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# Electromagnetic knots from de Sitter space

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We find all analytic SU(2) Yang–Mills solutions on de Sitter space by reducing the field equations to Newton's equation for a particle in a particular 3d potential and solving the latter in a special case. In contrast, Maxwell's equations on de Sitter space can be solved in generality, by separating them in hysperspherical coordinates. Employing a well-known conformal map between (half of) de Sitter space and (the future half of) Minkowski space, the Maxwell solutions are mapped to a complete basis of rational electromagnetic knot configurations. We discuss some of their properties and illustrate the construction method with two nontrivial examples given by rational functions of increasing complexity. The material is partly based on [1, 2].

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## 1. Description of de Sitter space

Four-dimensional de Sitter space is a one-sheeted hyperboloid (of radius $\ell$) in $\mathbb{R}^{1,4} \ni \{Z_0, Z_1, \ldots, Z_4\}$ given by

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \;=\; \ell^2 \;. \tag{1}$$

Constant $Z_0$ slices are 3-spheres of varying radius, yielding a parametrization of $\mathrm{dS}_4 \ni \{\tau, \omega_A\}$ as

$$\begin{aligned} &Z_0 \;=\; -\ell\,\cot\tau \qquad &&\text{and} \qquad Z_A \;=\; \frac{\ell}{\sin\tau}\,\omega_A \quad \text{for} \quad A = 1,\ldots,4 \\ \text{with} \qquad &\tau \in \mathcal{I} := (0,\pi) \qquad &&\text{and} \qquad \omega_A\omega_A = 1\;. \end{aligned} \tag{2}$$

The details of the embedding $\omega_A : (\chi,\theta,\phi) \ni S^3 \hookrightarrow \mathbb{R}^4$ are irrelevant. The Minkowski metric

$$\mathrm{d}s^2 \;=\; -\mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2 \tag{3}$$

induces on $\mathrm{dS}_4$ the metric

$$\mathrm{d}s^2 \;=\; \frac{\ell^2}{\sin^2\tau}\big(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\big) \qquad \text{with} \qquad \mathrm{d}\Omega_3^2 \quad \text{for} \quad S^3\;, \tag{4}$$

showing that $\mathrm{dS}_4$ is conformally equivalent to a finite cylinder $\mathcal{I} \times S^3$.

## 2. Reduction of Yang–Mills to matrix equations

We wish to find solutions to the Yang–Mills (and Maxwell) equations on de Sitter space. Due to their conformal invariance in four spacetime dimensions, we may also study the problem on the finite Minkowskian cylinder $\mathcal{I} \times S^3$.

The gauge potential taking values in a Lie algebra $\mathfrak{g}$ can always be chosen as

$$\mathcal{A} \;=\; X_a(\tau,\omega)\, e^a \qquad \text{on} \quad \mathcal{I} \times S^3 \tag{5}$$

where $X_a \in \mathfrak{g}$, and $\{e^a, a = 1,2,3\}$ is a basis of left-invariant one-forms on $S^3 \simeq \mathrm{SU}(2)$, with

$$\mathrm{d}e^a + \varepsilon^a_{\ bc}\, e^b \wedge e^c \;=\; 0 \qquad \text{and} \qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2\;. \tag{6}$$

There is no $\mathrm{d}\tau$ component because we picked the temporal gauge $\mathcal{A}_\tau = 0$. In terms of the $S^3$ coordinates $(a,i,j,k = 1,2,3$ and $B,C = 1,2,3,4)$ these one-forms can be constructed as

$$e^a \;=\; -\eta^a_{BC}\,\omega_B\,\mathrm{d}\omega_C \qquad \text{where} \quad \eta^i_{jk} = \varepsilon^i_{\ jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j\;. \tag{7}$$

Dual to the $e^a$ are the left-invariant vector fields

$$R_a \;=\; -\eta^a_{BC}\,\omega_B\,\frac{\partial}{\partial\omega_C} \qquad \Rightarrow \qquad [R_a, R_b] \;=\; 2\,\varepsilon_{abc}\,R_c \tag{8}$$

generating the right multiplication on SU(2), so that an arbitrary function $\Phi$ on $S^3$ obeys

$$\mathrm{d}\Phi(\omega) \;=\; e^a\,R_a\Phi(\omega)\;. \tag{9}$$

The full SO(4) isometry group of $S^3$ is generated by left-invariant $R_a$ and right-invariant $L_a$.

In this language, the gauge field two-form becomes ($\dot{X}_a \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} X_a$)

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{\tau a}\, e^\tau \wedge e^a + \tfrac{1}{2}\mathcal{F}_{bc}\, e^b \wedge e^c \\ &= \dot{X}_a\, e^\tau \wedge e^a + \tfrac{1}{2}\big(R_{[b}X_{c]} - 2\varepsilon^a_{bc} X_a + [X_b, X_c]\big)\, e^b \wedge e^c\,, \end{aligned} \tag{10}$$

where we define $R_{[b}X_{c]} = R_b X_c - R_c X_b$, and the Yang–Mills Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{8}\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;=\; -\tfrac{1}{4}\,\mathrm{tr}\mathcal{F}_{\tau a}\mathcal{F}_{\tau a} \;+\; \tfrac{1}{8}\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}_{ab} \\ &= -\tfrac{1}{2}\mathrm{tr}\big\{\tfrac{1}{2}\dot{X}_a\dot{X}_a - 2X_aX_a + \varepsilon_{abc}X_a D_{[b}X_{c]} - \tfrac{1}{4}(D_{[a}X_{b]})(D_{[a}X_{b]})\big\} \end{aligned} \tag{11}$$

