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Modave lecture notes: two useful topics in Lie algebras

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These notes are a write-up of a six-hour series of lectures given at the XVI Modave Summer School in Mathematical Physics, aimed at Ph.D. students in high-energy theoretical physics. They give a basic introduction to two important topics in the theory of Lie algebras: Lie algebra cohomology and Kac-Moody algebras.

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Introduction

In these lectures, I give a basic introduction to two topics in the theory of Lie algebras: Lie algebra cohomology and Kac-Moody algebras. These topics appear in many areas of theoretical physics and, while many details and important concepts are missing, my hope is that the reader will have

1. some understanding of the main message when these terms appear in a talk;

2. a basic entry point for further study.

These lectures are aimed at Ph.D. students in high-energy theoretical physics. They are highly encouraged to work out the thirteen provided exercises.

The first topic is the so-called *Chevalley-Eilenberg cohomology* of Lie algebras. As motivation, we will discuss some important physical applications: central extensions, projective representations and Lie algebra deformations. In particular, we will derive the uniqueness of the Virasoro algebra. This part of the lectures is based on references [1–3].

The second topic is *Kac-Moody algebras*. They generalize the structure of simple finite-dimensional Lie algebras to the infinite-dimensional case while preserving much of their rich mathematical structures, for example Dynkin diagrams, root lattices, highest weight representations etc. As such, they appear in many physical applications and provide one of the few tractable families of infinite-dimensional Lie algebras. We will begin by presenting the realisation of affine

Kac-Moody algebras in terms of extensions of loop algebras. Then, generic Kac-Moody algebras are defined in terms of generalized Cartan matrices and Dynkin diagrams, and the lectures finish with classification theorems for affine and hyperbolic Kac-Moody algebras. This second part of the lectures is based on [2, 4, 5].

I have mostly used references [1–5] while preparing the lectures, and they constitute an excellent starting point for further study. Beyond those, I have only listed a few more papers that I found useful; more complete references to the mathematical literature and some of the modern physical applications can be found in [1–5].

1. Lie algebra cohomology

1.1 Cohomology: generalities

Let's start with some general definitions before we go to Lie algebra cohomology. A *cochain complex* (C^*, d^*) is a collection of abelian groups C_n indexed by an integer number n (this integer is called the *degree*), along with group homomorphisms

$$d^n : C^n \to C^{n+1} \tag{1.1}$$

that increase the degree by one (so, if $c \in C^n$, then $d^n c \in C^{n+1}$). They are required to satisfy

$$d^{n+1} \circ d^n = 0 \tag{1.2}$$

for all $n \in \mathbb{Z}$. This is pictured as

$$\dots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots , \tag{1.3}$$

where going two steps to the right gives zero. This means that we always have

$$\operatorname{Im} d^{n-1} \subset \operatorname{Ker} d^n \subset C^n . \tag{1.4}$$

In these lectures, the C^n will be vector spaces and the d^n will be linear maps. The abelian group structure just given by the addition of vectors. A few more words of vocabulary:

- elements of C^n are called *n-cochains*;
- the maps d^n are the *coboundary maps*, or sometimes more simply the *differentials*;
- elements of $\operatorname{Ker} d^n$, i.e. n-cochains which satisfy $d^n c = 0$, are called *n-cocycles*; and
- elements of $\operatorname{Im} d^{n-1}$, i.e. n-cochains $c \in C^n$ which are of the form $c = d^{n-1} b$ for some $b \in C^{n-1}$, are *n-coboundaries*.

Often the superscript n on the differential will be omitted. However, in the general case it is important to keep in mind that each of the d^n could be defined in very different ways for different n.

Coboundaries are automatically cocycles, but they are the 'uninteresting ones': cohomology is about finding the non-trivial ones. Accordingly, the n-th cohomology group H^n is defined as the quotient

$$H^n = \frac{\operatorname{Ker} d^n}{\operatorname{Im} d^{n-1}} . \tag{1.5}$$

Elements of H^n are equivalence classes of cocycles, where two cocycles are considered equivalent if they differ by a coboundary:

$$[c] \in H^n \quad \Leftrightarrow \quad d^n c = 0\,, \; c \sim c + d^{n-1} b\,. \tag{1.6}$$

So for example, all elements of the form $d^{n-1}b$ are equivalent to (or, in the same equivalence class as) the zero element.

For completeness, let us mention the group structure of the cohomology groups (it will not be too important in these lectures, but it justifies the name 'cohomology *group*'). It is just the addition that comes from the abelian group structure of the C^n: so, the addition of two cohomology classes $[a]$ and $[b]$ is defined as

$$[a] + [b] = [a + b]\,. \tag{1.7}$$

This is well defined. Indeed, if $da = db = 0$, then $d(a + b) = da + db = 0$ by the homomorphism property of the differential. Moreover, if we had started with other elements $a' = a + df$, $b' = b + dg$ that are equivalent to a and b, we would have gotten the same class:

$$\begin{aligned}[a'] + [b'] = [a' + b'] &= [a + df + b + dg] = [(a + b) + (df + dg)] \\ &= [(a + b) + d(f + g)] = [a + b] \end{aligned} \tag{1.8}$$

so $[a'] + [b'] = [a] + [b]$: the definition really depends only on the equivalence class and not on the choice of representative (note that the abelian property of the group is crucial in this proof).

There are two well-known examples:

- The main example of cohomology groups is given by the de Rham cohomology of smooth manifolds. There, the cochains are the differential forms and d is the de Rham differential: for a manifold M of dimension n, the complex looks like

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \Omega^n(M) \longrightarrow 0 \tag{1.9}$$

  where $\Omega^p(M)$ is the vector space of p-forms on M. Cohomology groups are equivalence classes of closed forms modulo exact ones; they carry important topological information about the manifold M.

- Another familiar example to physicists is BRST cohomology. There, n is called the 'ghost number' and the C^n are the space of functionals in the fields, ghosts, antifields etc. of total ghost number n. The cohomology of the BRST differential s then encodes important field-theoretic quantities, for example conserved currents ($n = -1$), counterterms ($n = 0$) and anomalies ($n = 1$).

We will define a third one in section 1.3.

1.2 Motivations

As a motivation for this part of the lectures, we'll be looking at some algebraic problems in the theory of Lie groups and algebras. They have some trivial solutions that we want to discard and, in fact, we will even want the freedom of redefining solutions by adding a trivial part. It is then useful to reformulate this as cohomology, i.e. define suitable spaces C^n and differentials d^n such that the algebraic problem can be written as $d^n c = 0$, and the trivial solutions are of the form $c = d^{n-1} b$. This enables us to see the underlying structure, and use all the mathematician's work on cohomology to understand them instead of reinventing the wheel.

Projective representations

In quantum mechanics, normalised states are defined only up to a phase. Accordingly, when we have a symmetry group G of the system, the condition for a representation T,

$$T(f) \circ T(g) = T(fg)\,, \tag{1.10}$$

is too strong: one should only ask for this *up to a phase*, i.e.

$$T(f) \circ T(g) = \omega(f,g)\, T(fg) \tag{1.11}$$

where $\omega(f,g)$ is some complex number satisfying $|\omega(f,g)| = 1$. Such representations are called *projective* representations, and are those relevant in quantum mechanics. As the notation suggests, the phase appearing in (1.11) can depend on the group elements, so we are dealing with a map

$$\omega : G \times G \to U(1)\,. \tag{1.12}$$

Equivalently, one can work with the "local exponents"

$$\xi : G \times G \to \mathbb{R} \tag{1.13}$$

defined modulo 2π by

$$\omega(f,g) = e^{i\xi(f,g)}\,. \tag{1.14}$$

What conditions should these maps satisfy? Since the $T(g)$ are linear operators acting on a Hilbert space, associativity of composition still holds,

$$(T(f) \circ T(g)) \circ T(h) = T(f) \circ (T(g) \circ T(h))\,. \tag{1.15}$$

Combined with (1.11) and the associativity of group multiplication, $(fg)h = f(gh)$, this implies that ξ should satisfy

$$\xi(f,g) + \xi(fg,h) = \xi(g,h) + \xi(f,gh) \qquad \forall f,g,h \in G\,. \tag{1.16}$$

Such a map is called a ($\mathbb{R}$-valued) 2-cocycle. There is a class of maps that satisfy this automatically: the ξ's of the form

$$\xi(f,g) = K(fg) - K(f) - K(g) \tag{1.17}$$

for some $K : G \to \mathbb{R}$, as can easily be checked. But when ξ takes this form, it can be redefined away: take the operators

$$\tilde{T}(g) = e^{iK(g)} T(g) \tag{1.18}$$

instead of $T(g)$: they satisfy

$$\tilde{T}(g) \circ \tilde{T}(h) = e^{i(K(g)+K(h))} T(g) \circ T(h) = e^{iK(gh)} T(gh) = \tilde{T}(gh) \tag{1.19}$$

and so define a representation of the usual kind.

So, we look for maps satisfying the "cocycle condition" (1.16). However, we want to discard the trivial solutions of the form (1.17), and also consider two solutions that differ by a trivial one as equivalent (since this difference corresponds to redefining the operators of the projective representation by an irrelevant phase). This is exactly a cohomology problem.

