1} ds.$$

By the integral representation (2), we expect that $F_B(a_1, \ldots, a_m, b_1, \ldots, b_m, c; x_1, \ldots, x_m)$ and

$$\prod x_k^{-b_k} \cdot F_A\Big(\sum b_k - c + 1, b_1, \ldots, b_m, b_1 - a_1 + 1, \ldots, b_m - a_m + 1; \frac{1}{x_1}, \ldots, \frac{1}{x_m}\Big).$$

have similar properties; see also Remark 2.1.

#### 2.2.3 Euler-type integral representation of $F_C$

By considering the power series expansion, we expect the integral representation

$$\begin{aligned} &F_C(a, b, c_1, \ldots, c_m; x) \\ &= \frac{\Gamma(1-a)}{\prod \Gamma(1-c_k) \cdot \Gamma(\sum c_k - a - m + 1)} \cdot \int_{\Delta} \prod t_k^{-c_k} \cdot \left(1-\sum t_k\right)^{\sum c_k - a - m} \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} dt. \end{aligned} \tag{3}$$

However, if we put $\Delta = \Delta^m$, the power series expansion of $(1 - \sum \frac{x_k}{t_k})^{-b}$ is divergent. A "twisted cycle" $\Delta$ such that this integral representation holds has been constructed in [8]. In Section 4.2, we will explain the construction. Note that the zero set of the factor $(1 - \sum \frac{x_k}{t_k})$ is not a hyperplane, while those of the factors in the integral representations of $F_A$, $F_B$ and $F_D$ are hyperplanes.

#### 2.2.4 Euler-type integral representations of $F_D$

There are two types of integral representations of $F_D$. The first one is expressed as a one-dimensional integral; if $\mathrm{Re}(a)$, $\mathrm{Re}(c-a) > 0$, then

$$F_D(a, b_1, \ldots, b_m, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \cdot \int_0^1 t^{a-1}(1-t)^{c-a-1} \prod_{i=1}^{m} (1-x_i t)^{-b_i} \, dt. \tag{4}$$

The second one is expressed as an $m$-dimensional integral; if $\mathrm{Re}(b_k)$, $\mathrm{Re}(c - \sum b_k) > 0$, then

$$\begin{aligned} &F_D(a, b_1, \ldots, b_m, c; x) \\ &= \frac{\Gamma(c)}{\Gamma(c - \sum b_k) \prod \Gamma(b_k)} \cdot \int_{\Delta^m} \prod t_k^{b_k - 1} \cdot \left(1 - \sum t_k\right)^{c - \sum b_k - 1} \left(1 - \sum x_k t_k\right)^{-a} dt. \end{aligned}$$

### 2.3 Differential equations

Lauricella's $F_A$, $F_B$, $F_C$ and $F_D$ are solutions to the regular holonomic systems $E_A$, $E_B$, $E_C$ and $E_D$ of linear differential equations, respectively.

#### 2.3.1 Differential equations for $F_A$

We set $\partial_i = \frac{\partial}{\partial x_i}$. Lauricella's $F_A(a, b_1, \ldots, b_m, c_1, \ldots, c_m; x)$ satisfies

$$\Big( x_k(1-x_k)\partial_k^2 - x_k \sum_{\substack{1 \le i \le m \\ i \ne k}} x_i \partial_k \partial_i + (c_k - (a + b_k + 1)x_k)\partial_k - b_k \sum_{\substack{1 \le i \le m \\ i \ne k}} x_i \partial_i - a b_k \Big) f(x) = 0$$

for $k = 1, \ldots, m$. Let $E_A(a, b_1, \ldots, b_m, c_1, \ldots, c_m)$ be the system of differential equations generated by them. It is known that the rank of $E_A$ is $2^m$, and the set of singular points (called the singular locus) is

$$S_A = \left( \prod_{k=1}^{m} x_k \cdot \prod_{\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}} \left(1 - \sum_{p=1}^{r} x_{i_p}\right) = 0 \right) \subset \mathbb{C}^m.$$

For example, $S_A = (x_1 x_2 (1-x_1)(1-x_2)(1-x_1-x_2) = 0)$ if $m = 2$. Let $x \notin S_A$ be a point near to the origin. If $c_k \notin \mathbb{Z}$, then a basis of the local solution space of $E_A$ is given by

$$f_A^{i_1 \cdots i_r} = \prod_{p=1}^{r} x_{i_p}^{1 - c_{i_p}} \cdot F_A\Big(a + r - \sum_{p=1}^{r} c_{i_p}, b^I, c^I; x\Big) \quad (I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}), \tag{5}$$

where the vectors $b^I = (b_1^I, \ldots, b_m^I)$ and $c^I = (c_1^I, \ldots, c_m^I)$ are defined by

$$\begin{cases} b_{i_p}^I = b_{i_p} + 1 - c_{i_p}, \\ b_j^I = b_j, \end{cases} \qquad \begin{cases} c_{i_p}^I = 2 - c_{i_p}, \\ c_j^I = c_j \end{cases} \qquad (i_p \in I, \; j \notin I).$$

Note that if $I = \emptyset$, then $f_A^{\emptyset} = F_A(a, b_1, \ldots, b_m, c_1, \ldots, c_m; x)$.

#### 2.3.2 Differential equations for $F_B$

Lauricella's $F_B(a_1,\dots,a_m,b_1,\dots,b_m,c;x)$ satisfies

$$\Big(x_k(1-x_k)\partial_k^2+\sum_{\substack{1\le i\le m\\ i\neq k}} x_i\partial_k\partial_i+(c-(a_k+b_k+1)x_k)\partial_k-a_kb_k\Big)f(x)=0$$

for $k=1,\dots,m$. Let $E_B(a_1,\dots,a_m,b_1,\dots,b_m,c)$ be the system of differential equations generated by them.

*Remark* 2.1. In fact, by setting $x_k=\frac{1}{\xi_k}$ ($k=1,\dots,m$), we have

$$\begin{aligned}&f(x)\text{ is a solution to }E_B(a_1,\dots,a_m,b_1,\dots,b_m,c)\\ \Longleftrightarrow\ &\prod\xi^{b_k}\cdot f(\xi)\text{ is a solution to }E_A\Big(\sum b_k-c+1,b_1,\dots,b_m,b_1-a_1+1,\dots,b_m-a_m+1\Big).\end{aligned}$$

Thus, some results for $F_B$ are obtained from those for $F_A$.