with the short-hand $D_a := R_a + X_a$. The Yang–Mills equations using (8) then take the form

$$\begin{aligned} \ddot{X}_a &= -4\,X_a + 2\,\varepsilon_{abc}R_{[b}X_{c]} + R_bR_{[b}X_{a]} + 3\,\varepsilon_{abc}\,[X_b, X_c] \\ &\quad + 2[X_b, R_bX_a] - [X_b, R_aX_b] - [X_a, R_bX_b] - \big[X_b, [X_a, X_b]\big] \\ &= -4\,X_a + 2\,\varepsilon_{abc}R_bX_c + R_bR_bX_a - R_aR_bX_b + 3\,\varepsilon_{abc}\,[X_b, X_c] \\ &\quad + 2[X_b, R_bX_a] - [X_b, R_aX_b] - [X_a, R_bX_b] - \big[X_b, [X_a, X_b]\big] \end{aligned} \tag{12}$$

with the Gauss law

$$R_a\dot{X}_a + [X_a, \dot{X}_a] \;=\; 0\,. \tag{13}$$

## 3. Yang–Mills configurations on de Sitter space

The simplest Yang–Mills solutions are most symmetric. To obtain them, let us impose SO(4) symmetry by setting $X_a(\tau,\omega) = X_a(\tau)$. The Yang–Mills equations then become ordinary matrix differential equations [3–5],

$$\ddot{X}_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \big[X_b, [X_a, X_b]\big] \qquad \text{and} \qquad [X_a, \dot{X}_a] \;=\; 0\,. \tag{14}$$

These three coupled ordinary differential equations for the three matrix functions $X_a(\tau)$ are still too complicated. However, for the gauge group SU(2), these equations admit some analytic solutions. So let us choose a spin-$j$ representation of $\mathfrak{g} = su(2)$ and introduce the three SU(2) generators $T_a$,

$$[T_b, T_c] \;=\; 2\,\varepsilon^a_{bc}T_a \qquad \text{and} \qquad \mathrm{tr}(T_aT_b) \;=\; -4C(j)\,\delta_{ab} \qquad \text{for} \quad C(j) = \tfrac{1}{3}\,j(j{+}1)(2j{+}1)\,. \tag{15}$$

A simple ansatz for the matrices $X_a$ is

$$X_1 = \Psi_1T_1\,, \quad X_2 = \Psi_2T_2\,, \quad X_3 = \Psi_3T_3 \qquad \text{with} \quad \Psi_a = \Psi_a(\tau) \in \mathbb{R}\,. \tag{16}$$

The resulting simplification of Yang–Mills Lagrangian density,

$$\mathcal{L} \;=\; 4\,C(j)\,\big\{\tfrac{1}{4}\dot{\Psi}_a\dot{\Psi}_a - (\Psi_1 - \Psi_2\Psi_3)^2 - (\Psi_2 - \Psi_3\Psi_1)^2 - (\Psi_3 - \Psi_1\Psi_2)^2\big\}\,, \tag{17}$$

suggests an interpretation of $\{\Psi_a\}$ as the coordinates of a Newtonian particle in $\mathbb{R}^3$ moving in a potential

$$\tfrac{1}{2}\mathcal{V}(\Psi) \;=\; (\Psi_1 - \Psi_2\Psi_3)^2 \;+\; (\Psi_2 - \Psi_3\Psi_1)^2 \;+\; (\Psi_3 - \Psi_1\Psi_2)^2\,. \tag{18}$$