Central extensions of groups

Using a cocycle ξ, we can define a new group $\widehat{G}$ which is called a *central extension* of G. As a set it is given by $\widehat{G} = G \times \mathbb{R}$, and the group operation is

$$(g, a) * (h, b) = (gh, a + b + \xi(g, h)) . \tag{1.20}$$

If ξ is trivial (in the sense of equation (1.17)), then this is isomorphic to the direct product $G \times \mathbb{R}$ with group operation $(g, a) \bullet (h, b) = (gh, a + b)$, where ξ does not appear (so, ξ can be 'redefined away').

Exercise 1. Using the cocycle condition for ξ, prove that $\hat{G}$ is indeed a group. Prove its isomorphism with $G \times \mathbb{R}$ when ξ is trivial.

Now comes the crucial bit: *all projective representations of G come from usual representations of* $\widehat{G}$. Indeed, if we have a projective representation of G as in (1.11), then the map $\mathbf{T}$ defined by

$$\mathbf{T}[(g, a)] = e^{ia} T(g) \tag{1.21}$$

gives a genuine representation of $\widehat{G}$, i.e.

$$\mathbf{T}[(g, a)] \circ \mathbf{T}[(h, b)] = \mathbf{T}[(g, a) * (h, b)] . \tag{1.22}$$

(Checking this is your **exercise 2**.) This is why central extensions are important: for the symmetries of a quantum system, one should be looking at representations of the centrally extended group instead of representations of the group itself.

Central extensions come in two kinds. Topological central extensions have to do with the global structure of the group; for example, the difference between $SO(3)$ and $SU(2)$ is responsible for fermionic representations. In these lectures we will focus on the algebraic, local kind: then, this information is already contained in the Lie algebra.

Central extensions of Lie algebras

So, we turn to the notion of central extensions of Lie algebras. As a vector space, the centrally extended Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, with bracket

$$[(x, \lambda), (y, \mu)]_{\hat{\mathfrak{g}}} = ([x, y]_{\mathfrak{g}}, c(x, y)) \tag{1.23}$$

for some linear map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Writing a basis of $\hat{\mathfrak{g}}$ as $\{T^a, Z\}$, where $T^a = (t^a, 0)$ with $\{t^a\}$ a basis of $\mathfrak{g}$ and $Z = (0, 1)$, this gives

$$[T^a, T^b]_{\hat{\mathfrak{g}}} = f^{ab}{}_c T^c + c^{ab} Z , \qquad [T^a, Z]_{\hat{\mathfrak{g}}} = 0 . \tag{1.24}$$

Here, $f^{ab}{}_c$ are the original structure constants, $[t^a, t^b]_{\mathfrak{g}} = f^{ab}{}_c t^c$, and the c^{ab} are the components of c in this basis, $c^{ab} = c(t^a, t^b)$. So, the new 'central element' Z commutes with everything, but appears in an extra term in the original commutators. Now, this new bracket should be a Lie bracket: this implies that the map c is antisymmetric and satisfies

$$c([x,y]_{\mathfrak{g}},z)+c([y,z]_{\mathfrak{g}},x)+c([z,x]_{\mathfrak{g}},y)=0 \tag{1.25}$$

as a consequence of the Jacobi identity. This is the cocycle condition. Again, we want to discard trivial extensions: those occur when c takes the form

$$c(x,y)=\alpha([x,y]_{\mathfrak{g}}) \tag{1.26}$$

for some linear function $\alpha:\mathfrak{g}\to\mathbb{R}$, since then (1.25) is clearly satisfied automatically. Moreover, when c is trivial in this sense, one can redefine the generators to get rid of the extra term in (1.24): the function $f(X,a)=(X,a+\alpha(X))$ provides an isomorphism between the central extension $\hat{\mathfrak{g}}$ and the non-centrally extended algebra $\mathfrak{g}\oplus\mathbb{R}$. This is again a cohomology problem.

Exercise 3. Differentiate correctly the discussion of central extensions of groups and make the link with central extensions of Lie algebras.

Lie algebra deformations

Another important mathematical question is the following [6]: starting with a given Lie algebra $(\mathfrak{g},[\cdot,\cdot])$, can the Lie algebra structure be deformed? This means that on the same vector space $\mathfrak{g}$, we look for another bracket $\mu_t(\cdot,\cdot)$ depending on a continuous parameter $t\in]-\epsilon,\epsilon[$ such that

1. $\mathfrak{g}$ equipped with this new bracket is a Lie algebra (for all values of t); and
2. we recover the original Lie algebra at $t=0$, i.e. $\mu_0(x,y)=[x,y]$.

As is often the case, it is useful to look at the infinitesimal version of the problem, writing

$$\mu_t=\mu_0+t\,\varphi+O(t^2)\,. \tag{1.27}$$

for some map $\varphi:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. The second condition is then automatic, and the first condition becomes the two equations

$$\varphi(x,y)=-\varphi(y,x) \tag{1.28}$$

$$\sum_{\text{cyclic}}\big([\varphi(x,y),z]+\varphi([x,y],z)\big)=0\,. \tag{1.29}$$

The first is antisymmetry; the second is again called the cocycle condition. Here too, there is a notion of trivial deformation, which leads to trivial solutions of these constraints and the idea of cohomology.

A deformation is trivial if there is an isomorphism between the deformed algebra and the original one, i.e. if the deformation can be undone by a change of basis. This isomorphism would be a linear, invertible map $f:\mathfrak{g}\to\mathfrak{g}$ satisfying

$$f(\mu_t(x,y))=[f(x),f(y)]\,. \tag{1.30}$$

Again, we look at the infinitesimal version and write $f=\mathrm{id}_{\mathfrak{g}}-t\beta+O(t^2)$: at first order, this condition is

$$\varphi(x,y)=\beta([x,y])-[\beta(x),y]-[x,\beta(y)]\,. \tag{1.31}$$

Using the Jacobi identity, it is easy to check that maps φ of this form satisfies the cocycle condition automatically.

1.3 The Chevalley-Eilenberg complex

We now come to the general definition of Lie algebra cohomology. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ a representation of $\mathfrak{g}$ in a vector space V.[1] A (V-valued) n-cochain is a totally antisymmetric multilinear map

$$c : \underbrace{\mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g}}_{n \text{ times}} \to V \,. \tag{1.32}$$

The space of all n-cochains is written $C^n(\mathfrak{g}, V)$. Note that $C^0(\mathfrak{g}, V) \simeq V$ (maps with no arguments are just defined by their image), and that $C^n(\mathfrak{g}, V) = 0$ for $n > \dim \mathfrak{g}$. The differential d is given by the collection of maps

$$d^n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V) \tag{1.33}$$

defined by

$$\begin{aligned}(d^n c)(x_1, x_2, \dots, x_{n+1}) = &\sum_{1\leq i<j\leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i)\,(c(x_1, \dots, \hat{x}_i, \dots, x_{n+1}))\,,\end{aligned} \tag{1.34}$$

where the hats mean that those elements are omitted from the list. This is clearly multilinear, and the right-hand side is indeed an element of V. Less easy to see are the following two properties:

1. the right-hand side of (1.34) is antisymmetric in $x_1, \dots, x_{n+1}$;

2. we have $d^{n+1} \circ d^n = 0$.

The proof of those two properties is **exercise 4**. The first one ensures that $d^n c \in C^{n+1}(\mathfrak{g}, V)$, and the second one means that we really defined a cochain complex, the *Chevalley-Eilenberg* complex. Now, the cohomology groups are defined as usual:

$$H^n(\mathfrak{g}, V) = \frac{\operatorname{Ker} d^n}{\operatorname{Im} d^{n-1}} \,. \tag{1.35}$$

Let's unpack the definition of d^n for the first few cases:

- $n = 0$: The map d^0 acts on 0-cochains, which are just elements of V. Then, the 1-cochain $d^0 c$ is
$$(d^0 c)(x) = \rho(x) c \,. \tag{1.36}$$

- $n = 1$: 1-cochains are maps $\mathfrak{g} \to V$, on which d^1 acts as
$$(d^1 c)(x, y) = -c([x, y]) + \rho(x)c(y) - \rho(y)c(x) \,. \tag{1.37}$$

[1] Reminder: this means that to each x in $\mathfrak{g}$, we associate a linear map $\rho(x) : V \to V$ and, moreover, we have the property $\rho([x, y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x)$ for all $x, y \in \mathfrak{g}$.

- $n = 2$: now

$$(d^2c)(x,y,z) = -c([x,y],z) + c([x,z],y) - c([y,z],x) + \rho(x)c(y,z) - \rho(y)c(x,z) + \rho(z)c(x,y) \,. \tag{1.38}$$

Note that we can shuffle terms around to get cyclic permutations with nicer signs:

$$(d^2c)(x,y,z) = \sum_{\text{cyclic}} \big(-c([x,y],z) + \rho(x)c(y,z) \big) \,. \tag{1.39}$$

With these formulas, we can check explicitly $d^{n+1} \circ d^n = 0$ at low degrees: on a 0-cochain we get

$$\begin{aligned}(d^1d^0c)(x,y) &= -(d_0c)([x,y]) + \rho(x)(d^0c)(y) - \rho(y)(d^0c)(x) \\ &= -\rho([x,y])c + \rho(x)\rho(y)c - \rho(y)\rho(x)c\end{aligned} \tag{1.40}$$

which is zero because ρ is a representation, and on a 1-cochain we have

$$\begin{aligned}(d^2d^1c)(x,y,z) &= \sum_{\text{cyclic}} \big(-(d^1c)([x,y],z) + \rho(x)(d^1c)(y,z) \big) \\ &= \sum_{\text{cyclic}} \big(c([[x,y],z]) - \rho([x,y])c(z) + \rho(z)c([x,y]) \\ &\qquad - \rho(x)c([y,z]) + \rho(x)\rho(y)c(z) - \rho(x)\rho(z)c(y) \big) \,,\end{aligned} \tag{1.41}$$

which vanishes because of the Jacobi identity and the fact that ρ is a representation.

