Updated Standard Model Prediction for $K \to \pi\nu\bar{\nu}$ and ϵ_K

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The rare $K \to \pi\nu\bar{\nu}$ decay modes and the parameter ϵ_K that measures CP violation in Kaon mixing are sensitive probes of physics beyond the standard model. In this article we provide the updated standard-model prediction for the rare decay modes in detail, and summarise the status of standard-model prediction of ϵ_K. We find $\mathrm{BR}(K^+ \to \pi^+\nu\bar{\nu}) = 7.73(61) \times 10^{-11}$ and $\mathrm{BR}(K_L \to \pi^0\nu\bar{\nu}) = 2.59(29) \times 10^{-11}$. The uncertainties are dominated by parametric input.

1. Introduction

The rare kaon decays $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$ together with indirect CP violation in the neutral kaon system, parameterised by ϵ_K, are among the cleanest probes of physics beyond the standard model (SM). The reason is the exceptional control over short- and long-distance SM contributions.

The rare kaon decay modes are generated by highly virtual electroweak box and Z-penguin diagrams that can be calculated to high precision in perturbation theory. Light-quark contributions are strongly suppressed by the GIM mechanism, and the decay matrix elements can be extracted from precisely measured semi-leptonic kaon decays using approximate isospin symmetry. The GIM suppression also implies that the decay modes are dominated by internal top-quark exchanges, which makes these decay modes very sensitive to new sources of flavour violation.

The NA62 collaboration recently reported [1, 2] the measurement of the branching ratio $\text{BR}(K^+ \to \pi^+\nu\bar{\nu}) = (10.6^{+3.4}_{-3.4}|_{\text{stat}} \pm 0.9_{\text{syst}}) \times 10^{-11}$ that also includes data analysed in previous runs [2] (see Refs. [3, 4] for the older Brookhaven results). The best upper bound for the neutral decay mode $\text{BR}(K^+ \to \pi^+\nu\bar{\nu}) \leq 3.0 \times 10^{-9}$ was obtained by the JPARC-KOTO [5, 6] experiment.

On the other hand, the size of indirect CP violation in the neutral kaon system is experimentally well-known since many years; it is given by the parameter $\epsilon_K = (2.228 \pm 0.011)e^{i(43.5\pm0.5)^\circ} \times 10^{-3}$ [7]. A precise SM prediction of ϵ_K was long impeded by a non-converging perturbation series for the charm-quark contribution; this problem has been solved in Ref. [8].

In light of the recent measurement it is timely to update the theory prediction of the so called golden rare kaon decays and discuss the recent progress in the prediction of the parameter ϵ_K. In Sec. 2 we provide the updated SM prediction for the rare kaon decays. We discuss in detail how the numerical values are obtained, and compare to other recent SM predictions. The theory status of ϵ_K is briefly summarised in Sec. 3.

2. Standard Model Update of $K \to \pi\nu\bar{\nu}$

The effective Hamiltonian relevant for the two rare $K \to \pi\nu\bar{\nu}$ decays is given by [9]

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2\theta_w} \sum_{\ell=e,\mu,\tau} \left(\lambda_c X^\ell + \lambda_t X_t\right)(\bar{s}_L\gamma_\mu d_L)(\bar{\nu}_{\ell L}\gamma^\mu \nu_{\ell L}) + \text{h.c.} . \tag{1}$$

Here, G_F denotes the Fermi constant, α the electromagnetic coupling constant, and $\sin\theta_w$ the sine of the weak mixing angle. The Cabibbo-Kobayashi-Maskawa (CKM) matrix elements are contained in the parameters $\lambda_i = V_{is}^* V_{id}$. The left-handed fermion fields are denoted by $f_L \equiv (1 - \gamma_5)/2f$. The loop functions X^ℓ and X_t are discussed below.