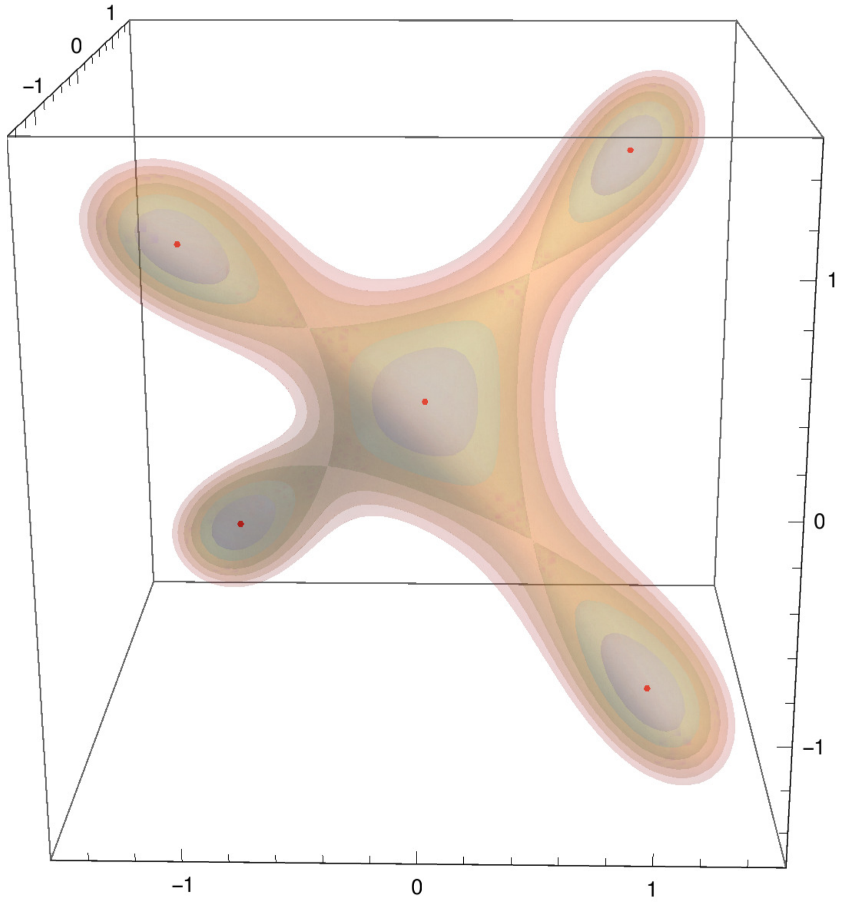


**Figure 1:** Contours of the Newtonian potential in (18).



The only analytic nonabelian solutions come from

$$\Psi_1 = \Psi_2 = \Psi_3 =: \Psi \qquad \text{with} \qquad \ddot{\Psi} = 16\,\Psi\,(\Psi-1)(2\Psi-1)\,, \tag{19}$$

leading to elliptic functions $\Psi(\tau)$, except for the special cases $\Psi(\tau) = 0$ or 1 (the vacuum), $\Psi(\tau) = \frac{1}{2}$ (the sphaleron), and the bounce solution in the double-well potential. The corresponding gauge potential takes the simple form

$$\mathcal{A} \;=\; \Psi(\tau)\,g^{-1}\mathrm{d}g \qquad \text{for} \quad g : S^3 \xrightarrow{1:1} \mathrm{SU}(2)\,, \tag{20}$$

and the SU(2) color electric and magnetic fields are

$$\mathcal{E}_a \;=\; \mathcal{F}_{\tau a} \;=\; \dot{\Psi}\,T_a \qquad \text{and} \qquad \mathcal{B}_a \;=\; \tfrac{1}{2}\varepsilon_{abc}\mathcal{F}_{bc} \;=\; 2\,\Psi\,(\Psi-1)\,T_a\,. \tag{21}$$

Their total de Sitter energy and action is finite and proportional to double-well energy. These analytic Yang–Mills configurations are related to Minkowski-space solutions found in the seventies [6–8] (for a review from this period, see [9]). Their stability, however, has been analyzed only recently [10].

## 4. All Maxwell solutions on de Sitter space

The other analytic solutions to (12) and (13) are abelian, i.e. excite only a single direction in isospin space. In this case we can drop the matrix valuedness and treat the $X_a$ as real functions. Dropping all commutator terms, the Yang–Mills equations (12) turn into the linear Mawell equations,

$$\ddot{X}_a \;=\; (R^2-4)\,X_a + 2\,\varepsilon_{abc}R_b X_c \tag{22}$$

where $R^2 \equiv R_b R_b$ is the laplacian on $S^3$, and we refined the temporal gauge to the Coulomb gauge

$$\mathcal{A}_\tau = 0 \qquad \text{and} \qquad R_a X_a = 0\,, \tag{23}$$

which takes care of the Gauss law.

The coupled wave equations (22) may be completely solved by separation of variables. Seeking factorized complex basis solutions [1]

$$X_a(\tau,\omega) \;=\; Z_a(\omega)\,\mathrm{e}^{\mathrm{i}\Omega\tau}\,, \tag{24}$$

one learns that the frequency $\Omega$ only depends on the SO(4) spin $2j \in \mathbb{N}_0$,

$$-\,R^2\,Z_a^j(\omega) = 2j(2j{+}2)\,Z_a^j(\omega) \qquad \Rightarrow \qquad \big((\Omega^j)^2-4(j{+}1)^2\big)\big((\Omega^j)^2-4j^2\big) = 0\,, \tag{25}$$

where the second factor appears only for $j{\ge}1$. The basis solutions $Z_a^j$ to the linear system come in two types and carry two further labels $m$ and $n$ [1]:

- type I : $\quad j{\ge}0\,,\quad m=-j,\ldots,+j\,,\quad n=-j{-}1,\ldots,j{+}1\,,\quad \Omega^j=\pm2(j{+}1)$

$$\begin{aligned} Z_+^{j;m,n} &= \sqrt{(j{-}n)(j{-}n{+}1)/2}\;Y_{j;m,n+1} \\ Z_3^{j;m,n} &= \sqrt{(j{-}n{+}1)(j{+}n{+}1)}\;Y_{j;m,n} \\ Z_-^{j;m,n} &= -\sqrt{(j{+}n)(j{+}n{+}1)/2}\;Y_{j;m,n-1} \end{aligned} \tag{26}$$