#### 2.3.3 Differential equations for $F_C$

Lauricella's $F_C(a,b,c_1,\dots,c_m;x)$ satisfies

$$\begin{aligned}\Big(x_k(1-x_k)\partial_k^2-x_k\sum_{\substack{1\le i\le m\\ i\neq k}}x_i\partial_i\partial_k-\sum_{\substack{1\le i,j\le m\\ i\neq k}}x_ix_j\partial_i\partial_j+(c_k-(a+b+1)x_k)\partial_k\\ -(a+b+1)\sum_{\substack{1\le i\le m\\ i\neq k}}x_i\partial_i-ab\Big)f(x)=0\end{aligned}$$

for $k=1,\dots,m$. Let $E_C(a,b,c_1,\dots,c_m)$ be the system of differential equations generated by them. It is known that the rank of $E_C$ is $2^m$, and the singular locus is

$$S_C=\Big(\prod_{k=1}^m x_k\cdot\prod_{\varepsilon_1,\dots\varepsilon_m=\pm1}\Big(1+\sum_k\varepsilon_k\sqrt{x_k}\Big)=0\Big)\subset\mathbb{C}^m.$$

For example, $S_C=(x_1x_2(x_1^2+x_2^2-2x_1x_2-2x_1-2x_2+1)=0)$ if $m=2$. Let $x\notin S_C$ be a point near to the origin. If $c_k\notin\mathbb{Z}$, then a basis of the local solution space of $E_C$ is given by

$$f_C^{i_1\cdots i_r}=\prod_{p=1}^r x_{i_p}^{1-c_{i_p}}\cdot F_C\Big(a+r-\sum_{p=1}^r c_{i_p},b+r-\sum_{p=1}^r c_{i_p},c^I;x\Big)\quad(I=\{i_1,\dots,i_r\}\subset\{1,\dots,m\}),\tag{6}$$

where the vector $c^I=(c_1^I,\dots,c_m^I)$ is defined by

$$\begin{cases}c_{i_p}^I=2-c_{i_p} & (i_p\in I),\\ c_j^I=c_j & (j\notin I).\end{cases}$$

Note that if $I=\emptyset$, then $f_C^{\emptyset}=F_C(a,b,c_1,\dots,c_m;x)$.

#### 2.3.4 Differential equations for $F_D$

Lauricella's $F_D(a, b_1, \ldots, b_m, c; x)$ satisfies

$$\left(x_k(1-x_k)\partial_k^2 + (1-x_k)\sum_{\substack{1\le i\le m\\ i\ne k}} x_i\partial_i\partial_k + (c-(a+b_k+1)x_k)\partial_k - b_k\sum_{\substack{1\le i\le m\\ i\ne k}} x_i\partial_i - ab_k\right)f(x) = 0$$

for $k = 1, \ldots, m$, and

$$\left((x_i - x_j)\partial_i\partial_j - b_j\partial_i + b_i\partial_j\right)f(x) = 0$$

for $1 \le i < j \le m$. Let $E_D(a, b_1, \ldots, b_m, c)$ be the system of differential equations generated by them. It is known that the rank of $E_D$ is $m + 1$, and the singular locus is

$$S_D = \left(\prod_{k=1}^{m} x_k(1-x_k) \cdot \prod_{1\le i<j\le m}(x_i - x_j) = 0\right) \subset \mathbb{C}^m.$$

For example, $S_D = (x_1x_2(1-x_1)(1-x_2)(x_1-x_2) = 0)$ if $m = 2$.

We do not have a basis of the local solution space, which are expressed by $F_D$. On the other hand, $F_D$ is well-studied from the view point of the integral representation (4), because twisted (co)homology theory for a one-dimensional integral is easier than the higher dimensional cases. In [15], the study of $F_D$ in the framework of the intersection theory for twisted (co)homology groups is explained.

## 3. Twisted homology and cohomology groups for $F_A$ and $F_C$

In this article, we focus on $F_A$ and $F_C$. We study them by using twisted (co)homology groups that are associated with the Euler-type integral representations.

*Assumption* 3.1. Hereafter, we assume that the parameters satisfy some non-integral conditions.

- For $F_A$, we assume

$$b_1, \ldots, b_m,\ c_1 - b_1, \ldots, c_m - b_m,\ a - \sum_{p=1}^{r} c_{i_r} \notin \mathbb{Z} \quad (\text{for any } \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}), \tag{7}$$

$$c_1, \ldots, c_m \notin \mathbb{Z}. \tag{8}$$

- For $F_C$, we assume

$$a - \sum_{p=1}^{r} c_{i_r},\ \ b - \sum_{p=1}^{r} c_{i_r} \notin \mathbb{Z} \quad (\text{for any } \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}), \tag{9}$$

$$c_1, \ldots, c_m \notin \mathbb{Z}. \tag{10}$$

The system $E_A$ (resp. $E_C$) is irreducible when the condition (7) (resp. (9)) holds. Under the condition (8) (resp. (10)), the series solutions (5) (resp. (6)) form a basis of local solution space of $E_A$ (resp. $E_C$).

For basic ideas of twisted (co)homology groups and intersection theory, refer to [15].

### 3.1 Twisted homology and cohomology groups

Recall the integral representations (2) and (3). Up to $\Gamma$-factors, $F_A$ and $F_C$ are expressed by the integrals

$$\int_{(0,1)^m} \prod \left(t_k^{b_k}(1-t_k)^{c_k-b_k-1}\right) \cdot \left(1-\sum x_k t_k\right)^{-a} \frac{dt}{\prod t_k}, \tag{11}$$

$$\int_{\Delta} \prod t_k^{1-c_k+b} \cdot \left(1-\sum t_k\right)^{\sum c_k-a-m+1} \cdot w_x(t)^{-b} \frac{dt}{\prod t_k(1-\sum t_k)}, \tag{12}$$

respectively, where we set $w_x(t) = \prod t_k \cdot \left(1-\sum \frac{x_k}{t_k}\right)$ which is a polynomial of degree $m$. Let us consider twisted (co)homology theory for the multi-valued functions

$$U_A(t) = \prod \left(t_k^{b_k}(1-t_k)^{c_k-b_k-1}\right) \cdot \left(1-\sum x_k t_k\right)^{-a},$$
$$U_C(t) = \prod t_k^{1-c_k+b} \cdot \left(1-\sum t_k\right)^{\sum c_k-a-m+1} \cdot w_x(t)^{-b}$$

defined on

$$T_A = \mathbb{C}^m - \left(\bigcup (t_k=0) \cup \bigcup (1-t_k=0) \cup (1-\sum x_k t_k = 0)\right),$$
$$T_C = \mathbb{C}^m - \left(\bigcup (t_k=0) \cup (1-\sum t_k = 0) \cup (w_x(t)=0)\right),$$

respectively. Here, we fix $x = (x_1, \ldots, x_m)$ and regard $t = (t_1, \ldots, t_m)$ as variables.

