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Semileptonic $b \to u$ decays and $|V_{ub}|$

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The Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{ub}|$ describes the coupling between u and b quarks in the weak interaction, and is one of the fundamental parameters of the Standard Model. $|V_{ub}|$ is the focus of a longstanding puzzle, as the world-average values derived from inclusive and exclusive B-meson decays show a tension of a few standard deviations.

Semileptonic decays can be used to extract CKM elements by combining a lattice QCD calculation of the form factors and experimental branching fractions. In this report we will focus on the recent progress in lattice QCD calculations and the current status of $|V_{ub}|$.

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1. Introduction

The Cabibbo–Kobayashi–Maskawa (CKM) matrix element $|V_{ub}|$ describes the coupling between u and b quarks in the weak interaction. It is one of the fundamental parameters of the Standard Model, and a focus of a longstanding puzzle: a tension between inclusive and exclusive determinations. The inclusive determination includes all final states X that can occur in the $b \to u$ semileptonic decay process $B \to X\ell\nu$, whereas in the exclusive determination one restricts the analysis to a specific final state, for example $X = \pi$.

Both leptonic and semileptonic decays can be used to extract $|V_{ub}|$, and using different processes can give us valuable information by providing independent determinations of the CKM element. However, in this report we will focus on the exclusive semileptonic decays and the status of Lattice QCD calculations of the form factors.

2. Semileptonic decays

Several semileptonic decays can be used to extract the CKM element. For $B \to \pi\ell\nu$ we have good experimental data from BaBar [9, 10] and Belle [11, 12], but the light valence quarks make the lattice calculations fairly expensive. $B_s \to K\ell\nu$ is easier to calculate on the lattice than $B \to \pi\ell\nu$ due to the heavier spectator quark (strange quark), but at the moment there is only one measurement by LHCb with 2 wide bins [8]. The third possible process is $B_c \to D\ell\nu$, but we do not have any experimental data yet. However, LHCb expects to measure $B_c^+ \to D^0\mu^+\nu_\mu$ with sufficient accuracy to enable a determination of $|V_{ub}|$ [15].

In the following subsections we will briefly review the theory of semileptonic transitions with pseudoscalar initial and final states, before reporting on recent progress in lattice QCD calculations in section 3. In section 4 we will discuss extracting $|V_{ub}|$ by combining the form factors from lattice QCD and experimental data, and summarise the current status in section 5.

2.1 Form factors and matrix elemets

Two form factors, f_+ and f_0, are associated with a pseudoscalar to pseudoscalar semileptonic decay. They are defined through the vector current matrix element

$$\langle \pi(k_\pi)|V^\mu|B(p_B)\rangle = f_+(q^2)\left[(p_B + k_\pi)^\mu - \frac{M_B^2 - M_\pi^2}{q^2}q^\mu\right] + f_0(q^2)\frac{M_B^2 - M_\pi^2}{q^2}q^\mu. \quad (1)$$

Here p_B is the B meson four-momentum, k_π is the pion four-momentum and $q^\mu = p_B^\mu - k_\pi^\mu$ is the momentum transfer. We use $B \to \pi\ell\nu$ here as an example of a pseudoscalar to pseudoscalar decay — for the process $B_s \to K\ell\nu$ one would just replace B with B_s and π with K in this and the following equations, and similarly for $B_c \to D\ell\nu$.

Another useful and widely used parametrisation uses the parallel and perpendicular form factors $f_\parallel$ and $f_\perp$:

$$\langle \pi(k_\pi)|V^\mu|B(p_B)\rangle = 2\sqrt{M_B}\left[f_\parallel(E_\pi)v^\mu + k^\mu_{\pi,\perp}f_\perp(E_\pi)\right], \quad (2)$$

where $v^\mu = p_B/M_B$ is the heavy quark velocity, $E_\pi = v \cdot k_\pi = (M_B^2 + M_\pi^2 - q^2)/(2M_B)$ is the energy of the pion and $k^\mu_{\pi,\perp} = k^\mu_\pi - (v \cdot k_\pi)v^\mu$ is the projection of the pion momentum in the direction perpendicular to v^μ.