The branching ratio of the charged mode is given by

$$\text{Br}\left(K^+ \to \pi^+\nu\bar{\nu}(\gamma)\right) = \kappa_+(1 + \Delta_{\text{EM}})\left[\left(\frac{\text{Im}\lambda_t}{\lambda^5}X_t\right)^2 + \left(\frac{\text{Re}\lambda_c}{\lambda}\left(P_c + \delta P_{c,u}\right) + \frac{\text{Re}\lambda_t}{\lambda^5}X_t\right)^2\right]. \tag{2}$$

Here, X_t is a function of $x_t = m_t(\mu_t)^2/M_W^2$ and has been calculated including next-to-leading order QCD [10, 11] and electroweak [12] corrections. Hence, the top-quark mass, m_t, and the W-boson

mass, M_W, depend on the QCD and electroweak renormalization schemes. In fact, M_W is not a primary input and has to be calculated as a function of the Z-boson mass, M_Z, the Higgs-boson mass, M_h, and the strong and the electromagnetic coupling constants α_s and α, respectively (see Ref. [13] for more details). The $\overline{\text{MS}}$ scheme is the natural choice regarding QCD. We obtained the numerical value $m_t(m_t) = 162.83(67)$ GeV from the top-quark pole mass (see Tab. 1) by converting it to QCD-$\overline{\text{MS}}$ at three-loop accuracy, using RunDec [14]. The electroweak corrections are minimized if all masses are renormalized on-shell with respect to the electroweak interactions [12]; we adopt this scheme for our numerics. We obtain our numerical value for X_t by calculating a mean value of the QCD contribution, $X_t^{\text{QCD,avg}}$, by varying $\mu_t \in [60, 320]$ GeV in the expression

$$X_t^{\text{QCD}} = X_t^{(0)}(x_t(\mu_t)) + \frac{\alpha_s(\mu_t)}{4\pi} X_t^{(1)}(x_t(\mu_t)), \tag{3}$$

and taking the average of the smallest and largest value of X_t^{QCD}. Here, $X_t^{(0)}$ and $X_t^{(1)}$ denote the leading-order (LO) and next-to-leading-order (NLO) QCD contributions to X_t, respectively. Electroweak corrections are then taken into account by including the fit function r_X, which is valid for electroweak-onshell masses and given in Ref. [12]. In total

$$X_t = X_t^{\text{QCD,avg}} + [r_X(m_t(m_t)) - 1] X_t^{(0)}(m_t(m_t)). \tag{4}$$

The theory uncertainty associated with the QCD corrections is given by the difference of the central value $X_t^{\text{QCD,avg}}$ and the minimal / maximal value in the μ_t interval. The uncertainty associated to the electroweak corrections is $\pm 0.00134 \times X_t$ [12]. In total, we find

$$X_t = 1.462 \pm 0.017_{\text{QCD}} \pm 0.002_{\text{EW}}. \tag{5}$$

The parameter $P_c = \lambda^{-4}(\frac{2}{3}X^e + \frac{1}{3}X^\tau)$ comprises the charm-quark contribution and has been calculated at next-to-next-to-leading order (NNLO) in QCD [15] and at NLO in the electroweak interactions [16]. It is a function of $x_c = m_c(\mu_c)^2/M_W^2$ which, upon inclusion of the electroweak corrections, is defined as $x_c = \sqrt{2}\sin^2\theta_w G_F m_c^2(\mu_c)/(\pi\alpha)$. A fit formula for P_c and its theory uncertainty, including the NLO electroweak and NNLO QCD correction in dependence on the strong coupling and the charm-quark mass, has been presented in Ref. [16]. With the current PDG input, we find

$$P_c = \left(\frac{0.2255}{\lambda}\right)^4 \times (0.3604 \pm 0.0087). \tag{6}$$

The effects of dimension-eight operators at the charm threshold, as well as additional long-distance contributions arising from up- and charm-quarks have been estimated in Ref. [17], leading to the correction $\delta P_{c,u} = 0.04(2)$. These effects can be computed using lattice QCD in the future [18] (see Ref. [19] for preliminary results).

The hadronic matrix element is contained in the parameter

$$\kappa_+ = \left(\frac{0.231}{\sin^2\theta_w}\right)^2 \left(\frac{\alpha(M_Z)}{127.9^{-1}}\right)^2 \left(\frac{\lambda}{0.225}\right)^8 \times 0.5173(25) \times 10^{-10}, \tag{7}$$

extracted from $K_{\ell 3}$ decay including higher-order chiral corrections [20]. The NLO QED corrections [20] are parameterised by $\Delta_{\text{EM}} = -0.003$ in Eq. (2).