We recognize the formulas that appeared in the previous section:

1. For central extensions, the relevant representation is the trivial one, $V = \mathbb{R}$ and $\rho(x) = 0$. The cocycle condition is $d^2c = 0$ and the trivial solutions are $c = -d^1\alpha$. So, central extensions are classified by $H^2(\mathfrak{g}, \mathbb{R})$.

2. For infinitesimal deformations, the representation is the adjoint, $V = \mathfrak{g}$ and $\rho(x) = \mathrm{ad}_x$, i.e.

$$\rho(x)(y) = \mathrm{ad}_x(y) = [x,y] \,. \tag{1.42}$$

The cocycle condition is $d^2\varphi = 0$ and the trivial solutions are $\varphi = -d^1\beta$: infinitesimal deformations are classified by elements of $H^2(\mathfrak{g}, \mathfrak{g})$.

Remark: in the notation $H^n(\mathfrak{g}, V)$, the representation ρ does not appear. This can unfortunately lead to confusion when there exist several inequivalent representations in the same vector space V, so sometimes the heavier notation $H^n_\rho(\mathfrak{g}, V)$ is used. Here, we will stick to the notation $H^n(\mathfrak{g}, V)$; in particular, in these lectures $V = \mathbb{R}$ will always mean the trivial representation, and $V = \mathfrak{g}$ the adjoint.

Exercise 5. For infinitesimal deformations, spell out the conditions at order two. What is the relevant cohomology group?

1.4 General theorems and examples

In this section, we first give a few general theorems about Chevalley-Eilenberg cohomology, with some applications. Then we give two examples of non-trivial central extensions, leading to the Heisenberg and Virasoro algebras.

- First, there is a nice general interpretation of the 'zeroth' cohomology group $H^0(\mathfrak{g}, V)$: it is the space of *invariants*[2] in V,

$$H^0(\mathfrak{g}, V) \simeq \{v \in V \mid \rho(x)(v) = 0 \quad \forall x \in \mathfrak{g}\}\,. \tag{1.43}$$

Indeed, the cocycle condition is $d^0 c = 0$, i.e. $(d^0 c)(x) = \rho(x)c = 0$ for all $x \in \mathfrak{g}$. There are no trivial solutions since there is no space at degree -1.

In particular, if the representation is trivial (meaning $\rho(x) = 0$ for all x in $\mathfrak{g}$), we have $H^0(\mathfrak{g}, V) = V$ since everything is invariant. So, for any Lie algebra $\mathfrak{g}$, we have

$$H^0(\mathfrak{g}, \mathbb{R}) = \mathbb{R} \tag{1.44}$$

(where we mean the trivial representation $\rho = 0$ on $\mathbb{R}$, see remark above).

- Another generic statement we can make is Whitehead's lemmas: if $\mathfrak{g}$ is a finite-dimensional semisimple algebra[3] and ρ is a finite-dimensional representation, then

$$H^1(\mathfrak{g}, V) = 0 \qquad \text{(Whitehead's first lemma)} \tag{1.45}$$

and

$$H^2(\mathfrak{g}, V) = 0 \qquad \text{(Whitehead's second lemma)}. \tag{1.46}$$

There is no third lemma under these conditions; however, if we also assume that the representation is irreducible[4] and non-trivial, then actually

$$H^n(\mathfrak{g}, V) = 0 \qquad \forall n \geq 0\,. \tag{1.47}$$

Exercise 6. Prove the $n = 0$ case.

In particular, semisimple, finite-dimensional algebras are very rigid: they cannot be deformed, since $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, and don't admit central extensions, since $H^2(\mathfrak{g}, \mathbb{R}) = 0$. This is why we never consider deforming (or analysing the RG flow) of the structure constants in Yang-Mills theory (despite the fact that they are parameters in the Lagrangian), only the masses and couplings: such a deformation can always be absorbed by a change of basis.

[2] At the group level, the invariance condition would be $R(g)v = v$ for all $g \in G$, where R is now a group representation. Differentiating gives $\rho(x)v = 0$ for a Lie algebra representation.

[3] Reminder: a subset $\mathfrak{i} \subseteq \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if any commutator involving an element of $\mathfrak{i}$ is again an element of $\mathfrak{i}$, $[x, i] \in \mathfrak{i}$ for all $x \in \mathfrak{g}$ and $i \in \mathfrak{i}$. There are always 'trivial' ideals: $\{0\}$ and the whole of $\mathfrak{g}$. Now, an algebra is *simple* if it is non-abelian and has no non-trivial ideal. A *semisimple* algebra is a direct sum of simple ones.

[4] Reminder: a subspace $W \subseteq V$ is *invariant* if $\rho(x)w \in W$ for all $x \in \mathfrak{g}$ and all $w \in W$. Any representation has trivial invariant subspaces: $\{0\}$ and V. A representation is *irreducible* if it has no nontrivial invariant subspace.

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- As a last general theorem before we move on to examples, let us mention the link with the de Rham cohomology of the Lie group. First, define the cohomology groups $E^n(G)$ as de Rham cohomology $H^n_{\text{dR}}(G)$ restricted to left-invariant differential forms[5]. Then, we have the following theorem:

  Theorem. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then,*
$$H^n(\mathfrak{g}, \mathbb{R}) = E^n(G)\,. \tag{1.48}$$
  If in addition G is compact,
$$E^n(G) = H^n_{dR}(G)\,. \tag{1.49}$$

As application, we can give the cohomology of $su(2)$: since the group $SU(2)$ is compact and equal to S^3 as a manifold, the theorem gives
$$H^n(su(2), \mathbb{R}) = H^n_{\text{dR}}(S^3) = \begin{cases} \mathbb{R} & \text{if } n = 0, 3 \\ 0 & \text{if } n = 1, 2. \end{cases} \tag{1.50}$$
(The second equality is a well-known fact for the de Rham cohomology of spheres.) This is of course consistent with Whitehead's lemmas.

We now move on to examples of non-trivial central extensions, i.e. examples of non-vanishing second cohomology groups $H^2(\mathfrak{g}, \mathbb{R})$. To reiterate: we are looking for antisymmetric bilinear maps $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying the cocycle condition (1.25), up to trivial solutions of the form (1.26).

Heisenberg algebra. The easiest example with a central extension is the abelian algebra $\mathbb{R}^2$, which has $H^2(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}$. This can be seen easily from the explicit formulas (1.25) and (1.26): since all commutators are zero, any antisymmetric map is automatically a cocycle, and the only trivial cocycle is the zero map. But, up to a constant, there is only one antisymmetric map on $\mathbb{R}^2$: the volume form $c(x, y) = \varepsilon_{ij} x^i y^j$. Said equivalently, the space of antisymmetric 2×2 matrices is one-dimensional. This proves $H^2(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}$. In the basis $Q = (1, 0)$, $P = (0, 1)$ of $\mathbb{R}^2$, this leads to the Heisenberg algebra
$$[Q, P] = 0 + c(Q, P)\, Z = Z \tag{1.51}$$
with Z the central element.

Virasoro algebra. Another important example of central extension is the Virasoro algebra. This is the unique central extension of the algebra $\text{Vect}(S^1)$ of vector fields on a circle (the Witt algebra). In a Fourier basis
$$L_m = i\, e^{imt} \frac{d}{dt}\,, \quad m \in \mathbb{Z}\,, \tag{1.52}$$
the commutators of the Witt algebra are
$$[L_m, L_n] = (m - n) L_{m+n}\,. \tag{1.53}$$

[5]Left-invariant forms are defined by the property $L_g^* \omega = \omega$ where L_g^* is the pullback by the smooth map $L_g : G \to G : h \mapsto gh$ (left translation by g). Because of $d(L_g^* \omega) = L_g^*(d\omega)$, the exterior derivative preserves this property and it makes sense to look at $E^n(G)$. Note that this is really different from de Rham cohomology in general: it could be the case that a left-invariant ω can be written as $\omega = d\eta$, but where η cannot be chosen to be left-invariant. In that case $[\omega]$ is zero in $H^n_{\text{dR}}(G)$ but not in $E^n(G)$.