*Remark* 3.2. In (11) and (12), we take out the differential forms $\frac{dt}{\prod t_k}$ and $\frac{dt}{\prod t_k(1-\sum t_k)}$, respectively. If we choose other differential forms, structures of twisted cohomology groups are slightly changed. However, it is not essential in this article.

#### 3.1.1 Twisted homology groups

Let # be $A$ or $C$. We set

$$C_k = \left\{ \sum_{j:\text{finite}} a_j \cdot \Delta_j \otimes U_{\#,\Delta_j} \;\middle|\; a_j \in \mathbb{C},\ \Delta_j : k\text{-simplex} \right\},$$
$$C_k^{\mathrm{lf}} = \left\{ \sum_{j:\text{locally finite}} a_j \cdot \Delta_j \otimes U_{\#,\Delta_j} \;\middle|\; a_j \in \mathbb{C},\ \Delta_j : k\text{-simplex} \right\} \supset C_k,$$

where $\Delta \otimes U_{\#,\Delta}$ denotes the pair of the simplex $\Delta \subset T_\#$ and the branch $U_{\#,\Delta}(t)$ of $U_\#(t)$ on $\Delta$. We define twisted boundary operators $\partial^{U_\#} : C_k \to C_{k-1}$ and $\partial^{U_\#} : C_k^{\mathrm{lf}} \to C_{k-1}^{\mathrm{lf}}$ by

$$\partial^{U_\#}(\Delta \otimes U_{\#,\Delta}) = \partial\Delta \otimes U_{\#,\Delta}|_{\partial\Delta}.$$

Since we have $\partial^{U_\#} \circ \partial^{U_\#} = 0$, we define the $k$-th twisted homology group and the $k$-th locally finite twisted homology group by

$$H_k(T_\#, U_\#) = \ker(\partial^{U_\#} : C_k \to C_{k-1})/\partial^{U_\#}(C_{k+1}), \quad H_k^{\mathrm{lf}}(T_\#, U_\#) = \ker(\partial^{U_\#} : C_k^{\mathrm{lf}} \to C_{k-1}^{\mathrm{lf}})/\partial^{U_\#}(C_{k+1}^{\mathrm{lf}}),$$

respectively. An element of $\ker(\partial^{U_\#})$ is called a twisted cycle.

#### 3.1.2 Twisted cohomology groups

Let $\Omega^k(T_\#)$ be the space of the rational $k$-forms on $\mathbb{P}^k$ that have poles along $\mathbb{P}^k - T_\#$. We set

$$\omega_\# = \frac{dU_\#}{U_\#} \in \Omega^1(T_\#).$$

Since $\nabla_\# = d + \omega_\# \wedge : \Omega^k(T_\#) \to \Omega^{k+1}(T_\#)$ satisfies $\nabla_\# \circ \nabla_\# = 0$, we can define the $k$-th twisted cohomology group by

$$H^k(\Omega^\bullet(T_\#), \nabla_\#) = \ker(\nabla : \Omega^k(T_\#) \to \Omega^{k+1}(T_\#))/\nabla(\Omega^{k-1}(T_\#)).$$

Let $\mathcal{E}^k_c(T_\#)$ be the space of smooth $k$-forms on $T_\#$ with compact support. By using $\mathcal{E}^\bullet_c(T_\#)$ and $\nabla_\#$, we can also define the twisted cohomology group $H^k(\mathcal{E}^\bullet_c(T_\#), \nabla_\#)$ with compact support.

**Fact 3.3** (cf. [1], [3], [13])**.** *Let # be A or C. Under Assumption 3.1, we have the following.*

1. *If $k \neq m$, then $H_k(T_\#, U_\#) = 0$, $H^{\mathrm{lf}}_k(T_\#, U_\#) = 0$, $H^k(\Omega^\bullet(T_\#), \nabla_\#) = 0$ and $H^k(\mathcal{E}^\bullet_c(T_\#), \nabla_\#) = 0$.*
2. *A canonical map $H_m(T_\#, U_\#) \to H^{\mathrm{lf}}_m(T_\#, U_\#)$ is an isomorphism.*
3. *There exists an isomorphism $\jmath : H^m(\Omega^\bullet(T_\#), \nabla_\#) \to H^m(\mathcal{E}^\bullet_c(T_\#), \nabla_\#)$.*
4. $\dim H_m(T_\#, U_\#) = \dim H^m(\Omega^\bullet(T_\#), \nabla_\#) = 2^m$ (= *rank of* $E_\#$).

### 3.2 Intersection pairings

#### 3.2.1 Intersection pairing for twisted homology groups

By replacing $U_\#$ with $U_\#^{-1} = 1/U_\#$, we can also define $H_m(T_\#, U_\#^{-1})$ and $H^{\mathrm{lf}}_m(T_\#, U_\#^{-1})$. We define the intersection pairing $\mathcal{I}^h$ between $H_m(T_\#, U_\#)$ and $H^{\mathrm{lf}}_m(T_\#, U_\#^{-1}) \simeq H_m(T_\#, U_\#^{-1})$.

Let

$$\sigma = \sum_{i:\text{fin.}} a_i \cdot \Delta_i \otimes U_{\#,\Delta_i}(t), \quad \tau = \sum_j b_j \cdot \Delta'_j \otimes U_{\#,\Delta'_j}(t)^{-1}$$

be twisted cycles, where each pair $(\Delta_i, \Delta'_j)$ does not intersect or intersects transversally. Their intersection number is defined by

$$\mathcal{I}^h(\sigma, \tau) = \sum_{p \in \Delta_i \cap \Delta'_j} a_i b_j \cdot (\Delta_i, \Delta'_j)_p \cdot U_{\#,\Delta_i}(p) \cdot U_{\#,\Delta'_j}(p)^{-1},$$

where $(\Delta_i, \Delta'_j)_p$ is the topological intersection number of $\Delta_i$ and $\Delta'_j$ at $p$.

#### 3.2.2 Intersection pairing for twisted cohomology groups

We can define the intersection pairing $\mathcal{I}^c$ between $H^m(\Omega^\bullet(T_\#), \nabla_\#)$ and $H^m(\Omega^\bullet(T_\#), \nabla^\vee_\#)$, where $\nabla^\vee_\# = d - \omega_\# \wedge$.