The two form factor definitions are not independent, and they are related by

$$f_+(q^2) = \frac{1}{\sqrt{2M_B}} \left[f_\parallel(E_\pi) + (M_B - E_\pi) f_\perp(E_\pi) \right]$$
$$f_0(q^2) = \frac{\sqrt{2M_B}}{M_B^2 - M_\pi^2} \left[(M_B - E_\pi) f_\parallel(E_\pi) + (E_\pi^2 - M_\pi^2) f_\perp(E_\pi) \right]. \quad (3)$$

In lattice QCD calculations the B meson is often kept at rest, and the vector-current matrix elements calculated on the lattice can be directly related to the form factors:

$$f_\parallel(E_\pi) = \frac{\langle \pi(k_\pi)|V^0|B\rangle}{\sqrt{2M_B}},$$
$$f_\perp(E_\pi) = \frac{\langle \pi(k_\pi)|V^i|B\rangle}{\sqrt{2M_B}} \frac{1}{k_\pi^i}. \quad (4)$$

The differential decay rate is related to the vector form factors and the CKM element $|V_{ub}|$ through

$$\frac{d\Gamma(B \to \pi\ell\nu)}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} \frac{(q^2 - m_\ell^2)^2 \sqrt{E_\pi^2 - M_\pi^2}}{q^4 M_B^2} \times \quad (5)$$
$$\left[\left(1 + \frac{m_\ell^2}{2q^2}\right) M_B^2 \left(E_\pi^2 - M_\pi^2\right) |f_+(q^2)|^2 + \frac{3m_\ell^2}{8q^2} \left(E_\pi^2 - M_\pi^2\right)^2 |f_0(q^2)|^2 \right],$$

where G_F is Fermi's constant and m_ℓ is the lepton mass. If the lepton is an electron or a muon, the terms suppressed by m_ℓ^2 can be discarded (at least at the current precision), and the contribution from the form factor f_0 can be neglected. Thus the relation between the differential decay rate and the form factors is reduced to a much simpler form

$$\frac{\mathrm{d}\Gamma(B \to \pi\ell\nu)}{\mathrm{d}q^2} = \frac{G_F^2 |V_{ub}|^2}{24\pi^3} |k_\pi|^3 |f_+(q^2)|^2. \quad (6)$$

Using either one of these relations allows one to combine experimetal data (the differential decay rate) and a theoretical calculation of the form factors to extract $|V_{ub}|$.

2.2 Parametrising q^2 dependence: z-expansion

The z-parameter expansion is the most widely used way to parametrize the shape of the form factors. Especially for the B to π decay, the physical range of momentum transfer q^2 is large. Using

$$z = z(q^2, t_0) \equiv \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}, \quad t_\pm = (M_B \pm M_\pi)^2, \quad (7)$$

we transform q^2 to a small parameter z that can be used in a series expansion. Here t_0 is a free parameter, and the choice $t_0 = t_+(1 - \sqrt{1 - t_-/t_+})$ gives $|z| < 0.3$ (for a B to π decay). This mapping from q^2 to z is illustrated in Figure 1.

Two forms of the expansion, the BGL parametrisation and the BCL parametrisation, are widely used in parametrising form factors of semileptonic decays.

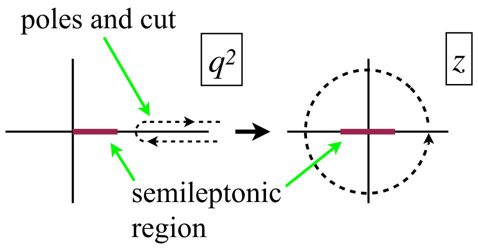

Figure 1: A sketch of the conformal mapping $q^2 \to z$, which maps the q^2-plane cut for $q^2 \geq t_+$ onto the disk $|z(q^2, t_0)| < 1$ in the z complex plane.