m_t^{Pole}[GeV]	172.4 ± 0.7	$\alpha_s^{\text{5fl}}(M_Z)$	0.1179 ± 0.0010
M_h[GeV]	125.10 ± 0.14	$\alpha^{-1}(M_Z)$	127.952 ± 0.009
M_Z[GeV]	91.1876 ± 0.0021	$\Delta\alpha^{\text{had}}$	0.02766 ± 0.00007
m_c[GeV]	1.27 ± 0.02	$s_{w,\text{ND}}^2(M_Z)$	0.23141 ± 0.00004
		$|\epsilon_K|$	$(2.228 \pm 0.011) \times 10^{-3}$
$\bar{\rho}$	0.141 ± 0.017	λ	0.22650 ± 0.00048
$\bar{\eta}$	0.357 ± 0.011	A	0.790 ± 0.017

Table 1: Parametric input used for our SM prediction of the $K \to \pi\nu\bar{\nu}$ branching rations; all values are taken from pdgLive [7]. Using this input, we find $m_t(m_t) = 162.83$ GeV and $M_W = 80.36$ GeV (see text for details).

The remaining parametric input is contained in the CKM factors λ_t and λ_c, defined above. We expand these parameters in λ, including the quadratic corrections, and find

$$
\begin{aligned}
\text{Im}\lambda_t &= A^2\bar{\eta}\lambda^5 + \frac{1}{2}A^2\bar{\eta}\lambda^7 + O(\lambda^9)\,, \\
\text{Re}\lambda_t &= A^2\lambda^5(\bar{\rho} - 1) + \frac{1}{2}A^2\lambda^7(2\bar{\eta}^2 + 2\bar{\rho}^2 - 3\bar{\rho} + 1) + O(\lambda^9)\,, \\
\text{Re}\lambda_c &= -\lambda + \frac{1}{2}\lambda^3 + O(\lambda^5)\,.
\end{aligned}
\tag{8}
$$

As the rare kaon decay modes do not enter the standard global CKM fit, we use the values obtained from the global fit as input parameters. All input values are taken from pdgLive [7] and are collected here in Tab. 1. We find the following prediction for the charged mode in the SM,

$$
\text{BR}(K^+ \to \pi^+\nu\bar{\nu}) = 7.73(16)(25)(54) \times 10^{-11}\,.
\tag{9}
$$

The errors in parentheses correspond to the remaining short-distance, long-distance, and parametric uncertainties, with all contributions added in quadrature. In more detail, the leading contributions to the uncertainty are

$$
\begin{aligned}
10^{11} \times \text{BR}(K^+ \to \pi^+\nu\bar{\nu}) = 7.73 &\pm 0.12_{X_t^{\text{QCD}}} \pm 0.01_{X_t^{\text{EW}}} \pm 0.11_{P_c} \pm 0.24_{\delta P_{cu}} \pm 0.04_{\kappa_+} \\
&\pm 0.13_\lambda \pm 0.46_A \pm 0.18_{\bar{\rho}} \pm 0.03_{\bar{\eta}} \pm 0.05_{m_t} \pm 0.15_{m_c} \pm 0.05_{\alpha_s}\,.
\end{aligned}
\tag{10}
$$

The branching ratio of the neutral mode is computed from

$$
\text{Br}\left(K_L \to \pi^0\nu\bar{\nu}\right) = \kappa_L r_{\epsilon_K} \left(\frac{\text{Im}\lambda_t}{\lambda^5}X_t\right)^2\,,
\tag{11}
$$

it depends to a good approximation only on the top-quark function X_t discussed above. The hadronic matrix element is contained in the parameter

$$
\kappa_L = \left(\frac{0.231}{\sin^2\theta_w}\right)^2 \left(\frac{\alpha(M_Z)}{127.9^{-1}}\right)^2 \left(\frac{\lambda}{0.225}\right)^8 \times 2.231(13) \times 10^{-10}\,,
\tag{12}
$$

again extracted from $K_{\ell 3}$ decay including higher-order chiral corrections [20].

At the current level of accuracy, also the small contribution of indirect CP violation [21] should be included. It is taken into account in Eq. (11) by the factor

$$r_{\epsilon_K} \equiv 1 - \sqrt{2}|\epsilon_K| \frac{1 + P_c/A^2 X_t - \rho}{\eta}, \tag{13}$$

where A, $\rho = \bar{\rho}/(1 - \lambda^2/2 + \dots)$, and $\eta = \bar{\eta}/(1 - \lambda^2/2 + \dots)$ are Wolfenstein parameters. The loop function X_t and all remaining parametric input has been discussed above in the context of the charged mode. Our SM prediction for the neutral mode then reads

$$\mathrm{BR}(K_L \to \pi^0 \nu \bar{\nu}) = 2.59(6)(2)(28) \times 10^{-11}. \tag{14}$$