Again, the cohomology group $H^2(\mathrm{Vect}(S^1), \mathbb{R})$ is one-dimensional. It is generated by the cocycle

$$c(L_m, L_n) = m^3 \delta_{m+n,0} \,. \tag{1.54}$$

The proof of this takes a few steps. First, we show that, up to a trivial redefinition, we can always take $c(L_0, L_m) = 0$. For $m = 0$, this is already the case by antisymmetry. A trivial redefinition gives

$$c'(L_0, L_m) = c(L_0, L_m) + \alpha([L_0, L_m]) = c(L_0, L_m) - m\alpha(L_m) \,. \tag{1.55}$$

Choosing $\alpha(L_m) = c(L_0, L_m)/m$ for all $m \neq 0$ then gives $c'(L_0, L_m) = 0$. (We drop the primes in what follows.) Using the notation $c_{m,n} = c(L_m, L_n)$, the cocycle condition in this basis is

$$\begin{aligned} 0 &= c([L_m, L_n], L_p) + c([L_n, L_p], L_m) + c([L_p, L_m], L_n) \\ &= (m-n)c_{m+n,p} + (n-p)c_{n+p,m} + (p-m)c_{p+m,n} \,. \end{aligned} \tag{1.56}$$

Choosing $p = 0$ and using $c_{m,0} = 0$ as well as $c_{n,m} = -c_{m,n}$, this reduces to

$$(m+n)c_{m,n} = 0 \,. \tag{1.57}$$

So, $c_{m,n}$ is non-zero only when $n = -m$, and we can write

$$c_{m,n} = f_m \, \delta_{m+n,0} \tag{1.58}$$

for some constants f_m which satisfy $f_{-m} = -f_m$. Then, evaluating (1.56) for $p = -m-1$ and $n = 1$ gives the recurrence relation

$$0 = (m-1)f_{m+1} - (m+2)f_m + (1+2m)f_1 \,. \tag{1.59}$$

Observe that this gives $0 = 0$ for $m = 1$: so, all f_m's for $m \geq 3$ are determined from the two "initial conditions" f_1 and f_2. The general solution to the recurrence is

$$f_m = Am + Bm^3 \qquad \forall m > 0 \,, \tag{1.60}$$

where A and B is some linear combination of f_1 and f_2. The f_m for negative m are determined from those using $f_{-m} = -f_m$. So, we found two independent cocycles, one linear in m and the other cubic. But the linear one is trivial: indeed, a redefinition acts as

$$c'_{m,-m} = c_{m,-m} + \alpha([L_m, L_{-m}]) = c_{m,-m} + 2m\, \alpha(L_0) \,. \tag{1.61}$$

Clearly, this can be used to redefine away the linear part, but not the cubic part. Observe also that this redefinition is independent from the previous one where $\alpha(L_0)$ was left arbitrary.

This proves the statement and leads to the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + m^3 \delta_{m+n,0}\, Z \,, \quad [L_m, Z] = 0 \,. \tag{1.62}$$

A more conventional way to write the first commutator is as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} m(m^2-1)\delta_{m+n,0} Z \,. \tag{1.63}$$

This is always allowed by a change of basis: the factor of $1/12$ can be absorbed in a redefinition of Z, and the cocycle $\tilde{c}(L_m, L_n) = m\delta_{m+n,0}$ is trivial. This algebra is very important in string theory and in two-dimensional conformal field theory; however, it is interesting to know that this result is purely algebraic and can be derived without quantum field techniques.

2. Kac-Moody algebras

We now come to the second topic: Kac-Moody algebras. The presentation is organized as follows:

1. First, we give the explicit construction of a family of infinite-dimensional algebras known as affine Kac-Moody algebras.
2. The second subsection is a short reminder about the presentation of simple, finite-dimensional Lie algebras starting from a Cartan matrix.
3. We then define generalized Cartan matrices, weakening the definition of the previous section, and list a few of their properties.
4. The generalized Cartan matrices are used to define Kac-Moody algebras; this definition reproduces that of section 2.2 when the Cartan matrix is of the usual kind (of 'finite type').
5. Then, we explain how the affine algebras of section 2.1 are understood from this point of view (generalized Cartan matrices of 'affine type').
6. We finish with the definition of hyperbolic Kac-Moody algebras and the presentation of their classification theorem.

The whole second part of these lectures will be over the complex numbers.

2.1 Affine Kac-Moody algebras

We begin with the construction and a few properties of loop algebras, and their extensions the affine Kac-Moody algebras.

Loop algebras

Start with a finite-dimensional Lie algebra $\mathfrak{g}$. It's *loop* extension is the space of analytic maps from the circle S^1 to $\mathfrak{g}$ (hence the name: the image of this map is a loop in $\mathfrak{g}$). So, using a parametrization of S^1 by $t \in \mathbb{R}$, $\alpha \in \mathfrak{g}_{\text{loop}}$ means that $\alpha(t) \in \mathfrak{g}$ for all $t \in \mathbb{R}$, with the periodicity condition

$$\alpha(t + 2\pi) = \alpha(t) \,. \tag{2.1}$$

Moreover, this map must be analytic, so we can expand it in Fourier modes.

There is a natural bracket on this space: take brackets pointwise,

$$[\alpha, \beta](t) = [\alpha(t), \beta(t)]_{\mathfrak{g}} \,. \tag{2.2}$$

(On the LHS, we define the *map* $[\alpha, \beta] \in \mathfrak{g}_{\text{loop}}$ by specifying its value at t; on the RHS, the bracket is the original one in $\mathfrak{g}$ between the elements $\alpha(t)$, $\beta(t) \in \mathfrak{g}$.) This makes the space $\mathfrak{g}_{\text{loop}}$ into an infinite-dimensional Lie algebra.

Using a Fourier basis

$$T^a_n(t) = t^a e^{int} \tag{2.3}$$

with the t^a a basis of the original Lie algebra $\mathfrak{g}$ and $n \in \mathbb{Z}$, this gives

$$[T^a_n, T^b_m] = f^{ab}{}_c \, T^c_{m+n} \tag{2.4}$$

in terms of the structure constants of $\mathfrak{g}$, $[t^a, t^b]_{\mathfrak{g}} = f^{ab}{}_c \, t^c$. Note that the generators T^a_0 span a subalgebra of $\mathfrak{g}_{\text{loop}}$ that is isomorphic to $\mathfrak{g}$. This is called the *zero-mode subalgebra* of the loop algebra.

Untwisted affine Kac-Moody algebras

From now on, we assume that $\mathfrak{g}$ is simple. Then, there is a unique central extension of the algebra above: it reads

$$[T^a_m, T^b_n] = f^{ab}{}_c \, T^c_{m+n} + m \, \delta_{m+n,\,0} \, \kappa^{ab} K \tag{2.5}$$

$$[K, T^a_n] = 0 \,, \tag{2.6}$$

where κ^{ab} are the components of the Killing form κ of the original finite-dimensional algebra $\mathfrak{g}$,

$$\kappa^{ab} = \kappa(t^a, t^b) = \mathrm{tr}(\mathrm{ad}_{t^a} \circ \mathrm{ad}_{t^b}) = f^{ac}{}_d \, f^{bd}{}_c \tag{2.7}$$

(in (2.7), the first two equalities hold by definition; the last one is **exercise 7**). Remember that κ^{ab} is non-degenerate since $\mathfrak{g}$ is simple (Cartan's criterion). As explained before, in a quantum theory one should always be looking at the centrally extended algebra instead of the original one.

This is not the last step, however. To get to the proper affine Kac-Moody algebra, one extra generator D is needed: it is called the "derivation" and its brackets with the other generators are simply

$$[D, T^a_m] = m \, T^a_m \,, \qquad [D, K] = 0 \,. \tag{2.8}$$

This finishes the construction of the "untwisted" affine Kac-Moody algebra based on $\mathfrak{g}$. It is written $\mathfrak{g}^{(1)}$: as a vector space it is

$$\mathfrak{g}^{(1)} = \mathfrak{g}_{\text{loop}} \oplus \mathbb{C}K \oplus \mathbb{C}D \,, \tag{2.9}$$

with basis $\{T^a_n, K, D\}$. Its brackets are given in (2.5), (2.6) and (2.8).

Exercise 8. Show that the bilinear form $(\cdot\,, \cdot) : \mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)} \to \mathbb{C}$ defined by

$$(T^a_m, T^b_n) = \kappa^{ab} \delta_{m+n,0} \qquad (K, D) = 1 \,, \qquad \text{others} = 0 \tag{2.10}$$

is invariant, i.e. $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathfrak{g}^{(1)}$, and non-degenerate. Two more (harder) exercises are: prove that it is unique up to a change of basis, and that no such form would exist if D had not been included.

Twisted affine Kac-Moody algebras

In the previous section we constructed the so-called "untwisted" affine Lie algebras. To get the "twisted" ones, we give up the periodicity condition and impose instead the twisted boundary conditions

$$\alpha(t + 2\pi) = \omega(\alpha(t)) \,, \tag{2.11}$$

for some automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ of finite order N. This means that the map ω satisfies the following three conditions:

1. it is linear and preserves the Lie bracket, $\omega([x,y]_{\mathfrak{g}}) = [\omega(x),\omega(y)]_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$;
2. it is invertible; and
3. applying it N times gives the identity,

$$\underbrace{\omega \circ \omega \circ \cdots \circ \omega}_{N \text{ times}} \equiv \omega^N = \mathrm{id}_{\mathfrak{g}} \,. \tag{2.12}$$

In particular, the first condition means that we can still define the bracket of maps as before: that definition is compatible with the twisted periodicity,

$$\begin{aligned}[][\alpha,\beta](t+2\pi) &= [\alpha(t+2\pi),\beta(t+2\pi)]_{\mathfrak{g}} = [\omega(\alpha(t)),\omega(\beta(t))]_{\mathfrak{g}} = \omega([\alpha(t),\beta(t)]_{\mathfrak{g}}) \\ &= \omega([\alpha,\beta](t)) \,. \end{aligned} \tag{2.13}$$