- type II : $\quad j{\ge}1\,,\quad m=-j,\ldots,+j\,,\quad n=-j{+}1,\ldots,j{-}1\,,\quad \Omega^j=\pm2j$

$$\begin{aligned} Z_+^{j;m,n} &= -\sqrt{(j{+}n)(j{+}n{+}1)/2}\;Y_{j;m,n+1} \\ Z_3^{j;m,n} &= \sqrt{(j{+}n)(j{-}n)}\;Y_{j;m,n} \\ Z_-^{j;m,n} &= \sqrt{(j{-}n)(j{-}n{+}1)/2}\;Y_{j;m,n-1} \end{aligned} \tag{27}$$

where $Z_\pm = (Z_1 \pm \mathrm{i}Z_2)/\sqrt{2}$, and the hyperspherical harmonics

$$Y_{j;m,n}(\omega) \qquad \text{with} \qquad m,n=-j,-j{+}1,\ldots,+j \qquad \text{and} \qquad 2j=0,1,2,\ldots \tag{28}$$

are characterized by [2]

$$-\,\tfrac14 R^2\,Y_{j;m,n} = j(j{+}1)\,Y_{j;m,n} \qquad \text{and} \qquad \tfrac{\mathrm{i}}{2}\,R_3\,Y_{j;m,n} = n\,Y_{j;m,n}\,. \tag{29}$$

[1] $Z_a(\omega)$ is not to be confused with the ambient-space coordinates $Z_A$.
[2] The label $m$ is the eigenvalue of $\frac{\mathrm{i}}{2}\,L_3$.



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Hence, the general real Maxwell solution $\mathcal{A} = X_a(\tau,\omega)\, e^a$ is a linear combination with

$$X_a(\tau,\omega) \;=\; \sum_{jmn}\Big\{c^{\mathrm{I}}_{j;m,n}\, Z^{j;m,n}_{a\,\mathrm{I}}(\omega)\,\mathrm{e}^{2(j+1)\mathrm{i}\tau} \;+\; c^{\mathrm{II}}_{j;m,n}\, Z^{j;m,n}_{a\,\mathrm{II}}(\omega)\,\mathrm{e}^{2j\,\mathrm{i}\tau} \;+\; \mathrm{c.c.}\Big\}\,. \tag{30}$$

Each complex solution yields two real ones (real part and imaginary part). We count $2(2j{+}1)(2j{+}3)$ real type-I solutions and $2(2j{+}1)(2j{-}1)$ real type-II solutions ($j{\ge}1$), which add up to $4(2j{+}1)^2$ solutions for $j{>}0$ and 6 solutions for $j{=}0$, as it should. Constant solutions ($\Omega = 0$) are not allowed; the simplest ones are $j{=}0$ type I or $j{=}1$ type II. The most general $j{=}0$ configuration is

$$X_a^{(j=0)} \;=\; \Big\{c_{0;0,-1}\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix} + c_{0;0,0}\begin{pmatrix}0\\0\\1\end{pmatrix} - c_{0;0,+1}\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix}\Big\}\,\mathrm{e}^{2\mathrm{i}\tau} \;+\; \mathrm{c.c.}\,. \tag{31}$$

The parity inversion, which interchanges left and right invariance, relates spin $j$ type I solutions with spin $j{+}1$ type II solutions, swopping labels $m$ and $n$. Finally, electromagnetic duality is realized by shifting $|\Omega^j|\tau$ by $\pm\frac{\pi}{2}$, which produces from a solution $\mathcal{A}$ a dual solution $\mathcal{A}_D$. We shall now see that this basis of Maxwell solutions relates to so-called electromagnetic knots in Minkowski space.

## 5. Conformal mapping to Minkowski space

The $Z_0{+}Z_4{<}0$ half of $\mathrm{dS}_4$ is also conformally related to future Minkowski space $\mathbb{R}^{1,3}_+ \ni \{t,x,y,z\}$,

$$\begin{gathered} Z_0 = \frac{t^2{-}r^2{-}\ell^2}{2\,t}\,,\quad Z_1 = \ell\,\frac{x}{t}\,,\quad Z_2 = \ell\,\frac{y}{t}\,,\quad Z_3 = \ell\,\frac{z}{t}\,,\quad Z_4 = \frac{r^2{-}t^2{-}\ell^2}{2\,t} \\ \text{with}\qquad x,y,z\in\mathbb{R}\qquad\text{and}\qquad r^2 = x^2+y^2+z^2\qquad\text{but}\qquad t\in\mathbb{R}_+\,, \end{gathered} \tag{32}$$

since $t\in[0,\infty]$ corresponds to $Z_0\in[-\infty,\infty]$ but $Z_0{+}Z_4 < 0$. In these Minkowski coordinates,

$$\mathrm{d}s^2 \;=\; \frac{\ell^2}{t^2}\left(-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)\,. \tag{33}$$

One may cover the entire $\mathbb{R}^{1,3}$ by gluing a second $\mathrm{dS}_4$ copy and using the patch $Z_0{+}Z_4 > 0$.