Again, the errors in parentheses correspond to the remaining short-distance, long-distance, and parametric uncertainties, with all contributions added in quadrature. In more detail, the leading contributions to the uncertainty are

$$\begin{aligned}
10^{11} \times \mathrm{BR}(K_L \to \pi^0 \nu \bar{\nu}) = 2.59 &\pm 0.06_{X_t^{\mathrm{QCD}}} \pm 0.01_{X_t^{\mathrm{EW}}} \pm 0.02_{\kappa_L} \\
&\pm 0.16_{\bar{\eta}} \pm 0.22_A \pm 0.04_\lambda \pm 0.02_{m_t}.
\end{aligned} \tag{15}$$

Next we discuss the differences between the theory prediction of Ref. [22] and our analysis. The largest discrepancy arises from the choice of numerical values for the CKM parameters. The choice of $|V_{ub}|$, $|V_{cb}|$ and γ in Ref. [22] implies central values for $\bar{\rho}_{[22]} = 0.119$ and $\bar{\eta}_{[22]} = 0.394$ that deviate from the PDG values for the SM CKM-fit by roughly 1- and 3-σ for $\bar{\rho}$ and $\bar{\eta}$, respectively. The difference in the numerical value for X_t has a milder impact on the branching ratios. Here, the most recent, improved measurement of M_t and α_s and the corresponding change of their central values results in a reduction of X_t compared to older determinations. The error due to unknown higher-order QCD corrections is estimated using a different range for varying the matching scale, μ_t, which also implies a slightly lower (0.7%) X_t value in our determination. The fact that only approximate results for the electroweak corrections have been included in Ref. [22] is negligible.

3. Status of ϵ_K

In this section, we give a brief overview of the recent progress in the SM prediction for indirect CP violation in the neutral kaon system. We define the parameter $\epsilon_K \equiv e^{i\phi_\epsilon} \sin \phi_\epsilon \frac{1}{2} \arg(-M_{12}/\Gamma_{12})$, where $\phi_\epsilon = \arctan(2\Delta M_K/\Delta \Gamma_K)$, with ΔM_K and $\Delta \Gamma_K$ the mass and lifetime differences of the weak eigenstates K_L and K_S. M_{12} and Γ_{12} are the Hermitian and anti-Hermitian parts of the Hamiltonian that determines the time evolution of the neutral kaon system. The evaluation of the matrix element $M_{12} = -\langle K^0 | \mathcal{L}_{f=3}^{\Delta S=2} | \bar{K}^0 \rangle / (2\Delta M_K)$ can, at leading order in the operator-product expansion, be factorised into short- and long-distance contributions that can be calculated in perturbation theory and on the lattice, respectively. Note that the ratio M_{12}/Γ_{12}, and hence ϵ_K, does not depend on the phase convention of the CKM matrix. To make this apparent, we factor out $1/(\lambda_u^*)^2$ and $1/(\lambda_u^*)$ from the $|\Delta S = 2|$ and $|\Delta S = 1|$ effective Lagrangians, respectively, and use CKM unitarity to express the effective three-flavor $|\Delta S = 2|$ Lagrangian in terms of the minimal number of independent CKM parameters. The resulting Lagrangian with *manifest CKM unitarity* [8],

$$\mathcal{L}_{f=3}^{\Delta S=2} = -\frac{G_F^2 M_W^2}{4\pi^2} \frac{1}{(\lambda_u^*)^2} Q_{S2} \left\{ f_1 C_1(\mu) + iJ \left[f_2 C_2(\mu) + f_3 C_3(\mu) \right] \right\} + \mathrm{h.c.} + \dots, \tag{16}$$

is written in terms of three real Wilson coefficients $C_i(\mu)$, $i = 1, 2, 3$, four real, independent, rephasing-invariant parameters J, f_1, f_2, and f_3 comprising the relevant CKM matrix elements, and the operator $Q_{S2} = (\bar{s}_L \gamma_\mu d_L) \otimes (\bar{s}_L \gamma^\mu d_L)$. Explicitly, we have $J = \text{Im}(V_{us} V_{cb} V_{ub}^* V_{cs}^*)$ and $f_1 = |\lambda_u|^4 + \ldots$, where the ellipsis denotes real terms that are suppressed by powers of λ. In this form it is evident that C_1, the Wilson coefficient relevant for ΔM_K, does not contribute to ϵ_K. In the PDG phase convention, the choice of $f_2 = 2\text{Re}(\lambda_t \lambda_u^*)$ and $f_3 = |\lambda_u|^2$ results in the effective Lagrangian