This defines the twisted loop algebra $\mathfrak{g}^{\omega}_{\text{loop}}$. Let us write the commutation relations in a basis. The maps are now $2\pi N$-periodic,

$$\alpha(t+2\pi N) = \omega^N(\alpha(t)) = \alpha(t)\,, \tag{2.14}$$

so a Fourier basis is now given by the

$$T^a_{n+j/N}(t) = t^a e^{i(n+j/N)t} \tag{2.15}$$

with $n \in \mathbb{Z}$ as before and $j = 0, \cdots, N-1$. Their commutation relations are

$$[T^a_{n+j/N}, T^b_{m+k/N}] = f^{ab}{}_c\, T^c_{n+m+(j+k)/N} \,. \tag{2.16}$$

Note that those basis elements satisfy

$$T^a_{n+j/N}(t+2\pi) = t^a e^{i(n+j/N)t} e^{i(2\pi j/N)}\,, \tag{2.17}$$

whereas

$$\omega(T^a_{n+j/N}(t)) = \omega(t^a) e^{i(n+j/N)t} \,. \tag{2.18}$$

This can be only compatible if the basis element t^a that appears in $T^a_{n+j/N}$ satisfies

$$\omega(t^a) = e^{i2\pi j/N} t^a \,. \tag{2.19}$$

In particular, the zero-mode subalgebra of $\mathfrak{g}^{\omega}_{\text{loop}}$ spanned by the T^a_0 is not isomorphic to $\mathfrak{g}$ anymore. Instead, since $j = 0$ in (2.19), it is the subalgebra $\mathfrak{g}_0$ of elements fixed by ω,

$$\mathfrak{g}_0 = \{\, x \in \mathfrak{g} \mid \omega(x) = x \,\} \subseteq \mathfrak{g} \,. \tag{2.20}$$

From there, we proceed as before by adding a central extension K and a derivation D. The central extension is again unique and takes the form

$$[T^a_{n+j/N}, T^b_{m+k/N}] = f^{ab}{}_c\, T^c_{n+m+(j+k)/N} + \left(n + \frac{j}{N}\right) \kappa^{ab} \delta_{n+m+(j+k)/N,0}\, K\,, \tag{2.21}$$

while the derivation acts as

$$[D, T^a_{n+j/N}] = \left(n + \frac{j}{N}\right) T^a_{n+j/N} \,. \tag{2.22}$$

This defines the twisted affine Kac-Moody algebra, written $\mathfrak{g}^{(N)}$ (where N is the order of ω). As a vector space it is

$$\mathfrak{g}^{(N)} = \mathfrak{g}^{\omega}_{\text{loop}} \oplus \mathbb{C}K \oplus \mathbb{C}D \,, \tag{2.23}$$

with basis $\{T^a_{n+j/N}, K, D\}$. Its non-zero brackets are given in (2.21) and (2.22).

Now, it can happen that for some automorphisms ω, the construction of the twisted algebra does not give anything new, and the resulting algebra is isomorphic to an untwisted one. It turns out that the automorphisms for which this construction yields truly new algebras come from symmetries of the Dynkin diagram (those define equivalence classes of 'outer' automorphisms). The Dynkin diagrams of simple finite-dimensional Lie algebras over the complex numbers are well known and displayed in figures 1 and 2. So, the algebras $A_n = sl(n+1, \mathbb{C})$, $D_n = so(2n, \mathbb{C})$ and the exceptional algebra E_6 all have an automorphism of order 2, and $D_4 = so(8, \mathbb{C})$ also has an automorphism of order 3. Therefore, from the classification of simple complex finite-dimensional Lie algebras we get the list of affine Kac-Moody algebras: the untwisted ones are

$$A_n^{(1)}\,,\; B_n^{(1)}\,,\; C_n^{(1)}\,,\; D_n^{(1)}\,,\; E_6^{(1)}\,,\; E_7^{(1)}\,,\; E_8^{(1)}\,,\; F_4^{(1)}\,,\; G_2^{(1)}\,, \tag{2.24}$$

and the twisted ones are

$$A_n^{(2)}\,,\; D_n^{(2)}\,,\; E_6^{(2)}\,,\; D_4^{(3)}\,. \tag{2.25}$$

In this notation, the exponent denotes the order of ω (this is $N = 1$ in the untwisted case, which can be seen as the degenerate twisted case with $\omega = \mathrm{id}_{\mathfrak{g}}$). Of course, at this stage the classification (2.24)–(2.25) is just a consequence of the construction, along with some theorems about automorphisms of simple Lie algebras. However, in section 2.4 we will see an independent definition of affine Kac-Moody algebras; it is a non-trivial fact that those two constructions actually coincide.

Finally, for the twisted algebras $\mathfrak{g}^{(N)}$ we list the zero-mode subalgebra (isomorphic to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ which is fixed under the automorphism ω, see (2.20)). Those are again simple Lie algebras of the Cartan classification:

$$\begin{array}{c|ccccccc} \mathfrak{g}^{(N)} & \mathfrak{g}^{(1)} & A_2^{(2)} & A_{2n}^{(2)} & A_{2n-1}^{(2)} & D_{n+1}^{(2)} & E_6^{(2)} & D_4^{(3)} \\ \hline \mathfrak{g}_0 & \mathfrak{g} & A_1 & B_n & C_n & B_n & F_4 & G_2 \end{array} \,. \tag{2.26}$$

The $u(1)$ case

For completeness, let us mention the algebra $\hat{u}(1)$, which by abuse of terminology is often called "$u(1)$ Kac-Moody". It is a central extension of the $u(1)$ loop algebra: as a vector space,

$$\hat{u}(1) = u(1)_{\text{loop}} \oplus \mathbb{C}K \,. \tag{2.27}$$

In the basis $\{t_m, K\}$, with $m \in \mathbb{Z}$, its brackets are

$$[t_m, t_n] = m\, \delta_{n+m,0}\, K \,, \qquad [t_n, K] = 0 \,. \tag{2.28}$$

2.2 Finite-dimensional case

Before going to the general notion of Kac-Moody algebra, we review some (maybe unfamiliar) facts about the finite-dimensional case.

A finite-dimensional, complex, simple Lie algebra of rank r is completely specified by its Cartan matrix A as the Lie algebra generated by the $3r$ generators

$$\{ h_i, e_i, f_i \mid i = 1, 2, \dots, r \} \tag{2.29}$$

and all their commutators, subject to the Chevalley-Serre relations[6]

$$[h_i, h_j] = 0 \qquad\qquad [e_i, f_j] = \delta_{ij} h_i \tag{2.30}$$
$$[h_i, e_j] = A_{ij} e_j \qquad\qquad [h_i, f_j] = -A_{ij} f_j \tag{2.31}$$
$$\mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0 \qquad\qquad \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) = 0 \tag{2.32}$$

(no sum over repeated indices) along with antisymmetry and the Jacobi identity. The relations of the last line are the *Serre relations*; they should be taken only for $i \neq j$.

What this means is that (2.29) does *not* give a basis of the algebra. To find a basis, one must take those and all possible commutators between them, then use the Chevalley-Serre relations to find the independent ones. In this presentation, the h_i's generate the Cartan subalgebra, the e_i's and their multicommutators are associated to positive roots, and the f_i's and their multicommutators to the negative ones.

In this case, the Cartan matrix is a square $r \times r$ matrix that satisfies the following properties:

a) $A_{ij} \in \mathbb{Z}$

b) $A_{ii} = 2$ (no sum on i),

c) $A_{ij} \leq 0$ for $i \neq j$,

d) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$,

e) $\det A > 0$,

f) A is "indecomposable": there is no renumbering of the indices that puts A in block-diagonal form.

If the last property is dropped, one simply gets a semi-simple Lie algebra, whose simple factors have Cartan matrices given by the diagonal blocks. The property $\det A > 0$ is equivalent to the requirement that the algebra obtained from the procedure described above (by generators and relations) is finite-dimensional. This is very non-trivial, and in general it is quite difficult to work out all the consequences of the Serre relations.

A Cartan matrix can be neatly encoded in a Dynkin diagram, defined as follows:

a) There is one node in the diagram for each $i = 1, \dots, r$;

[6] Unfortunately, two conventions exist. We follow here references [4, 5]; the book [2] has the opposite convention where the transpose of the Cartan matrix is used.

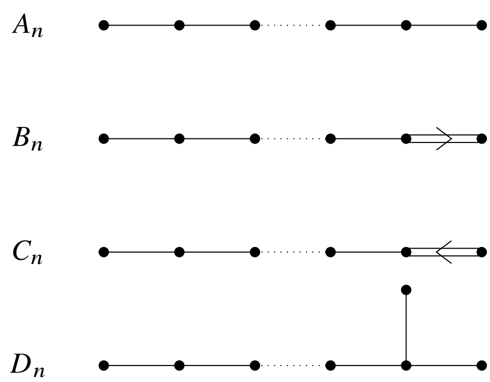

Figure 1: The Dynkin diagrams of the four infinite families of finite-dimensional complex simple Lie algebras. Each diagram contains n nodes. They are the algebras of the classical matrix groups over the complex numbers: $A_n \simeq \mathfrak{sl}(n+1, \mathbb{C})$, $B_n \simeq \mathfrak{so}(2n+1, \mathbb{C})$, $C_n \simeq \mathfrak{sp}(2n, \mathbb{C})$ and $D_n \simeq \mathfrak{so}(2n, \mathbb{C})$.

b) We draw a line between nodes i and j if $A_{ij} \neq 0$;

c) We write the pair of integers $(|A_{ij}|, |A_{ji}|)$ above that line.