We shall employ the direct relation between the cylinder and Minkowski coordinates,

$$\cot\tau = \frac{r^2{-}t^2{+}\ell^2}{2\,\ell\,t}\,,\quad \omega_1 = \gamma\,\frac{x}{\ell}\,,\quad \omega_2 = \gamma\,\frac{y}{\ell}\,,\quad \omega_3 = \gamma\,\frac{z}{\ell}\,,\quad \omega_4 = \gamma\frac{r^2{-}t^2{-}\ell^2}{2\,\ell^2}\,, \tag{34}$$

with the convenient abbreviation

$$\gamma \;=\; \frac{2\,\ell^2}{\sqrt{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}}\,. \tag{35}$$

Since $t = -\infty,0,\infty$ corresponds to $\tau = -\pi,0,\pi$, the cylinder gets doubled to $2\mathcal{I}\times S^3$, and full Minkowski space is covered by the cylinder patch $\omega_4 \le \cos\tau$. The cylinder time $\tau$ is a regular smooth function of $(t,x,y,z)$, but more useful will be

$$\exp(2\mathrm{i}\,\tau) \;=\; \frac{\big[(\ell+\mathrm{i}t)^2+r^2\big]^2}{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}\,. \tag{36}$$

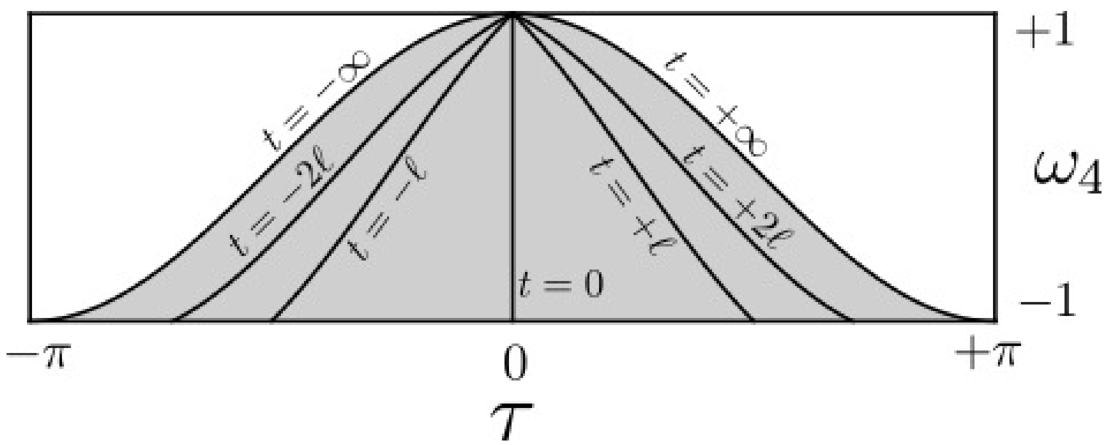


**Figure 2:** An illustration of the map between a cylinder $2\mathcal{I}\times S^3$ and Minkowski space $\mathbb{R}^{1,3}$. The Minkowski coordinates cover the shaded area. Its boundary is given by the curve $\omega_4 = \cos\tau$. Each point is a two-sphere spanned by $\omega_{1,2,3}$, which is mapped to a sphere of constant $r$ and $t$.

A slightly lengthy computation yields the Minkowski-coordinate expressions for the one-forms [1],

$$\begin{aligned} e^0 &= e^0_\mu\,\mathrm{d}x^\mu = \frac{\gamma^2}{\ell^3}\Big(\tfrac12(t^2+r^2+\ell^2)\,\mathrm{d}t - t\,x^k\mathrm{d}x^k\Big)\,, \\ e^a &= e^a_\mu\,\mathrm{d}x^\mu = \frac{\gamma^2}{\ell^3}\Big(t\,x^a\mathrm{d}t - \big(\tfrac12(t^2-r^2+\ell^2)\,\delta^a_k + x^a x^k + \ell\,\varepsilon^a_{\ jk}x^j\big)\,\mathrm{d}x^k\Big)\,, \end{aligned} \tag{37}$$

with the notation

$$(x^i) = (x,y,z) \qquad \text{and} \qquad (x^\mu) = (x^0,x^i) = (t,x,y,z)\,. \tag{38}$$

Due to the conformal invariance of the Maxwell equations, our oscillatory solutions on the cylinder $2\mathcal{I}\times S^3$ may be transferred to a basis of Maxwell solutions on Minkowski space (with certain fall-off properties). To accomplish this task, we only have to effect the coordinate change [3]

$$\text{from} \qquad (\tau,\omega) \sim (\tau,\chi,\theta,\phi) \qquad \text{to} \qquad x \equiv (x^\mu) = (t,x,y,z) \sim (t,r,\theta,\phi)\,, \tag{39}$$

so that

$$\mathcal{A} = X_a(\tau(x),\omega(x))\,e^a(x) = A_\mu(x)\,\mathrm{d}x^\mu \qquad \text{yielding} \quad A_\mu(x) \quad \text{with} \quad A_t \neq 0\,, \tag{40}$$

$$\mathrm{d}\mathcal{A} = \dot{X}_a\,e^0\wedge e^a - \varepsilon^a_{\ bc}X_a\,e^b\wedge e^c = \tfrac12 F_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \qquad \text{yielding} \quad F_{\mu\nu}(x)\,. \tag{41}$$

From this, we obtain electric and magnetic fields $E_i = F_{i0}$ and $B_i = \frac12\varepsilon_{ijk}F_{jk}$. For the computation it is helpful to recognize that $\exp(2\mathrm{i}\tau)$ is a rational function of $t$ and $r$. It follows that all physical quantities (and the gauge potential) are rational functions of the Minkowski coordinates!

[3] The $S^2$ angular coordinates $(\theta,\phi)$ on both sides can be identified. The map $(\tau,\chi)\mapsto(t,r)$ realizes the Penrose diagram of Minkowski space [2].