By using the isomorphism $\jmath : H^m(\Omega^\bullet(T_\#), \nabla_\#) \xrightarrow{\sim} H^m(\mathcal{E}^\bullet_c(T_\#), \nabla_\#)$ in Fact 3.3, we can define the intersection number by

$$\mathcal{I}^c(\varphi, \psi) = \int_T \jmath(\varphi) \wedge \psi \quad (\phi \in H^m(\Omega^\bullet(T_\#), \nabla_\#),\ \psi \in H^m(\Omega^\bullet(T_\#), \nabla^\vee_\#)).$$

Note that we regard $T_\# \subset \mathbb{P}^k$ as a $2k$-dimensional real manifold.

#### 3.2.3 Twisted period relations

By using the intersection pairings, we can obtain twisted period relations. In this section, we give only results. For details, see [15].

Let $\{\tau_j\}_{j=1,\dots,2^m} \subset H_m(T_\#, U_\#)$ and $\{\psi_j\}_{j=1,\dots,2^m} \subset H^m(\Omega^\bullet(T_\#), \nabla_\#)$ be bases. We set

$$H = \left(\mathcal{I}^h(\tau_i, \tau_j^\vee)\right)_{i,j=1,\dots,2^m}, \quad C = \left(\mathcal{I}^c(\psi_i, \psi_j)\right)_{i,j=1,\dots,2^m},$$
$$\Pi_+ = \left(\int_{\tau_j} U_\# \psi_i\right)_{i,j=1,\dots,2^m}, \quad \Pi_- = \left(\int_{\tau_j^\vee} U_\#^{-1} \psi_i\right)_{i,j=1,\dots,2^m},$$

where $\tau_j^\vee \in H_m(T_\#, U_\#^{-1}) \simeq H_m^{\mathrm{lf}}(T_\#, U_\#^{-1})$ is a twisted cycle defined by the same manner as $\tau_j$ with respect to $U_\#^{-1}$. Thus, we obtain the twisted period relation

$$\Pi_+ \, {}^t H^{-1} \, {}^t \Pi_- = C.$$

Note that the entries of $\Pi_\pm$ are hypergeometric integrals.

If the intersection matrix $H$ is diagonal, then its $(i, j)$-entry gives a simple relation:

$$\sum_{k=1}^{2^m} \frac{1}{\mathcal{I}^h(\tau_k, \tau_k^\vee)} \int_{\tau_k} U_\# \psi_i \int_{\tau_k^\vee} U_\#^{-1} \psi_j = \mathcal{I}^c(\psi_i, \psi_j). \tag{13}$$

In this article, we construct a basis of the twisted homology group such that the intersection matrix is diagonal, and we rewrite this relation in terms of $F_A$ or $F_C$.

## 4. Twisted cycles and twisted period relations for $F_A$ and $F_C$

As mentioned in the previous section, if the homology intersection matrix is diagonal, then we can obtain simple quadratic relations. In fact, cycles corresponding to series solutions (5) and (6) satisfy this property in our cases. In this section, we explain how to construct such cycles and rewrite the twisted period relation (13) in terms of $F_A$ or $F_C$.

### 4.1 Regularization of the $m$-simplex

Before constructing the twisted cycles for $F_A$ and $F_C$, we recall the regularization of the $m$-simplex $\Delta^m$. The regularization of a locally finite twisted cycle is a twisted cycle that represents the inverse image of the canonical isomorphism $H_m \to H_m^{\mathrm{lf}}$ (for $F_A$ and $F_C$, the isomorphism is mentioned in Fact 3.3). When $m = 1$, the regularization of the open interval $\Delta^1 = (0, 1) \subset \mathbb{R}$ is explained in [15]. Similarly to $m = 1$, we can construct the regularization of the $m$-simplex for $m \geq 2$.

As an example, we consider the case when $m = 2$. We set the multi-valued function $U(t) = t_1^{d_1} t_2^{d_2} (1 - t_1 - t_2)^{d_3}$ on $T = \mathbb{C}^2 - ((t_1 = 0) \cup (t_2 = 0) \cup (1 - t_1 - t_2 = 0))$. For general cases, refer to [1]. Let us construct the regularization of the locally finite twisted cycle $\Delta^2 \otimes U \in H_2^{\mathrm{lf}}(T, U)$ associated with the 2-simplex

$$\Delta^2 = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 > 0,\ t_2 > 0,\ 1 - t_1 - t_2 > 0\}.$$

As in the right side of Figure 1, let $\triangle$ be the small triangle included in $\Delta^2$ and $I_i$ $(i = 1, 2, 3)$ be its boundary. We denote by $S_i$ $(i = 1, 2)$ a positively oriented circle in the $t_i$-space starting from the projection of $I_i$ to this space and surrounding the divisor $(t_i = 0)$, and denote by $S_3$ a positively oriented circle with a small radius in the orthogonal complement of the divisor $(1 - t_1 - t_2 = 0)$ starting from the projection of $I_3$ to this space and surrounding the divisor. The twisted cycle $\Delta^2 \otimes U$ is equal to a finite cycle

$$\Delta^2_{\mathrm{reg}} = \triangle \otimes U + \sum_{i=1}^{3} \frac{(S_i \times I_i) \otimes U}{1 - \delta_i} + \sum_{(i,j)=(1,2),(2,3),(3,1)} \frac{(S_i \times S_j) \otimes U}{(1 - \delta_i)(1 - \delta_j)}$$

in $H_2^{\mathrm{lf}}(T, U)$, where $\delta_i = e^{2\pi\sqrt{-1}d_i}$. Thus, $\Delta^2_{\mathrm{reg}} \in H_2(T, U)$ gives the regularization of $\Delta^2 \otimes U$.

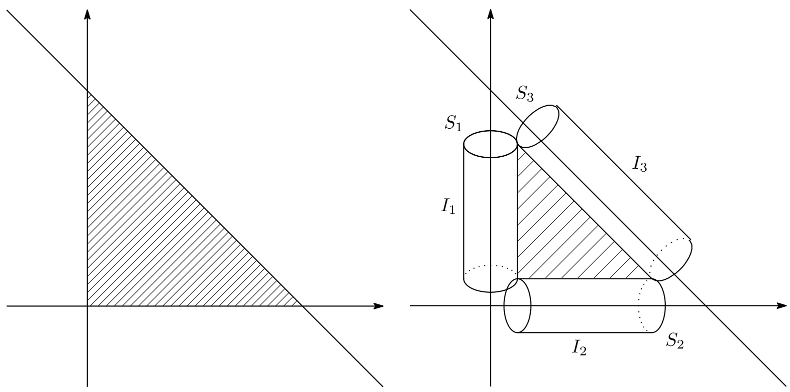


**Figure 1:** $\Delta^2$ and its regularization

### 4.2 $F_C$

We construct a basis of the twisted homology group that corresponds to the basis (6) of the solution space of $E_C$, which are expressed by $F_C$. As an application, we give some quadratic relations between $F_C$.