The BGL (Boyd, Grinstein, Lebed) parametrisation [13] was originally developed for $b \to c$ decays, using dispersion relations and analyticity but without recourse to heavy quark symmetry. It has the form

$$f_i = \frac{1}{\mathcal{P}_i(q^2)\phi_i(q^2, t_0)} \sum_{n=0}^{N_z} a_n^i z^n \text{ for } i \in \{+, 0\}, \tag{8}$$

where the pole at M_{B^*} in the vector form factor is taken care of by the Blaschke factor $\mathcal{P}_+ = z(q^2, M_{B^*}^2)$, and $\mathcal{P}_0 = 1$. $\phi_i(q^2, t_0)$ are analytic functions that obey the dispersive bounds, and the calculation includes perturbative QCD results as well as non-perturbative corrections (see [13]). $\phi_0(q^2, t_0) = 1$ is often used for the scalar form factor. Experimental groups have used this parametrisation when analysing B to π semileptonic decays.

The BCL (Bourrely, Caprini, Lellouch) parametrisation [14], on the other hand, uses

$$f_+ = \frac{1}{1 - q^2/M_{B^*}^2} \sum_{n=0}^{N_z - 1} b_n^+ \left[z^n - (-1)^{n - N_z} \frac{n}{N_z} z^{N_z} \right]; \quad f_0 = \sum_{n=0}^{N_z} b_n^0 z^n. \tag{9}$$

Here the pole is taken into account by the term $1/\left(1 - q^2/M_{B^*}^2\right)$, and the second term in parentheses ensures that the form factor satisfies the appropriate asymptotic form near the $B\pi$ threshold. Also this parametrisation satisfies unitarity, analyticity and perturbative QCD scaling requirements, and is often favoured by the lattice QCD community.

3. Form factors: Recent progress lattice QCD calculations

As we have seen, it is essential to know the form factors (or at least the vector form factor f_+) from theory. There are several new lattice QCD calculations or updates to existing calculations that aim to improve the accuracy and precision at which we know the form factor(s).

3.1 $B \to \pi \ell \nu$

The JLQCD Collaboration have calculated the form factors using Möbius Domain Wall fermions [16, 17]. The key feature of this calculation is the treatment of heavy quarks: the same action is used for all quarks, from light to heavy (both valence and sea quarks), and the treatment is fully relativistic. The form factors are calculated at various heavy quark masses, starting from charm quark mass towards the b quark mass, and the result is extrapolated to the physical point. This avoids any issues with matching to perturbative theories as in Non-Relativistic QCD

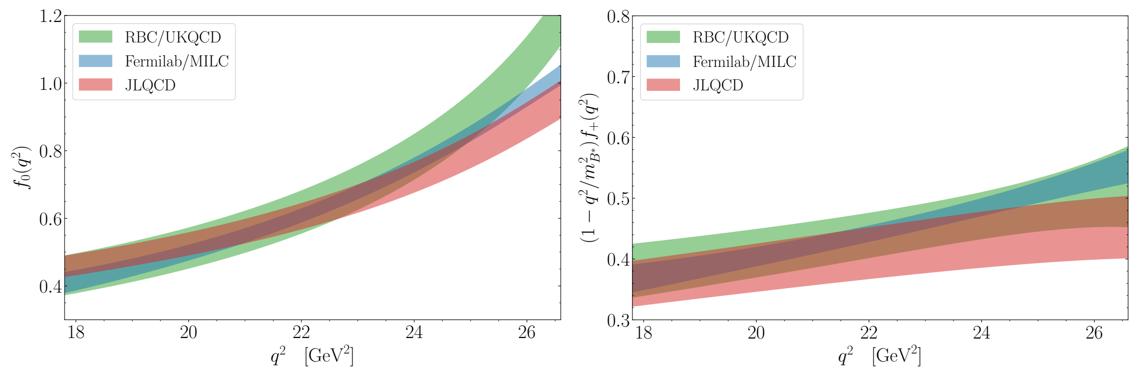

Figure 2: Comparison of form factors f_0 and f_+ from different lattice calculations for the process $B \to \pi\ell\nu$: RBC/UKQCD Collaboration [3], Fermilab Lattice and MILC collaborations [5], and JLQCD Collaboration [16, 17].

or effective heavy quark actions, but the price one pays is the additional extrapolation in the heavy quark mass (on top of the chiral and continuum extrapolations) and discretisation effects that grow like $(am_Q)^2$. However, these systematic effects can be quantified and controlled.