## 6. All knot solutions on Minkowski space

As we shall see below, the simplest ($j$=0) solutions neatly reproduces the celebrated Hopf-Rañada electromagnetic knot [11, 12]. From our construction, some general features of all knot solutions can be inferred.

Firstly, at spatial infinity (for $t$ fixed) all field strengths decay like $r^{-4}$, but they fall off only as $(t{\pm}r)^{-1}$ along the light-cone. Hence, the asymptotic energy flow is concentrated on past and future null infinity and peaks on the light-cone of the spacetime origin. Secondly, the "knot basis" forms a complete set of finite-action configurations. Of course, it does not contain plane waves. Thirdly, the obvious conserved (in Minkowski time) quantities are helicity and energy,

$$h \;=\; \tfrac12 \int_{\mathbb{R}^3} \bigl(A\wedge F + A_D\wedge F_D\bigr) \qquad\text{and}\qquad E \;=\; \tfrac12 \int_{\mathbb{R}^3} \mathrm{d}^3x\;\bigl(\vec{E}^2+\vec{B}^2\bigr)\,, \tag{42}$$

where the spatial integration is done at fixed $t$. Their common scale is determined by the amplitude of the solution, but their ratio is fixed for the basis configurations. Both quantities are best computed in the "sphere frame" at $t=\tau=0$,

$$F \;=\; \mathcal{E}_a\, e^a{\wedge}e^0 + \tfrac12 \mathcal{B}_a\, \varepsilon^a_{\ bc}\, e^b{\wedge}e^c\ . \tag{43}$$

Let us focus on type I solutions of a fixed spin $j$ and suppress these indices. For those one finds

$$\mathcal{E}_a \;=\; -\mathrm{i}\Omega \sum_{mn} c_{m,n}\, Z_a^{m,n}\, \mathrm{e}^{\mathrm{i}\Omega\tau} \;+\; \text{c.c.} \qquad\text{and}\qquad \mathcal{B}_a \;=\; -\Omega \sum_{mn} c_{m,n}\, Z_a^{m,n}\, \mathrm{e}^{\mathrm{i}\Omega\tau} \;+\; \text{c.c.}\,, \tag{44}$$

which yields

$$\tfrac12\bigl(\mathcal{E}_a\mathcal{E}_a + \mathcal{B}_a\mathcal{B}_a\bigr) \;=\; 2\Omega^2 \bigl|\textstyle\sum_{m,n} c_{m,n}\, Z_a^{m,n}(\omega)\bigr|^2\ . \tag{45}$$

The Minkowski energy at $t$=0 is easily pulled back to the cylinder frame and evaluated by exploiting the orthogonality properties of the hyperspherical harmonics [2],

$$E \;=\; \tfrac{1}{2\ell} \int_{S^3} \mathrm{d}^3\Omega_3\; (1{-}\omega_4)\, \bigl(\mathcal{E}_a\mathcal{E}_a + \mathcal{B}_a\mathcal{B}_a\bigr) \;=\; \tfrac{1}{\ell}\,(2j{+}1)\,\Omega^3 \sum_{m,n} |c_{m,n}|^2\ . \tag{46}$$

A similar computation produces an expression for the helicity. It turns out that single-spin solutions (of both types) have a universal energy-to-helicity ratio $E/h = |\Omega|/\ell$.

Fourthly, so-called null fields are easily characterized,

$$\vec{E}^2{-}\vec{B}^2 = 0 = \vec{E}\cdot\vec{B} \qquad\Leftrightarrow\qquad (\vec{E}\pm\mathrm{i}\vec{B})^2 = 0 \qquad\Leftrightarrow\qquad \sum_a (\mathcal{E}_a \pm \mathrm{i}\mathcal{B}_a)^2 = 0\ . \tag{47}$$

For fixed spin $j$ and type I we infer from above that

$$\mathcal{E}_a + \mathrm{i}\mathcal{B}_a \;=\; -2\mathrm{i}\,\Omega \sum_{mn} c_{m,n}\, Z_a^{m,n}(\omega)\, \mathrm{e}^{\mathrm{i}\Omega\tau} \qquad (\text{no c.c.!})\,, \tag{48}$$

hence in such a sector we have [2]

$$F_{\mu\nu}\quad \text{null} \qquad\Leftrightarrow\qquad \sum_a \Bigl(\sum_{mn} c_{m,n}\, Z_a^{m,n}(\omega)\Bigr)^2 = 0\ . \tag{49}$$

Given the known form of the functions $Z_a^{m,n}(\omega)$ we can expand this expression in hyperspherical harmonics and arrive at $\frac{1}{6}(4j{+}1)(4j{+}2)(4j{+}3)$ homogeneous quadratic equations for $(2j{+}1)(2j{+}3)$ complex parameters $c_{m,n}$. This system is vastly overdetermined, but only $4j^2{+}6j{+}1$ equations are independent, and thus we are still left with $2j{+}2$ free complex parameters for the solution manifold, which is explicitly parametrized as follows [2],[4]

$$c_{m,n}(w,\vec{z}) \;=\; \sqrt{\tbinom{2j+2}{j+1-n}}\; w^{\frac{j+1-n}{2j+2}}\; \mathrm{e}^{2\pi\mathrm{i}k_m\frac{j+1-n}{2j+2}}\; z_m \qquad \text{with} \quad w\in\mathbb{C}^* \quad \text{and} \quad \vec{z}\equiv\{z_m\}\in\mathbb{C}^{2j+1} \tag{50}$$