#### 4.2.1 Twisted cycles corresponding to series solutions

Assume that $x_1, \ldots, x_m$ are sufficiently small positive real numbers. We construct a twisted cycle $\Delta_{i_1 \cdots i_r}$ such that the integral $\int_{\Delta_{i_1 \cdots i_r}} U_C \frac{dt}{\prod t_k (1 - \sum t_k)}$ coincides with the series solution

$$f_C^{i_1 \cdots i_r} = \prod_{p=1}^{r} x_{i_p}^{1 - c_{i_p}} \cdot F_C\Big(a + r - \sum_{p=1}^{r} c_{i_p}, b + r - \sum_{p=1}^{r} c_{i_p}, c^I; x\Big) \quad (I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}),$$

up to $\Gamma$-factors.

*Remark* 4.1. Because of $f_C^{\emptyset} = F_C(a, b, c_1, \ldots, c_m; x)$, $\Delta_\emptyset$ is nothing but the twisted cycle $\Delta$ mentioned in Section 2.2.3.

We fix $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$, and put $\{j_1, \ldots, j_{m-r}\} = \{1, \ldots, m\} - I$. In the discussion below, the index $p$ (resp. $q$) runs from 1 to $r$ (resp. from 1 to $m - r$). We consider the change of variables

$$t_{i_p} = \frac{x_{i_p}}{s_{i_p}}, \quad t_{j_q} = s_{j_q} \tag{14}$$

in the integral

$$\int \prod t_k^{-c_k} \cdot \left(1 - \sum t_k\right)^{\sum c_k - a - m} \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} dt,$$

formally. Then we have

$$\begin{aligned} &\prod_p x_{i_p}^{1-c_{i_p}} \cdot \int \prod_p s_{i_p}^{c_{i_p}-2} \cdot \prod_q s_{j_q}^{-c_{j_q}} \\ &\cdot \left(1 - \sum_p \frac{x_{i_p}}{s_{i_p}} - \sum_q s_{j_q}\right)^{\sum c_k - a - m} \left(1 - \sum_p s_{i_p} - \sum_q \frac{x_{j_q}}{s_{j_q}}\right)^{-b} ds. \end{aligned}$$

We set

$$U_{C,I}(s) = \prod_p s_{i_p}^{c_{i_p}-2} \cdot \prod_q s_{j_q}^{-c_{j_q}} \cdot \left(1 - \sum_p \frac{x_{i_p}}{s_{i_p}} - \sum_q s_{j_q}\right)^{\sum c_k - a - m} \left(1 - \sum_p s_{i_p} - \sum_q \frac{x_{j_q}}{s_{j_q}}\right)^{-b}.$$

If we construct a twisted cycle $\tilde{\Delta}_{i_1 \cdots i_r}$ (in $s$-coordinates) such that

$$\int_{\tilde{\Delta}_{i_1 \cdots i_r}} U_{C,I}(s)\, ds = (\text{constant}) \cdot F_C\left(a + r - \sum_p c_{i_p}, b + r - \sum_p c_{i_p}, c^I; x\right),$$

then its image $\Delta_{i_1 \cdots i_r} \in H_m(T_C, U_C)$ (in $t$-coordinates) under the map (14) gives a desired one.

In $\mathbb{R}^m \subset \mathbb{C}^m$, the set

$$s_k > 0,\ 1 - \sum \frac{x_{i_p}}{s_{i_p}} - \sum s_{j_q} > 0,\ 1 - \sum s_{i_p} - \sum \frac{x_{j_q}}{s_{j_q}} > 0$$

is bounded region which includes the direct product

$$\sigma_{i_1 \cdots i_r} = \left\{ (s_1, \dots, s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} s_{i_p} > \varepsilon,\ 1 - \sum s_{i_p} > \varepsilon, \\ s_{j_q} > \varepsilon,\ 1 - \sum s_{j_q} > \varepsilon \end{array} \right\}$$

of an $r$-simplex and an $(m-r)$-simplex, for some $\varepsilon > 0$. We construct a twisted cycle $\tilde{\Delta}_{i_1 \cdots i_r}$ by using $\sigma_{i_1 \cdots i_r}$ and "$\varepsilon$-neighborhood" of $(s_1 = 0), \dots, (s_m = 0), (1 - \sum s_{i_p} = 0), (1 - \sum s_{j_q} = 0)$.

*Example* 4.2 (Example for $m = 2$).

- If $I = \emptyset$, $\tilde{\Delta}_\emptyset = \tilde{\Delta}$ is given as Figure 2. In this case, we have $t_1 = s_1$, $t_2 = s_2$ and $\tilde{\Delta} = \Delta$. Precisely, it is written as

$$\begin{aligned} \Delta = \sigma \otimes U_C &+ \frac{(S_1 \times I_1) \otimes U_C}{1 - \gamma_1^{-1}} + \frac{(S_2 \times I_2) \otimes U_C}{1 - \gamma_2^{-1}} + \frac{(S_3 \times I_3) \otimes U_C}{1 - \gamma_1 \gamma_2 \alpha^{-1}} \\ &+ \frac{(S_1 \times S_2) \otimes U_C}{(1 - \gamma_1^{-1})(1 - \gamma_2^{-1})} + \frac{(S_2 \times S_3) \otimes U_C}{(1 - \gamma_2^{-1})(1 - \gamma_1 \gamma_2 \alpha^{-1})} + \frac{(S_3 \times S_1) \otimes U_C}{(1 - \gamma_1 \gamma_2 \alpha^{-1})(1 - \gamma_1^{-1})}, \end{aligned}$$

  where $\alpha = e^{2\pi\sqrt{-1}a}$, $\gamma_k = e^{2\pi\sqrt{-1}c_k}$ and the radius of the circle $S_i$ is $\varepsilon$.