The Fermilab Lattice and MILC collaborations are pursuing a similar strategy using Highly Improved Staggered Quarks (HISQ) [6, 7]. In their earlier calculation of $B \to \pi\ell\nu$ form factors [5] they used different actions for light and heavy quarks: specifically, for the b quark they used the Fermilab interpretation of the Wilson and Sheikholeslami-Wohlert actions applied to nonrelativistic fermions. In their ongoing calculation the HISQ action is used for all quarks, and they calculate the form factors at multiple values of the heavy quark mass in order to extrapolate to the physical b quark mass.

The RBC/UKQCD Collaboration are updating their calculation [3] by adding a finer lattice spacing and improving the analysis of systematic effects [4] (see also these proceedings). They use $2+1$ flavor Domain Wall fermion and Iwasaki gauge actions. For the b quark they use the Columbia version of the Relativistic Heavy Quark (RHQ) action. This avoids the large discretisation effects usually introduced by a large b quark mass on the lattice, am_b. The RHQ parameters are tuned to reproduce the physical B_s meson mass and hyperfine splitting.

Results from different collaborations' calculations of the $B \to \pi\ell\nu$ form factors are compared in Fig. 2. The form factors agree within estimated errors (statistical and systematic) in the q^2 range 18–24 GeV^2, while the shape of the $f_0(q^2)$ form factor seems to be less steep in the JLQCD Collaboration's calculation especially near $q^2_{\rm max}$ (i.e. at small pion energy). This may be due to different systematics in the respective calculations, though the statistical significance is limited.

3.2 $B_s \to K\ell\nu$

Another process that can be used to extract $|V_{ub}|$, once better experimental data is available, is $B_s \to K\ell\nu$. As part of the project to calculate form factors of semileptonic decays, the Fermilab Lattice and MILC collaborations are pursuing a calculation of the $B_s \to K\ell\nu$ form factors alongside the $B \to \pi\ell\nu$ form factors. They use the same "heavy-HISQ" framework as for the $B \to \pi\ell\nu$ process [6, 7]. This uses the same action for light and heavy quarks, and relies on extrapolation to get the result at physical b quark mass. On the finest lattices one can actually reach this mass, but the discretisation effects have to be well understood and quantified, so that they can be taken into account.

The RBC/UKQCD Collaboration calculated also $B_s \to K\ell\nu$ form factors in their 2015 paper [3], and they are now updating these results alongside with the $B \to \pi\ell\nu$ form factors ([4], see also these proceedings). The lattice actions and analysis methods are the same for these two processes.

3.3 $B_c \to D\ell\nu$

Anticipating experimental results from the LHCb in the future [15], HPQCD has calculated the form factors for the process $B_c \to D\ell\nu$ [1] (see also [2]). They use the HISQ action for all quarks and the well established “heavy-HISQ" approach to extract the result at the physical b quark mass. Determining $|V_{ub}|$ via this exclusive semileptonic process will offer another data point, once experimental data becomes available.

4. Extracting $|V_{ub}|$

To extract $|V_{ub}|$, form factors from lattice QCD are fitted together with experimental results, integrating over each q^2 bin i:

$$\Delta\Gamma_i(B \to \pi\ell\nu) = \int_{q^2_{i,\mathrm{min}}}^{q^2_{i,\mathrm{max}}} \frac{G_F^2 |V_{ub}|^2}{24\pi^3} |k_\pi|^3 |f_+(q^2)|^2 \mathrm{d}q^2. \tag{10}$$

Here we focus on the $B \to \pi\ell\nu$ decay, where we have differential branching fractions in q^2 bins from several experiments. One can use the BCL parametrisation given in Eq. (9) (or some other parametrisation) for the form factors, and add $|V_{ub}|$ as an additional fit parameter. However, before extracting $|V_{ub}|$ we look at the shape of the form factor.