Clearly, indecomposable Cartan matrices correspond to connected Dynkin diagrams. Often[7], we have $A_{ij}A_{ji} \leq 4$: then, the last two rules are replaced by

b') The nodes i and j are connected by $n = \max\left(|A_{ij}|, |A_{ji}|\right)$ lines;

c') There is an arrow pointing to node i if $|A_{ij}| > |A_{ji}|$.

In this way, the classification of simple, finite-dimensional complex Lie algebras is reduced to the problem of classifying all Cartan matrices, or all corresponding Dynkin diagrams. The result (the Killing-Cartan classification) is well known and is displayed in figures 1 and 2.

Let's give two simple examples. The simplest Cartan matrix is the 1×1 matrix $A = (2)$, with Dynkin diagram consisting of just one node. The corresponding algebra is generated by the triple $\{h, e, f\}$, and the Chevalley-Serre relations reduce to

$$[h, e] = 2e\,, \quad [h, f] = -2f\,, \quad [e, f] = h\,. \tag{2.33}$$

This is just the algebra $sl(2, \mathbb{C})$ in the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.34}$$

Note that in the general case, for each fixed i the triple $\{h_i, e_i, f_i\}$ generates an $sl(2, \mathbb{C})$ subalgebra; the other relations tell us how these various $sl(2, \mathbb{C})$'s interact.

Another example is the rank 2 Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{2.35}$$

[7]Actually (and this is a non-trivial thing to prove), this is always the case here because of the condition $\det A > 0$, but the same set of rules will also be used for the more general Cartan matrices introduced in the next section.

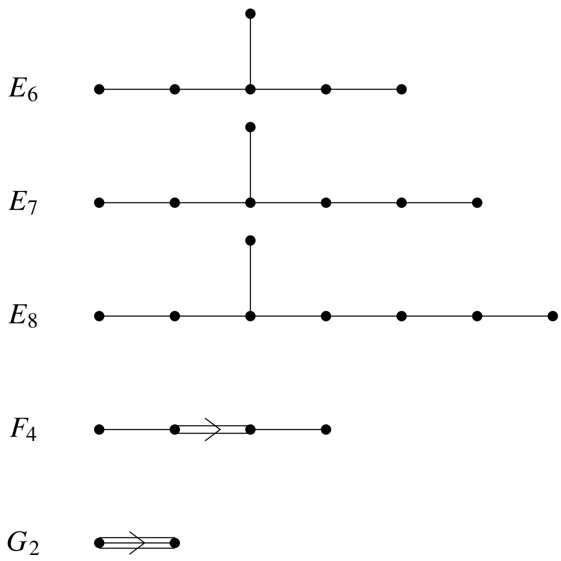

Figure 2: The Dynkin diagrams of the five exceptional simple Lie algebras E_6, E_7, E_8, F_4 and G_2.

with Dynkin diagram

 . (2.36)

The algebra is generated by $\{h_1, h_2, e_1, e_2, f_1, f_2\}$ and all their commutators. As before, the triples with the same index give two $sl(2,\mathbb{C})$ subalgebras. The other Chevalley-Serre relations (with mixed indices) are

$$\begin{aligned} &[h_1, h_2] = [e_1, f_2] = [e_2, f_1] = 0 \\ &[h_1, e_2] = -e_2\,, \quad [h_1, f_2] = f_2\,, \quad [h_2, e_1] = -e_1\,, \quad [h_2, f_1] = f_1 \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} [e_1, [e_1, e_2]] &= [e_2, [e_2, e_1]] = 0 \\ [f_1, [f_1, f_2]] &= [f_2, [f_2, f_1]] = 0\,. \end{aligned} \tag{2.38}$$

The first set of relations show that, among all multi-brackets, it is sufficient to consider those only containing e_i's or only containing f_i's, but not mixed ones or any containing h_i's. For example,

$$\begin{aligned} [f_1, [e_1, e_2]] &= -[e_1, [e_2, f_1]] - [e_2, [f_1, e_1]] = 0 + [e_2, h_1] = -A_{12} e_2 \\ &= e_2\,. \end{aligned} \tag{2.39}$$

This is indeed the case in general (a recursive proof of this fact, for any Cartan matrix, is left as **exercise 9**).

Then, the Serre relations (2.38) show that any such bracket with twice the same element vanishes; therefore, the algebra is 8-dimensional and spanned (as a vector space) by

$$\{h_1, h_2, e_1, e_2, [e_1, e_2], f_1, f_2, [f_1, f_2]\}\,. \tag{2.40}$$

A matrix realization of this basis, satisfying all required relations, is

$$\begin{gathered}
h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
[e_1, e_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [f_1, f_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{gathered} \tag{2.41}$$

This is the algebra $sl(3,\mathbb{C})$. The two $sl(2,\mathbb{C})$ subalgebras spanned by $\{h_1, e_1, f_1\}$ and $\{h_2, e_2, f_2\}$ are clearly visible; compare with (2.34).

With these formulas at hand, we can also prove explicitly a claim made in the previous section. (This can be skipped on a first reading.) There is a symmetry in the Dynkin diagram (2.36) of $sl(3,\mathbb{C})$ which exchanges the two nodes. This corresponds to an automorphism ω of the algebra exchanging the indices 1 and 2,

$$\begin{aligned}
\omega(h_1) &= h_2 \quad & \omega(e_1) &= e_2 \quad & \omega(f_1) &= f_2 \\
\omega(h_2) &= h_1 \quad & \omega(e_2) &= e_1 \quad & \omega(f_2) &= f_1 .
\end{aligned} \tag{2.42}$$

The morphism property then gives, for the two remaining basis elements,

$$\omega([e_1, e_2]) = [\omega(e_1), \omega(e_2)] = [e_2, e_1] = -[e_1, e_2] \tag{2.43}$$

and, likewise, $\omega([f_1, f_2]) = -[f_1, f_2]$. This is the Dynkin diagram automorphism[8] of order 2 appearing in the construction of the twisted algebra $A_2^{(2)}$. In table 2.26, we claimed that the subalgebra that is fixed by ω is isomorphic to $A_1 = sl(2,\mathbb{C})$. Indeed, that algebra is clearly given by linear combinations of $h_1 + h_2$, $e_1 + e_2$ and $f_1 + f_2$, and it is easily checked that

$$H = 2(h_1 + h_2), \quad E = \sqrt{2}(e_1 + e_2), \quad F = \sqrt{2}(f_1 + f_2) \tag{2.44}$$

satisfy the defining relations (2.33) of $sl(2,\mathbb{C})$.

2.3 Generalized Cartan matrices

The idea behind Kac-Moody algebras is to keep the same Chevalley-Serre generators and relations, but dropping the condition $\det A > 0$ on the Cartan matrix. The most striking consequence is that the algebras obtained this way, when $\det A \leq 0$, are infinite-dimensional: the Serre relations do not restrict the multicommutators to a finite number.

So, a *generalized Cartan matrix* is a $r \times r$ matrix such that

[8]The general definition of such automorphisms should be clear from this example: for a permutation $\dot\omega \in S_r$ of the r nodes that leave the diagram invariant, i.e. $A^{\dot\omega(i)\dot\omega(j)} = A^{ij}$ for all $i, j = 1, \ldots, r$, one can define an automorphism ω of the algebra by the formula

$$\omega(h_i) = h_{\dot\omega(i)}, \quad \omega(e_i) = e_{\dot\omega(i)}, \quad \omega(f_i) = f_{\dot\omega(i)}.$$

a) $A_{ij} \in \mathbb{Z}$

b) $A_{ii} = 2$ (no sum on i),

c) $A_{ij} \leq 0$ for $i \neq j$,

d) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$.

As before, we will often ask that A is also indecomposable. Dynkin diagrams can be associated to generalized Cartan matrices as above.

Generalized Cartan matrices come in three types (proving this is a fun exercise in linear algebra; see [5], chapter 4). For a vector $v = (v_1, v_2, \ldots, v_r) \in \mathbb{R}^r$, we use the notation $v > 0$ if all components satisfy $v_i > 0$ ($v \geq 0$ and $v < 0$ are defined in the same manner).

Theorem. *Let A be an indecomposable generalized Cartan matrix. Then, one and only one of the following holds, both for A and its transpose:*

1. $\exists u > 0$ *such that* $Au > 0$. *This is 'finite type'.*
2. $\exists u > 0$ *such that* $Au = 0$, *and u is unique (up to a positive factor). This is 'affine type'.*
3. $\exists u > 0$ *such that* $Au < 0$. *This is 'indefinite type'.*

Moreover, one often considers *symmetrizable* Cartan matrices: they are those which can be written as

$$A = DS\,, \tag{2.45}$$

where D is diagonal with strictly positive entries and S is symmetric. As in the finite-dimensional case (where A is given in terms of simple roots by the formula $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$), this property has to do with a scalar product on root space, which however is not Euclidean in the general case.

Theorem. *Cartan matrices of finite and affine type are symmetrizable.*

One can also prove the following characterization of finite and affine generalized Cartan matrices:

1. A is of finite type if and only if $\det A > 0$;
2. A is of affine type if and only if $\det A = 0$ and $\det A_{\{i\}} > 0$, where $A_{\{i\}}$ is the matrix obtained from A by removing the ith line and the ith column.