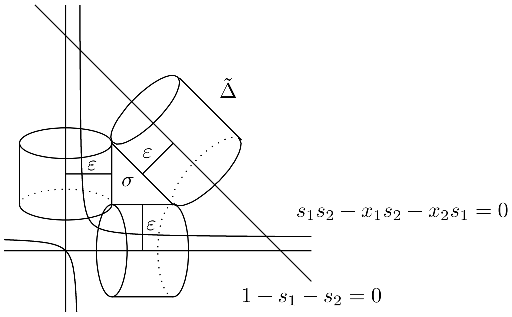


**Figure 2:** $\tilde{\Delta}$ for $m = 2$

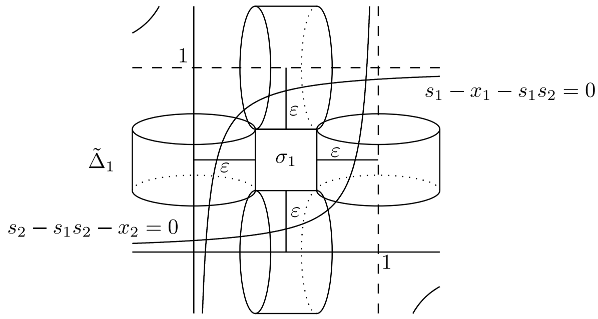


**Figure 3:** $\tilde{\Delta}_1$ for $m = 2$

- If $I = \{1\}$, $\tilde{\Delta}_1$ is given as Figure 3.

In any cases, some circles may surround two divisors.

**Proposition 4.3** ([3])**.**

$$\int_{\tilde{\Delta}_{i_1\cdots i_r}} U_{C,I}(s)\, ds = \frac{\prod_p \Gamma(c_{i_p} - 1) \cdot \prod_q \Gamma(1 - c_{j_q}) \cdot \Gamma(\sum c_k - a - m + 1)\Gamma(1 - b)}{\Gamma(\sum_p c_{i_p} - a - r + 1)\Gamma(\sum_p c_{i_p} - b - r + 1)} \cdot F_C\Big(a + r - \sum_p c_{i_p}, b + r - \sum_p c_{i_p}, c^I; x\Big).$$

*Proof.* Consider the power series expansion of the left-hand side with respect to $x_1, \dots, x_m$. By our construction, this expansion converges uniformly. As a coefficient of $x_1^{n_1} \cdots x_m^{n_m}$ in the power series expansion, the integral

$$\int_{\tilde{\Delta}_{i_1\cdots i_r}} \prod_p s_{i_p}^{c_{i_p} - n_{i_p} - 2} \prod_q s_{j_q}^{-c_{j_q} - n_{j_q}} \cdot \Big(1 - \sum_p s_{i_p}\Big)^{-b - \sum n_{j_q}} \Big(1 - \sum_q s_{j_q}\Big)^{\sum c_k - a - m - \sum n_{i_p}} ds \tag{15}$$

appears. If we regard $\tilde{\Delta}_{i_1\cdots i_r}$ as a twisted cycle loading this integrand, each circle in $\tilde{\Delta}_{i_1\cdots i_r}$ surrounds only one divisor. Thus, $\tilde{\Delta}_{i_1\cdots i_r}$ is nothing but the regularization of the direct product of two simplices $\Delta^r$ (in $(s_i)_{i\in I}$-coordinates) and $\Delta^{m-r}$ (in $(s_j)_{j\in J}$-coordinates). The integral (15) is equal to

$$\int_{\Delta^r} \prod_p s_{i_p}^{c_{i_p} - n_{i_p} - 2} \Big(1 - \sum_p s_{i_p}\Big)^{-b - \sum n_{j_q}} ds_I \cdot \int_{\Delta^{m-r}} \prod_q s_{j_q}^{-c_{j_q} - n_{j_q}} \Big(1 - \sum_q s_{j_q}\Big)^{\sum c_k - a - m - \sum n_{i_p}} ds_{I^c}$$
$$= \frac{\prod_p \Gamma(c_{i_p} - n_{i_p} - 1) \cdot \Gamma(-b - \sum_q n_{j_q} + 1)}{\Gamma(-b + \sum_p c_{i_p} - \sum n_k - r + 1)} \cdot \frac{\prod_q \Gamma(-c_{j_q} - n_{j_q} + 1) \cdot \Gamma(\sum c_k - a - m - \sum_p n_{i_p} + 1)}{\Gamma(\sum_p c_{i_p} - a - \sum n_k - r + 1)},$$

where $ds_I = ds_{i_1} \wedge \cdots \wedge ds_{i_r}$, $ds_{I^c} = ds_{j_1} \wedge \cdots \wedge ds_{j_{m-r}}$. By using $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ and $(a)_n = \Gamma(a + n)/\Gamma(a)$, we obtain the proposition. □

Therefore, we obtain a twisted cycle corresponding to the solution $f_C^{i_1\cdots i_r}$.

**Theorem 4.4** ([3]). *Let $\Delta_{i_1\cdots i_r}$ be the twisted cycle in $T_C$ which is obtained as the image of $\tilde{\Delta}_{i_1\cdots i_r}$ under the correspondence $t_{i_p} = \dfrac{x_{i_p}}{s_{i_p}}$, $t_{j_q} = s_{j_q}$. Then we have*

$$\int_{\Delta_{i_1\cdots i_r}} \prod t_k^{-c_k} \cdot (1-\sum t_k)^{\sum c_k - a - m} \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} dt$$
$$= \frac{\prod_p \Gamma(c_{i_p}-1)\cdot \prod_q \Gamma(1-c_{j_q})\cdot \Gamma(\sum c_k - a - m + 1)\Gamma(1-b)}{\Gamma(\sum_p c_{i_p} - a - r + 1)\Gamma(\sum_p c_{i_p} - b - r + 1)} \cdot f_C^{i_1\cdots i_r}.$$

*Proof.* Consider the change of variables (14), and use Proposition 4.3. □

**Theorem 4.5** ([3]). *We put $\alpha = e^{2\pi\sqrt{-1}a}$, $\beta = e^{2\pi\sqrt{-1}b}$, $\gamma_k = e^{2\pi\sqrt{-1}c_k}$.*

1. $I \neq I' \Longrightarrow \mathcal{I}^h(\Delta_I, \Delta^\vee_{I'}) = 0$;
2. $\mathcal{I}^h(\Delta_{i_1\cdots i_r}, \Delta^\vee_{i_1\cdots i_r}) = (-1)^r \cdot \dfrac{\prod_q \gamma_{j_q}\cdot(\alpha - \prod_p \gamma_{i_p})(\beta - \prod_p \gamma_{i_p})}{\prod_k(\gamma_k - 1)\cdot(\alpha - \prod_k \gamma_k)(\beta - 1)}$.

*Sketch of the proof.* 1. Since the function $f_C^I$ corresponding to $\Delta_I$ has the same monodromy property as $\prod_{i\in I} x_i^{1-c_i}$ around $x = (0,\ldots,0)$, the claim follows from the monodromy invariance of $\mathcal{I}^h$.

2. The self-intersection number of $\Delta_{i_1\cdots i_r}$ coincides with that of $\tilde{\Delta}_{i_1\cdots i_r}$. By using the results in [9], we can evaluate it.