To compare the shape of the form factors from experiment and lattice, one can fit each data set on its own and compare the ratios of the coefficients of the z-expansions, b_1^+/b_0^+ and b_2^+/b_0^+. This does not require any knowledge of $|V_{ub}|$, as it cancels in the ratio. The authors of [16, 17] used the BCL parametrisation given in Eq. (9) with $N_z = 3$. The results are illustrated in Figure 3, which is taken from the same paper. The fits to BaBar and Belle data show that BaBar 2010 [9] (untagged 6-bin) data has a slightly different shape than BaBar 2012 [10] (tagged 12-bin), Belle 2010 [11] and Belle 2013 [12] data. Therefore the authors do a simultaneous fit to all four expermental data sets (labelled as “All expts.”) as well as a fit that excludes the BaBar 2010 data set (labelled as “Excl. BaBar 2010”). For comparison, the plot also shows the shape of the lattice form factors from JLQCD collaboration’s calculation [16, 17]. Overall, the shapes of the form factors agree well.

We now come back to extracting $|V_{ub}|$ by fitting the differential branching fractions from experiments as well as form factors from lattice QCD. Using the expression given in Eq. (10), JLQCD collaboration integrate over q^2 bins and fit their lattice data and all four experimental data sets simultaneously [16, 17]. They obtain a good fit using $N_z = 4$ and get $|V_{ub}| = 3.91(38) \times 10^{-3}$. This fit is illustrated in Fig. 4, where the experimental branching fractions are plotted alongside with the lattice QCD data points as well as the fit results for different N_z ($N_z = 4$ being their preferred fit). They find that excluding BaBar 2010 data from this fit has minimal effect on the value of $|V_{ub}|$. This is the only calculation so far that includes form factors that are calculated using

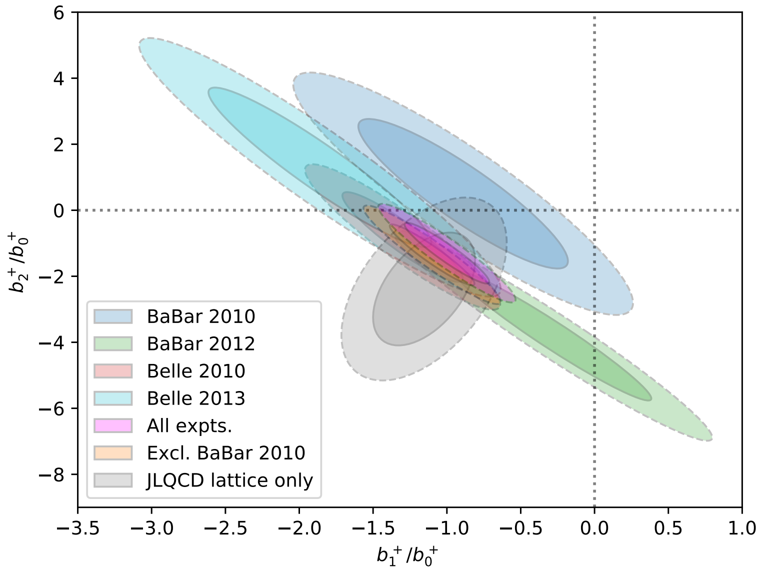

Figure 3: Contour plots for the shape parameters b_1^+/b_0^+ and b_2^+/b_0^+ for the process $B \to \pi\ell\nu$ using the BCL paremetrisation with $N_z = 3$. Each of the experimental data sets is fitted on its own, as well as fitting all of the experimental data sets simultaneously. The label 'Excl. BaBar 2010' refers to a simulatneous fit to other experimental data sets but excluding the BaBar 2010 data. The motivation to do this fit is to test how much the 'BaBar 2010' data set affects the fit, as it is observed to have a slightly different shape than the other data experimental sets (for more details see [16, 17]). For comparison the result of a similar fit to the lattice QCD form factors calculated by the JLQCD Collaboration is also shown. Here 68% confidence regions are shown with a solid outline, and 95% regions with a dashed outline. This figure is from [17].

a fully relativistic treatment of the b quark on the lattice. The value $|V_{ub}| = 3.91(38) \times 10^{-3}$ can be compared to earlier lattice QCD results: $|V_{ub}| = 3.72(16) \times 10^{-3}$ by the Fermilab Lattice and MILC collaborations [5], and $|V_{ub}| = 3.61(32) \times 10^{-3}$ by the RBC/UKQCD Collaboration [3].