The signature of S is then $(+ + \cdots +)$ (finite) or $(0 + \cdots +)$ (affine). Notice that, at the level of Dynkin diagrams, this characterization of affine generalized Cartan matrices means that removing any node leaves a diagram of finite type (or a disjoint union of them). Since those are classified (figures 1 and 2), this enables a complete classification of all affine generalized Cartan matrices. The corresponding Dynkin diagrams are displayed in figures 3 and 4.

Exercise 10. Study the generalized Cartan matrix

$$A_{p,q} = \begin{pmatrix} 2 & -p \\ -q & 2 \end{pmatrix} \tag{2.46}$$

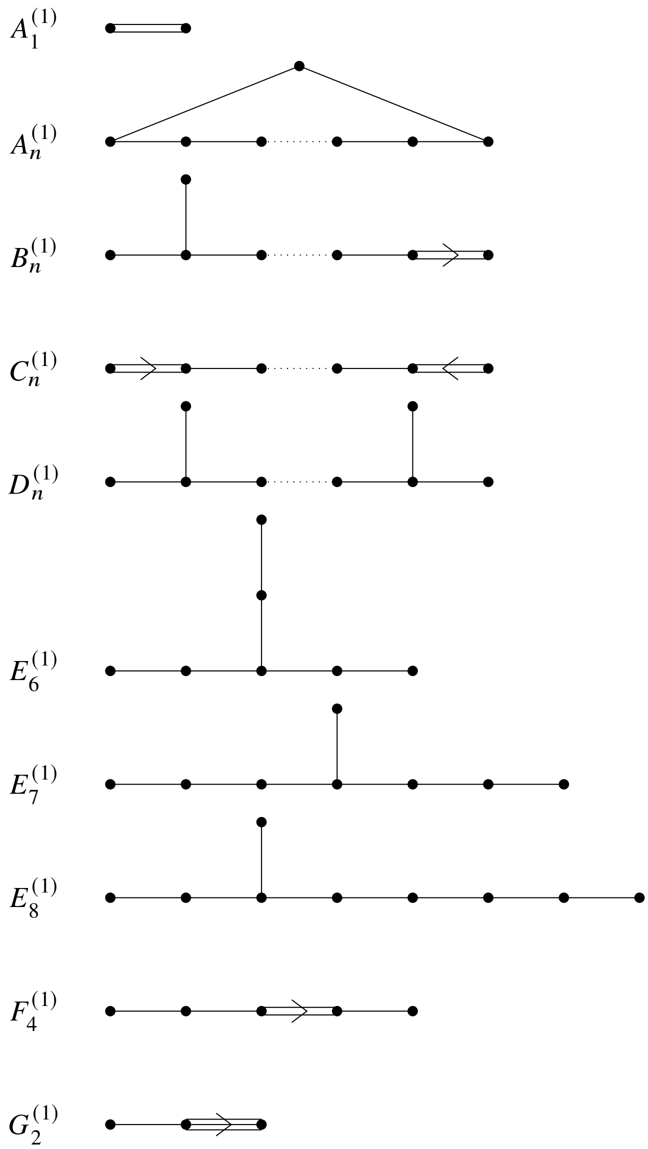

Figure 3: The Dynkin diagrams of the untwisted affine Kac-Moody algebras $\mathfrak{g}^{(1)}$. They contain one more node than the Dynkin diagram of $\mathfrak{g}$ (so, these diagrams have $n + 1$ nodes, where n is the rank of $\mathfrak{g}$).

for p, q strictly positive integers. This means the following questions: For which values of p and q is it of finite type? Affine? Indefinite? Draw the Dynkin diagrams in every case. When is it symmetrizable? What is the signature of S?

2.4 Definition of Kac-Moody algebras

We now come to the definition of the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix. (The definition in the general case is more complicated; see [5].) First, notice that the definition of the algebra from the Chevalley-Serre generators and relations in section 2.2 does not involve the condition $\det A > 0$: therefore, it also makes sense for generalized Cartan

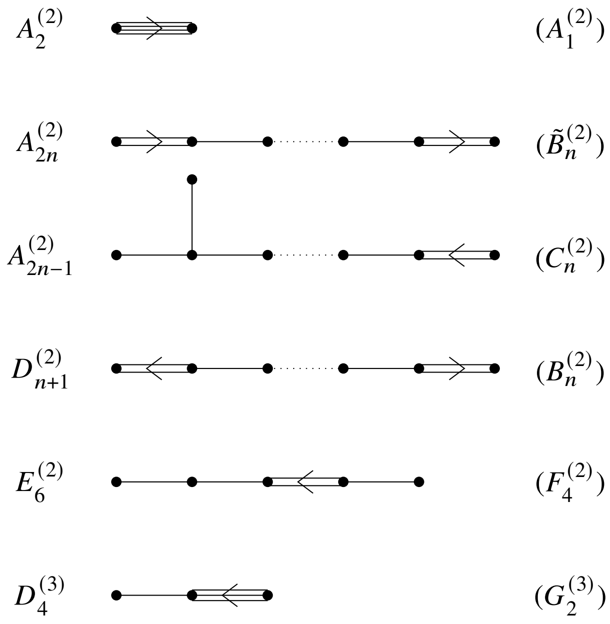

Figure 4: The Dynkin diagrams of the twisted affine Kac-Moody algebras $\mathfrak{g}^{(N)}$. The zero-mode subalgebra $\mathfrak{g}_0$ (see table 2.26) is obtained by removing the leftmost node: again, the number of nodes of these diagrams is $n+1$, where n is the rank of $\mathfrak{g}_0$. Sometimes, the notation in parenthesis is used for the same algebra to emphasize the sub-diagram corresponding to $\mathfrak{g}_0$.

matrices. We denote the algebra obtained this way by $\mathfrak{g}'(A)$. As alluded to before, one can prove the following:

Theorem. *Let A be a generalized Cartan matrix. The algebra $\mathfrak{g}'(A)$ is finite dimensional if and only if A is of finite type.*

As before, independent multicommutators only involve e_i's or f_i's, but not both, and never any h_i. Accordingly, $\mathfrak{g}'(A)$ has the following useful 'triangular' decomposition (as a vector space):

$$\mathfrak{g}'(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \,, \tag{2.47}$$

where as basis of $\mathfrak{h}$ is given by the h_i's, a basis of $\mathfrak{n}_+$ is given by the e_i's and all their commutators $[e_i, [e_j, \ldots, [e_k, e_l] \ldots]]$, and a basis of $\mathfrak{n}_-$ by the f_i's and their multicommutators (which of those commutators survive is determined by the Serre relations, as we saw before). Moreover, it is easy to see that the adjoint action of $\mathfrak{h}$ on $\mathfrak{n}_\pm$ is diagonal. Many of the interesting properties of finite-dimensional simple Lie algebras, such as root systems and highest-weight representations, come from this triangular structure and generalize to this infinite-dimensional case.

Let us now finally come to the definition of Kac-Moody algebras. There are two cases to consider. First, if $\det A \neq 0$, the Kac-Moody algebra $\mathfrak{g}(A)$ is just defined via the generators and relations we saw, i.e. as

$$\mathfrak{g}(A) = \mathfrak{g}'(A) \qquad (\det A \neq 0) \,. \tag{2.48}$$

The following important theorem holds:

Theorem. *Let A be an indecomposable generalized Cartan matrix with* $\det A \neq 0$*. Then, the Kac-Moody algebra* $\mathfrak{g}(A)$ *is simple.*

In the case $\det A = 0$, the definition of $\mathfrak{g}(A)$ involves a number of extra derivations, one per zero eigenvalue of S. We will give the definition in the affine case, in which there is only one new generator η: as a vector space, the affine Kac-Moody algebra is then

$$\mathfrak{g}(A) = \mathfrak{g}'(A) \oplus \mathbb{C}\eta \qquad \text{(affine case)}, \tag{2.49}$$

and the extra commutators involving η are

$$[\eta, h_i] = 0\,, \quad [\eta, e_i] = \delta_{1i} e_1\,, \quad [\eta, f_i] = -\delta_{1i} f_1\,. \tag{2.50}$$

This algebra is not simple. Indeed, let u be the non-zero left eigenvector of A, $\sum_{i=1}^{r} u_i A_{ij} = 0$, which is unique up to a factor since A is affine. Then, the linear combination

$$c = \sum_{i=1}^{r} u_i h_i \tag{2.51}$$

commutes with everything (**exercise 11**: check this assertion). The center of $\mathfrak{g}(A)$ is in fact one-dimensional and is spanned by c; the only proper ideals of $\mathfrak{g}(A)$ are then $\mathfrak{g}'(A)$ and $\mathbb{C}c$.

2.5 Affine Kac-Moody algebras, revisited

We now have two definitions of affine Kac-Moody algebras: there is the construction of section 2.1 based on loop extensions of finite-dimensional simple Lie algebras, and the definition of section 2.4 based on affine Cartan matrices. It is now a non-trivial fact that those two constructions give indeed isomorphic algebras. The Dynkin diagrams of the untwisted algebras listed in (2.24) are presented in figure 3, and those of the twisted ones listed in (2.25) appear in figure 4.

We will not present the general proof. Instead, we go quite explicitly through the simplest example, that of the algebra $A_1^{(1)}$ with Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{2.52}$$

(The trusting reader can skip the remainder of this subsection on a first reading.) This Cartan matrix can be read off the corresponding graph of figure 3. It is indeed an affine Cartan matrix, with zero determinant and a single zero eigenvalue, with corresponding eigenvector $u = (1, 1)$. We want to check that the corresponding Kac-Moody algebra $\mathfrak{g}(A)$ and the untwisted algebra $sl(2, \mathbb{C})^{(1)}$ constructed in section 2.1 are isomorphic.