□

#### 4.2.2 Cohomology intersection numbers

Except for $m = 2$ (Appell's $F_4$), we do not have so many results for intersection numbers of the twisted cohomology groups.

**Theorem 4.6** ([3]). *We put*

$$\varphi = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod_k t_k \cdot (1 - \sum_k t_k)}, \qquad \varphi' = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1-\sum t_k)\cdot(1 - \sum \frac{x_k}{t_k})}.$$

*Then we have*

$$\mathcal{I}^c(\varphi, \varphi') = 0,$$

$$\mathcal{I}^c(\varphi,\varphi) = \left(2\pi\sqrt{-1}\right)^m \left(\frac{1}{\sum c_k - a - m + 1} + \frac{1}{b + m - \sum c_k}\right) \sum_{\{I^{(r)}\}} \prod_{r=1}^{m-1} \frac{1}{b + r - \sum c_{i_p^{(r)}}},$$

*where $\{I^{(r)}\}$ is a sequence of subsets of $\{1,\ldots,m\}$ such that*

$$\{1,\ldots,m\} \supsetneq I^{(m-1)} \supsetneq \cdots \supsetneq I^{(2)} \supsetneq I^{(1)} \neq \emptyset,$$

*and we write $I^{(r)} = \{i_1^{(r)},\ldots,i_r^{(r)}\}$.*

Roughly speaking, intersection numbers are evaluated as

$$\sum_{P:\text{ intersection point of } m \text{ divisors}} \mathrm{Res}_{t=P}(\text{some differential form}),$$

if the pole divisor of $\omega_C = d\log U_C$ are normally crossing ([12]). In this case, we need to blow up $\mathbb{C}^m$. For detailed calculations, refer to [3].

#### 4.2.3 Twisted period relations for $F_C$

Since the intersection matrix $H$ with respect to the basis $\{\Delta_I\}_I \subset H_m(T_C, U_C)$ is diagonal by Theorem 4.5, we obtain twisted period relations (13) as follows:

$$\mathcal{I}^c(\varphi, \varphi') = \sum_I \frac{1}{\mathcal{I}^h(\Delta_{i_1\cdots i_r}, \Delta^\vee_{i_1\cdots i_r})} \cdot \int_{\Delta_{i_1\ldots i_r}} U_C\,\varphi \cdot \int_{\Delta^\vee_{i_1\ldots i_r}} U_C^{-1}\varphi',$$
$$\mathcal{I}^c(\varphi, \varphi) = \sum_I \frac{1}{\mathcal{I}^h(\Delta_{i_1\cdots i_r}, \Delta^\vee_{i_1\cdots i_r})} \cdot \int_{\Delta_{i_1\ldots i_r}} U_C\,\varphi \cdot \int_{\Delta^\vee_{i_1\ldots i_r}} U_C^{-1}\varphi.$$

By Theorem 4.4, these integrals are expressed by $F_C$. Thus, we obtain two quadratic relations

$$0 = \sum_I (-1)^r (a_{i_1\cdots i_r} - 1) \cdot F_C(a_{i_1\cdots i_r}, b_{i_1\cdots i_r}, c^{i_1\cdots i_r}; x) \cdot F_C(2 - a_{i_1\cdots i_r}, 1 - b_{i_1\cdots i_r}, \check{c}^{i_1\cdots i_r}; x),$$

and

$$\frac{(1-a+b)\cdot \prod(1-c_k)}{bb_{1\cdots m}} \cdot \sum_{\{I^{(r)}\}} \prod_{r=1}^{m-1} \frac{1}{b_{I^{(r)}}}$$
$$= \sum_I (-1)^r \frac{1 - a_{i_1\cdots i_r}}{b_{i_1\cdots i_r}} \cdot F_C(a_{i_1\cdots i_r}, b_{i_1\cdots i_r}, c^{i_1\cdots i_r}; x) \cdot F_C(2 - a_{i_1\cdots i_r}, -b_{i_1\cdots i_r}, \check{c}^{i_1\cdots i_r}; x),$$

where we put

$$a_{i_1\cdots i_r} = a + r - \sum c_{i_p}, \quad b_{i_1\cdots i_r} = b + r - \sum c_{i_p}, \quad \check{c}^{i_1\cdots i_r} = (2, \ldots, 2) - c^{i_1\cdots i_r}.$$

#### 4.2.4 Twisted period relations for Appell's $F_4$ ($m = 2$)

In the case of $m = 2$, we have more results. In [6], we put

$$\varphi_1 = \frac{dt_1 \wedge dt_2}{t_1 t_2(1 - t_1 - t_2)}, \quad \varphi_2 = \frac{dt_1 \wedge dt_2}{t_2(1 - t_1 - t_2)},$$
$$\varphi_3 = \frac{dt_1 \wedge dt_2}{t_1(1 - t_1 - t_2)}, \quad \varphi_4 = \frac{dt_1 \wedge dt_2}{(1 - t_1 - t_2)(t_1 t_2 - t_2 x_1 - t_1 x_2)}$$

and evaluate their intersection matrix. For example,

**Theorem 4.7** ([6]).

$$\mathcal{I}^c(\varphi_4, \varphi_4) = \frac{2\cdot\left(2\pi\sqrt{-1}\right)^2}{(c_1 + c_2 - a - 2)\cdot(-b)} \cdot \frac{1}{x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1}.$$

Note that $x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1$ is a factor of the defining polynomial of the singular locus $S_C$ (see Section 2.3.3). By using this intersection number, we obtain the following relation:

$$\begin{aligned}
&b(a-1)F_4(a, b+1, c_1, c_2; x) \cdot F_4(2-a, 1-b, 2-c_1, 2-c_2; x)\\
&- b_1(a_1 - 1)F_4(a_1, b_1 + 1, 2 - c_1, c_2; x) \cdot F_4(2 - a_1, 1 - b_1, c_1, 2 - c_2; x)\\
&- b_2(a_2 - 1)F_4(a_2, b_2 + 1, c_1, 2 - c_2; x) \cdot F_4(2 - a_2, 1 - b_2, 2 - c_1, c_2; x)\\
&+ b_{12}(a_{12} - 1)F_4(a_{12}, b_{12} + 1, 2 - c_1, 2 - c_2; x) \cdot F_4(2 - a_{12}, 1 - b_{12}, c_1, c_2; x)\\
&= \frac{2(1-c_1)(1-c_2)}{x_1^2 + x_2^2 - 2x_1x_2 - 2x_1 - 2x_2 + 1}.
\end{aligned}$$

### 4.3 $F_A$

We can apply a similar argument for $F_A$. We introduce some results, and omit detailed calculations.