and a choice of $2j{+}1$ integers $k_m\in\{0,1,\ldots,2j{+}1\}$ (one of which can be absorbed into $z_m$). Given that the overall scale of the solutions is irrelevant, the null fields form a complete-intersection projective variety of complex dimension $2j{+}1$ inside $\mathbb{C}P^{(2j+1)(2j+3)-1}$. The simplest example occurs for spin $j{=}0$, where the single null-field relation $c_{0,0}^2 = 2\,c_{0,-1}c_{0,1}$ defines a generic rank-3 quadric in $\mathbb{C}P^2$ or, alternatively, a cone over $\mathbb{C}P^1$ lying in $\mathbb{C}^3$.

## 7. Examples

We close with two concrete examples. First, the $j{=}0$ case represents SO(4)-symmetric Maxwell solutions in de Sitter space, meaning $X_a(\tau,\omega)=X_a(\tau)$ thus $R_aX_b=0$ and trivializing (22) to

$$\ddot{X}_a \;=\; -4\,X_a \qquad \Rightarrow \qquad X_a(\tau) \;=\; \xi_a\,\cos\bigl(2(\tau{-}\tau_a)\bigr)\,, \tag{51}$$

which describes an ellipse in $\mathbb{R}^3$.[5] We may always choose a frame where $\xi_3=0$ and $\tau_2=0$. The overall amplitude is irrelevant as all equations are linear, and solutions can be superposed at will. Specializing to

$$\xi_1=\xi_2=-\tfrac18 \quad \text{and} \quad \tau_1=\tfrac{\pi}{4} \qquad \Leftrightarrow \qquad c_{0;0,-1}=c_{0;0,0}=0 \quad \text{and} \quad c_{0;0,1}\in\mathrm{i}\mathbb{R}\,, \tag{52}$$

one has a null configuration with components

$$X_1(\tau)=-\tfrac18\sin 2\tau\,, \quad X_2(\tau)=-\tfrac18\cos 2\tau\,, \quad X_3(\tau)=0\,. \tag{53}$$

The result of a short computation yields

$$\vec{E}+\mathrm{i}\vec{B} \;=\; \frac{\ell^2}{\bigl((t-\mathrm{i}\ell)^2-r^2\bigr)^3}\begin{pmatrix}(x-\mathrm{i}y)^2-(t-\mathrm{i}\ell-z)^2\\ \mathrm{i}(x-\mathrm{i}y)^2+\mathrm{i}(t-\mathrm{i}\ell-z)^2\\ -2\,(x-\mathrm{i}y)\,(t-\mathrm{i}\ell-z)\end{pmatrix}. \tag{54}$$

This is the announced Hopf–Rañada electromagnetic knot [11, 12]. Our approach also yields its gauge potential.

Second, let us take the real part of the $(j;m,n)=(1;0,0)$ type I basis solution. Combining $\mathrm{e}^{4\mathrm{i}\tau}{+}\mathrm{e}^{-4\mathrm{i}\tau}=2\cos 4\tau$ and expressing $Y_{1;0,\star}$ from (26) in terms of $\omega_A$, we get

$$X_\pm \;=\; -\tfrac{\sqrt{3}}{\pi}\,(\omega_1{\pm}\mathrm{i}\omega_2)(\omega_3{\pm}\mathrm{i}\omega_4)\,\cos 4\tau \qquad \text{and} \qquad X_3 \;=\; -\tfrac{\sqrt{6}}{\pi}\,(\omega_1^2{+}\omega_2^2{-}\omega_3^2{-}\omega_4^2)\,\cos 4\tau\,. \tag{55}$$

[4] These are the generic solutions. There also exist special solutions with $c_{m,n}=0$ for $|n|\neq j{+}1$.

[5] Every solution $X_a(\tau)$ spontaneously breaks the SO(4) invariance by the choice of integration constants $(\xi_a,\tau_a)$.



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This solution takes the explicit form (putting $\ell$=1)