Let's take $\{t^a\} = \{H, E, F\}$ as a basis of $sl(2, \mathbb{C})$, with commutation relations as in (2.33). In that basis, the only non-zero components of the Killing form κ are

$$\kappa(E, F) = 1\,, \quad \kappa(H, H) = 2\,. \tag{2.53}$$

Then, the untwisted affine algebra $sl(2,\mathbb{C})^{(1)}$ as constructed in the previous section has basis $\{H_m, E_m, F_m, K, D\}$ with commutators

$$[H_m, E_n] = 2E_{m+n} \tag{2.54}$$
$$[H_m, F_n] = -2F_{m+n} \tag{2.55}$$
$$[E_m, F_n] = H_{m+n} + m\delta_{m+n,0}K \tag{2.56}$$
$$[H_m, H_n] = 2m\delta_{m+n,0}K \tag{2.57}$$
$$[D, E_m] = mE_m\,, \quad [D, F_m] = mF_m\,, \quad [D, H_m] = mH_m\,. \tag{2.58}$$

On the other hand, the Chevalley-Serre generators of $\mathfrak{g}(A)$ are

$$\{h_1, e_1, f_1, h_2, e_2, f_2, \eta\}\,. \tag{2.59}$$

The algebra is infinite-dimensional: indeed, the Serre relations

$$[e_1,[e_1,[e_1,e_2]]] = 0\,, \quad [e_2,[e_2,[e_2,e_1]]] = 0 \tag{2.60}$$

do not kill any of the multicommutators where e_1 and e_2 alternate,

$$\mathcal{E}^3_n = [e_1,[e_2,\ldots,[e_1,e_2]\ldots]] \qquad (n\ e_1\text{'s and } e_2\text{'s}) \tag{2.61}$$
$$\mathcal{E}^-_n = [e_1,[e_2,\ldots,[e_2,e_1]\ldots]] \qquad (n+1\ e_1\text{'s and } n\ e_2\text{'s}) \tag{2.62}$$
$$\mathcal{E}^+_n = [e_2,[e_1,\ldots,[e_1,e_2]\ldots]] \qquad (n\ e_1\text{'s and } n+1\ e_2\text{'s}) \tag{2.63}$$

for $n \geq 1$ (the same is true for multicommutators of f's, and $\mathcal{F}$'s are defined in the same way). In fact, those are the only multicommutators that survive and the set

$$\{h_1, e_1, f_1, h_2, e_2, f_2, \eta, \mathcal{E}^3_n, \mathcal{E}^-_n, \mathcal{E}^+_n, \mathcal{F}^3_n, \mathcal{F}^-_n, \mathcal{F}^+_n\} \qquad (n \geq 1) \tag{2.64}$$

is a basis of $\mathfrak{g}(A)$.

The isomorphism between the two algebras is reasonably easy to find explicitly, at least for the Chevalley-Serre generators. First, it is natural to identify the central element $c = \sum u_i h_i = h_1 + h_2$ appearing in (2.51) with K, and the derivation η with D. Another natural choice is to identify the zero-mode $sl(2,\mathbb{C})$ subalgebra spanned by $\{H_0, E_0, F_0\}$ with one of the two natural $sl(2,\mathbb{C})$'s in the Kac-Moody algebra, either $\{h_1, e_1, f_1\}$ or $\{h_2, e_2, f_2\}$. But the zero-mode subalgebra commutes with the derivation D; looking at (2.50) tells us to pick $\{h_2, e_2, f_2\}$. Then, $[\eta, e_1] = e_1$ and $[\eta, f_1] = -f_1$ tell us that e_1 and f_1 should be some linear combinations of generators at level 1 and -1 respectively. Combined with the $sl(2,\mathbb{C})$ commutators $[e_1, f_1] = h_1$, $[h_1, e_1] = 2e_1$ and $[h_1, f_1] = -2f_1$, the only solution is to identify e_1 with F_1 and f_1 with E_{-1}. So, we can define a Lie algebra homomorphism

$$\varphi : \mathfrak{g}(A) \to sl(2,\mathbb{C})^{(1)} \tag{2.65}$$

by its action on the Chevalley-Serre generators:

$$\begin{aligned}
&\varphi(h_1) = K - H_0\,, && \varphi(e_1) = F_1\,, && \varphi(f_1) = E_{-1}\,,\\
&\varphi(h_2) = H_0\,, && \varphi(e_2) = E_0\,, && \varphi(f_2) = F_0\,,\\
&\varphi(\eta) = D\,.
\end{aligned} \tag{2.66}$$

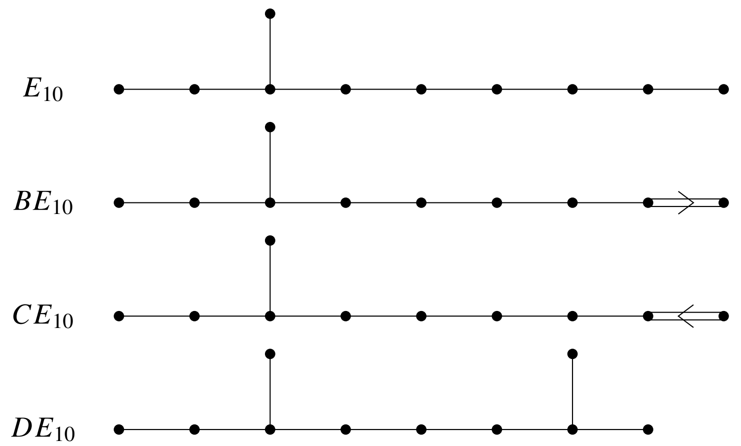

Figure 5: The Dynkin diagrams of the four hyperbolic Kac-Moody algebras of rank 10.

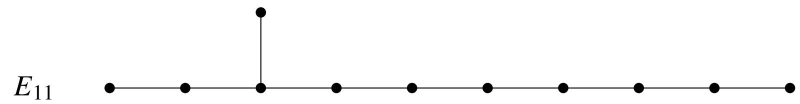

Figure 6: The Dynkin diagrams of the E_{11} algebra. It is Lorentzian but not hyperbolic.

This is extended to the rest of the algebra by linearity and the morphism property $\varphi([x,y]) = [\varphi(x), \varphi(y)]$. So, for example, further commutators give

$$\begin{aligned} \varphi([e_1, e_2]) &= [F_1, E_0] = -H_1 \,, \\ \varphi([e_2, [e_1, e_2]]) &= [E_0, -H_1] = -2E_1 \,, \\ \varphi([e_1, [e_2, [e_1, e_2]]]) &= [F_1, -2E_1] = 2H_2 \\ &\vdots \end{aligned} \tag{2.67}$$

In this way, it is easy to see that we have

$$\begin{aligned} &\varphi(\mathcal{E}^3_n) \sim H_n \,, &&\varphi(\mathcal{E}^+_n) \sim E_n \,, &&\varphi(\mathcal{E}^-_n) \sim F_{n+1} \\ &\varphi(\mathcal{F}^3_n) \sim H_{-n} \,, &&\varphi(\mathcal{F}^+_n) \sim F_{-n} \,, &&\varphi(\mathcal{F}^-_n) \sim E_{-n-1} \end{aligned} \tag{2.68}$$

for all positive n. This shows that the morphism φ is invertible and therefore an isomorphism.

Exercise 12. Check that all Chevalley-Serre relations are indeed satisfied in the $sl(2,\mathbb{C})^{(1)}$ algebra.

2.6 Hyperbolic case

As we saw above, the Dynkin diagrams of affine Kac-Moody algebras can be found by requiring that removing any node yields Dynkin diagrams corresponding to finite-dimensional algebras. The classification of affine algebras then nicely follows from the classification of finite-dimensional ones.

Likewise, *hyperbolic* Kac-Moody algebras are defined by the fact that removing any node only gives finite-dimensional or affine diagrams. One can prove that symmetrizable hyperbolic algebras are "Lorentzian": the signature of S is $(- + \cdots +)$. See figure 5 for some examples; by contrast, the algebra E_{11} of figure 6 is Lorentzian but not hyperbolic.

Hyperbolic algebras have also been completely classified [7–9]:

1. There is only one infinite family, of rank two: the algebras $\mathfrak{g}(A_{p,q})$ with Cartan matrix

$$A_{p,q} = \begin{pmatrix} 2 & -p \\ -q & 2 \end{pmatrix} \tag{2.69}$$

   and Dynkin diagram

$$\bullet \overset{(p,q)}{\longrightarrow\!\!\!\!-\!\!\!-\!\!\!-} \bullet\,, \tag{2.70}$$

   where p and q are strictly positive integers and $pq > 4$.

2. Apart from this infinite family, hyperbolic algebras are in finite number! They only appear in ranks ≤ 10, and their number is 238. This is reduced to 142 if one also imposes the symmetrizability condition.

This is in stark contrast to the finite and affine cases, where one finds infinite families with unbounded rank related to the classical matrix algebras. Here, the hyperbolic condition is stronger and, apart from the rank 2 family, one could say that they are all 'exceptional'. The bound $r \leq 10$ on the rank is closely related to the fact that exceptional finite-dimensional algebras stop with E_8.

Exercise 13. Classify all hyperbolic algebras of maximal rank 10. (You should find the algebras of figure 5.)

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