$$\mathcal{L}_{f=3}^{\Delta S=2} = -\frac{G_F^2 M_W^2}{4\pi^2} \left[\lambda_u^2 C_{S2}^{uu}(\mu) + \lambda_t^2 C_{S2}^{tt}(\mu) + \lambda_u \lambda_t C_{S2}^{ut}(\mu) \right] Q_{S2} + \text{h.c.} + \ldots, \tag{17}$$

with the Wilson coefficients $C_{S2}^{uu} \equiv C_1$, $C_{S2}^{tt} \equiv C_2$, and $C_{S2}^{ut} \equiv C_3$. Only the coefficients C_{S2}^{ut} and C_{S2}^{tt} are relevant for the prediction of ϵ_K. This choice of f_2 and f_3 results in their respective Wilson coefficients being free of the low-energy contributions of C_1 and hence they can be calculated with high precision in renormalisation-group improved perturbation theory. The higher-order corrections can be conveniently parameterized by the formally scale-independent parameters η_{tt} and η_{ut} that encode the higher-order QCD corrections to the LO Inami–Lim functions $S_{tt}(x_t)$ and $S_{ut}(x_c, x_t)$ (see Refs. [8, 23]). Their values are $\eta_{tt} = 0.55(2)$ and $\eta_{ut} = 0.402(5)$ [8].

The SM prediction for the absolute value of ϵ_K is then obtained via the phenomenological formula [9, 24, 25]

$$|\epsilon_K| = \kappa_\epsilon C_\epsilon \widehat{B}_K |V_{cb}|^2 \lambda^2 \bar{\eta} \times \left(|V_{cb}|^2 (1 - \bar{\rho}) \eta_{tt} S_{tt}(x_t) - \eta_{ut} S_{ut}(x_c, x_t) \right). \tag{18}$$

Here, the kaon bag parameter comprising the hadronic matrix element of the local $\Delta S = 2$ operators is given by $\widehat{B}_K = 0.7625(97)$ [26]. The phenomenological parameter $\kappa_\epsilon = 0.94(2)$ [25] comprises long-distance contributions beyond the lowest order in the operator-product expansions, which are not included in B_K, see also Ref. [27] for the calculation of dimension-eight operator matrix elements. The remaining parametric input is collected in the factor $C_\epsilon = (G_F^2 F_K^2 M_{K^0} M_W^2)/(6\sqrt{2}\pi^2 \Delta M_K)$.

The parameter ϵ_K is one of the main ingredients of the global CKM fit. Hence, we do not use the CKM parameters extracted from the global fit for its SM prediction, and instead directly use the PDG values of λ, V_{cb}, $\sin 2\beta$, as well as lattice input for the ratio of B-meson decay constants and bag factors ξ_s [26], see Ref. [8] for details. We find

$$|\epsilon_K| = \left(2.161 \pm 0.153_{\text{param.}} \pm 0.076_{\text{non-pert.}} \pm 0.065_{\text{pert.}} \right) \times 10^{-3}. \tag{19}$$

4. Conclusions

We have presented updated SM predictions of the branching ratios for the rare kaon decay modes, finding $\text{BR}(K^+ \to \pi^+ \nu \bar{\nu}) = 7.73(61) \times 10^{-11}$ and $\text{BR}(K_L \to \pi^0 \nu \bar{\nu}) = 2.59(29) \times 10^{-11}$ (all uncertainties have been added in quadrature), and briefly discussed the current theory status of the perturbative contribution to ϵ_K.

The perturbative uncertainties in the SM predictions can be further reduced by calculating the three-loop QCD corrections for the top-quark contributions to both the rare decays and ϵ_K, as well as electroweak corrections to the $|\Delta S = 2|$ effective Lagrangian. These projects are work in progress

by the authors. The improved SM theory prediction together with the current experimental progress will increase the sensitivity to physics beyond the SM. Given this sensitivity, it is interesting to note that contributions to the rare kaon decays and ϵ_K in a wide class of renormalizable theories beyond the SM have been presented in a general form in Refs. [28, 29].

Acknowledgements

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