$$
\begin{aligned}
(E+\mathrm{i}B)_x \;&=\; \frac{-2\mathrm{i}}{\big((t-\mathrm{i})^2-x^2-y^2-z^2\big)^5}\times \\
&\times\Big\{2y+3\mathrm{i}ty-xz+2t^2y+2\mathrm{i}txz-8x^2y-8y^3+4yz^2 \\
&\quad+\;4\mathrm{i}t^3y-6t^2xz-8\mathrm{i}tx^2y-8\mathrm{i}ty^3+4\mathrm{i}tyz^2+10x^3z+10xy^2z-2xz^3 \\
&\quad+\;2(\mathrm{i}txz+x^2y+y^3+yz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}ty-xz)(-t^2+x^2+y^2+z^2)^2\Big\}\,, \\
(E+\mathrm{i}B)_y \;&=\; \frac{2\mathrm{i}}{\big((t-\mathrm{i})^2-x^2-y^2-z^2\big)^5}\times \\
&\times\Big\{2x+3\mathrm{i}tx+yz+2t^2x-2\mathrm{i}tyz-8x^3-8xy^2+4xz^2 \\
&\quad+\;4\mathrm{i}t^3x+6t^2yz-8\mathrm{i}tx^3-8\mathrm{i}txy^2+4\mathrm{i}txz^2-10x^2yz-10y^3z+2yz^3 \\
&\quad+\;2(-\mathrm{i}tyz+x^3+xy^2+xz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}tx+yz)(-t^2+x^2+y^2+z^2)^2\Big\}\,, \\
(E+\mathrm{i}B)_z \;&=\; \frac{\mathrm{i}}{\big((t-\mathrm{i})^2-x^2-y^2-z^2\big)^5}\times \\
&\times\Big\{1+2\mathrm{i}t+t^2-11x^2-11y^2+3z^2+4\mathrm{i}t^3-16\mathrm{i}tx^2-16\mathrm{i}ty^2+4\mathrm{i}tz^2 \\
&\quad-\;t^4-2t^2x^2-2t^2y^2-2t^2z^2+11x^4+22x^2y^2+10x^2z^2+11y^4-10y^2z^2+3z^4 \\
&\quad+\;2\mathrm{i}t(t^2-3x^2-3y^2-z^2)(t^2-x^2-y^2-z^2)-(t^2+x^2+y^2-z^2)(-t^2+x^2+y^2+z^2)^2\Big\}\,.
\end{aligned}
\tag{56}
$$

Figures 3 and 4 below show $t$=0 energy density level surfaces and a particular magnetic field line.

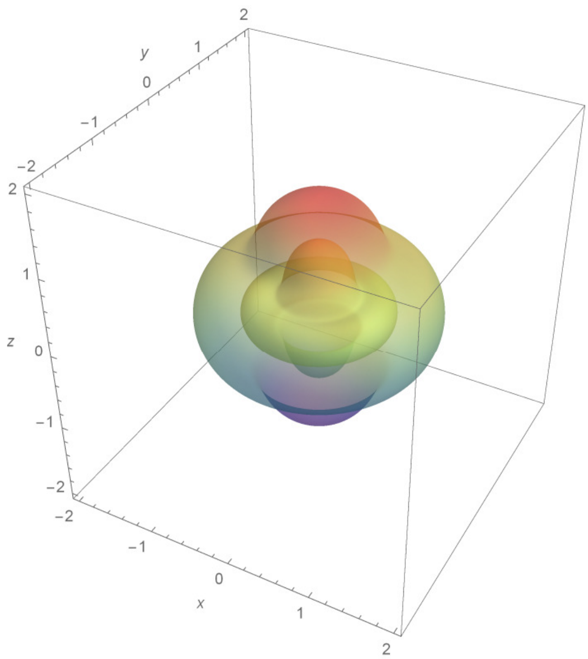


**Figure 3:** Energy density level surfaces at $t$=0 for the $(1;0,0)$ solution above.

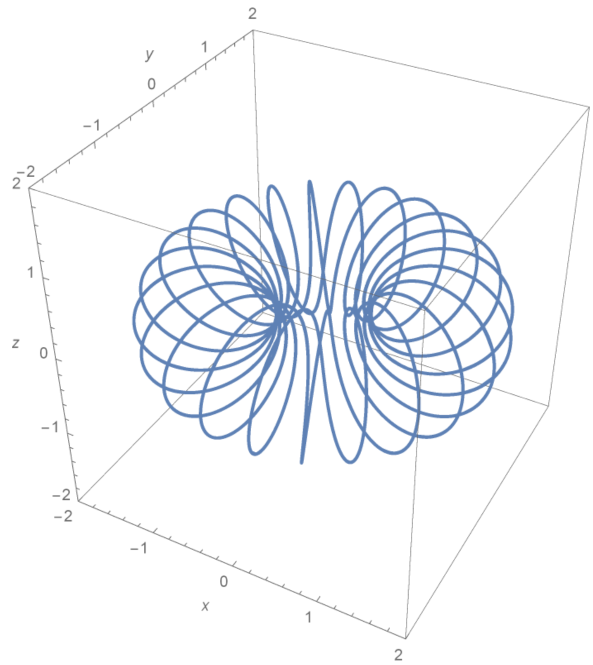


**Figure 4:** A particular magnetic field line for the $(1; 0, 0)$ solution above.

## 8. Summary and discussion

- Rational electromagnetic fields with nontrivial topology have been investigated since 1989
- We introduced a new construction method based on two insights:
  - the simplicity of solving Maxwell's equations on a temporal cylinder over a three-sphere
  - the conformal equivalence of a cylinder patch $\{\tau, \omega\}$ to Minkowski space $\{x\} \equiv \{t, \vec{x}\}$
- The gauge potential is transferred via $\mathcal{A} = X_\nu(\tau, \omega)\, e^\nu = X_\nu(\tau(x), \omega(x))\, e^\nu{}_\mu(x)\, \mathrm{d}x^\mu$
- Only finite-time $\tau \in (-\pi, +\pi)$ dynamics is required on the cylinder
- Our solutions have finite energy and action, by construction
- Energy and helicity are easily computed, null fields can be fully characterized
- A complete basis was constructed for sufficiently fast spatially and temporally decaying fields
- The non-Abelian extension couples different $j$ components of $X_a$ and will be harder to treat
- The method may be useful for numerics of Yang–Mills dynamics in Minkowski space

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