#### 4.3.1 Twisted cycles corresponding to series solutions

For $I = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}$, we put $\{j_1, \dots, j_{m-r}\} = \{1, \dots, m\} - I$. We consider the change of variables

$$t_{i_p} = \frac{s_{i_p}}{x_{i_p}}, \quad t_{j_q} = s_{j_q}$$

in the integral (11), formally. Similarly to Section 4.2.1, we can construct a twisted cycle $\tilde{\Delta}_{i_1 \cdots i_r}$ (in $s$-coordinates) such that

$$\int_{\tilde{\Delta}_{i_1 \cdots i_r}} \prod_p s_{i_p}^{c_{i_p}-2} \left(1 - \frac{x_{i_p}}{s_{i_p}}\right)^{c_{i_p}-b_{i_p}-1} \cdot \left(1 - \sum_p s_{i_p} - \sum_q x_{j_q} s_{j_q}\right)^{-a} \cdot \prod_q s_{j_q}^{b_{j_q}-1}(1 - s_{j_q})^{c_{j_q}-b_{j_q}-1} ds$$
$$= (\text{constant}) \cdot F_A\left(a + r - \sum_p c_{i_p}, b^I, c^I; x\right).$$

Then, we obtain the following theorems.

**Theorem 4.8** ([4]). *We can construct twisted cycles $\Delta_{i_1 \cdots i_r} \in H_m(T_A, U_A)$ corresponding to the series solutions:*

$$\int_{\Delta_{i_1 \cdots i_r}} \prod \left(t_k^{b_k} \cdot (1 - t_k)^{c_k - b_k - 1}\right) \cdot \left(1 - \sum x_k t_k\right)^{-a} dt$$
$$= e^{\pi\sqrt{-1}(\sum b_{i_p} - \sum c_{i_p} + r)} \cdot \frac{\Gamma(1-a)\prod_p \Gamma(c_{i_p} - 1)}{\Gamma(\sum_p c_{i_p} - a - r + 1)} \cdot \prod_q \frac{\Gamma(b_{j_q})\Gamma(c_{j_q} - b_{j_q})}{\Gamma(c_{j_q})} \cdot f_A^{i_1 \cdots i_r}.$$

**Theorem 4.9** ([4]). *We put $\alpha = e^{2\pi\sqrt{-1}a}$, $\beta_k = e^{2\pi\sqrt{-1}b_k}$, $\gamma_k = e^{2\pi\sqrt{-1}c_k}$.*

1. $I \neq I' \Longrightarrow \mathcal{I}^h(\Delta_I, \Delta_{I'}^\vee) = 0$;
2. $\mathcal{I}^h(\Delta_{i_1 \cdots i_r}, \Delta_{i_1 \cdots i_r}^\vee) = \dfrac{\alpha - \prod_p \gamma_{i_p}}{(\alpha - 1)\prod_p (1 - \gamma_{i_p})} \cdot \prod_q \dfrac{\beta_{j_q}(1 - \gamma_{j_q})}{(1 - \beta_{j_q})(\beta_{j_q} - \gamma_{j_q})}$.

*Remark* 4.10. In [11], our construction of twisted cycles corresponding to series solutions is generalized to a regular holonomic GKZ-hypergeometric system ($A$-hypergeometric system) which is associated with an integer matrix $A$. When $A$ has a unimodular triangulation, their intersection numbers have been also obtained in [11]. In the other cases, the intersection numbers are studied in [5].

#### 4.3.2 Cohomology intersection numbers

In [13], the intersection matrix with respect to the basis

$$\varphi_{i_1 \cdots i_r} = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod_p (t_{i_p} - 1) \cdot \prod_q t_{j_q}} \quad \begin{pmatrix} I = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}, \\ \{j_1, \dots, j_{m-r}\} = \{1, \dots, m\} - I \end{pmatrix}$$

of the twisted cohomology group $H^m(\Omega^\bullet(T_A), \nabla_A)$ is evaluated.

For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$, we put

$$A_{i_1\cdots i_r} = A_I = \sum_{\{I^{(l)}\}} \prod_{l=1}^{r} \frac{1}{a - \sum c_{i_p^{(l)}} + l},$$

where $\{I^{(l)}\}$ is a sequence of subsets of $I$ such that

$$I = I^{(r)} \supsetneq I^{(r-1)} \supsetneq \cdots \supsetneq I^{(2)} \supsetneq I^{(1)} \neq \emptyset.$$

**Fact 4.11** ([13]). *We have*

$$\mathcal{I}^c(\varphi_I, \varphi_{I'}) = \left(2\pi\sqrt{-1}\right)^m \cdot \sum_{N \subset \{1,\ldots,m\}} \left( A_N \prod_{n \notin N} \frac{\delta_{I,I'}(n)}{\tilde{b}_{I,I'}(n)} \right),$$

*where*

$$\delta_{I,I'}(n) = \begin{cases} 1 & (n \in (I \cap I') \cup (I^c \cap I'^c)), \\ 0 & (\text{otherwise}), \end{cases} \qquad \tilde{b}_{I,I'}(n) = \begin{cases} c_n - 1 - b_n & (n \in I \cap I'), \\ b_n & (n \in I^c \cap I'^c). \end{cases}$$

#### 4.3.3 Twisted period relations for $F_A$

We write one example of quadratic relations. The twisted period relation (13) with respect to $\mathcal{I}^c(\varphi_\emptyset, \varphi_{12\cdots m})$ and the basis $\{\Delta_I\} \subset H_m(T_A, U_A)$ is reduced into the relation

$$\frac{\prod(1-c_k)}{a} \cdot \sum_{\{I^{(l)}\}} \prod_{l=1}^{r} \frac{1}{a - \sum_p c_{i_p^{(l)}} + l} = \sum_I \frac{(-1)^r}{a_I} \cdot F_A(a_I, b^I, c^I; x) \cdot F_A(-a_I, \check{b}^I, \check{c}^I; x),$$

where a sequence $\{I^{(l)}\}$ is same as Section 4.2.2, and we put

$$a_I = a_{i_1\cdots i_r} = a + r - \sum_p c_{i_p},$$

$$\begin{cases} b^I_{i_p} = b_{i_p} + 1 - c_{i_p}, \\ b^I_j = b_j, \end{cases} \qquad \begin{cases} c^I_{i_p} = 2 - c_{i_p}, \\ c^I_j = c_j \end{cases} \qquad (i_p \in I,\ j \notin I),$$

$$\check{b}^I = (1, \ldots, 1) - b^I, \quad \check{c}^I = (2, \ldots, 2) - c^I.$$


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