4.1 Dispersive Matrix Method

The authors of [18] present an alternative method (to the z-expansion) of determining the form factors from Lattice QCD data using unitarity and analyticity. The Dispersive Matrix Method does not make any assumptions about the functional dependence of the form factors on the momentum transfer q^2. Instead it uses a lattice QCD calculation of the 2-point correlation functions of the quark currents and the form factors in the high-q^2 region to determine unitarity bounds of the form factors in the full kinematic q^2 range. The susceptibilities that are needed to implement the unitarity and analyticity bounds are calculated fully non-perturbatively by evaluating moments of the 2-point correlation functions, and the method does not rely on a series expansion or perturbative bounds. This approach could be beneficial for exclusive semileptonic B-meson decays, where the direct calculation of the form factors at low q^2 is particularly difficult due to large statistical fluctuations and discretisation effects. The form factors obtained by the Dispersive Matrix method can then be combined with experimental results to extract $|V_{ub}|$.

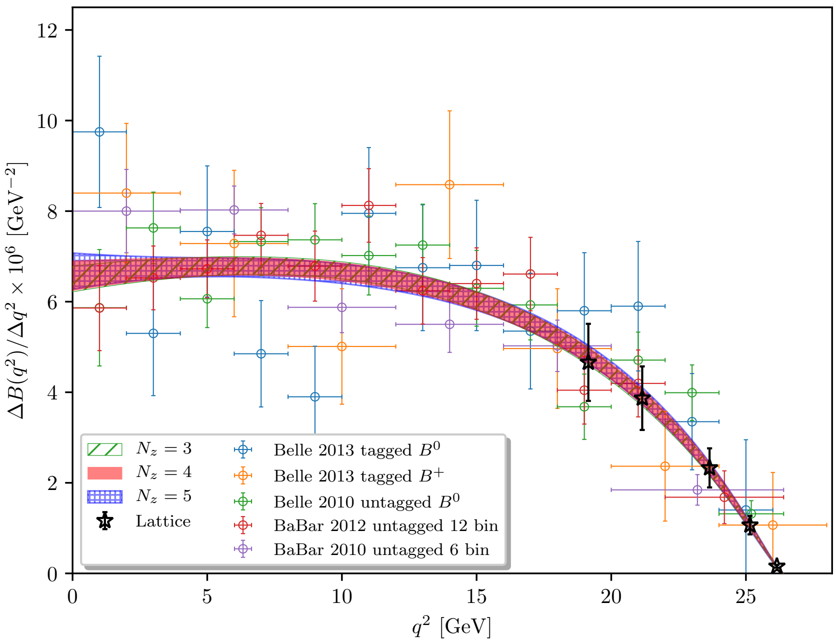

Figure 4: Fitting experimental branching fractions together with form factors from lattice QCD to extract $|V_{ub}|$. The error bands show JLQCD collaboration's fit results when N_z terms are included in the z-expansion. This figure is from [17].

5. Summary: $|V_{ub}|$

We give a summary of the current status of $|V_{ub}|$ in Figure 5, where we compare different lattice QCD calculations as well as exclusive and inclusive determinations by the Heavy Flavor Averaging Group (HFLAV). There are four values from the calculations of the form factors of the $B \to \pi \ell \nu$ process: from the Fermilab Lattice and MILC collaborations [5] (from 2015), from the RBC/UKQCD Collaboration [3] (from 2015), from the JLQCD Collaboration [17] (from 2021) and from the HPQCD Collaboration [22] (from 2007, using a nonrelativistic (NRQCD) action for the b quark). These lattice QCD determinations agree well, but the errors are still fairly large. There is also a determination from a lattice QCD calculation of the form factors of the Λ_b to p process [19] (from 2015), which gives a lower vaue for $|V_{ub}|$. The Flavour Lattice Averaging Group (FLAG) average is from the 2019 report [21], and the Heavy Flavor Averaging Group (HFLAV) exclusive and inclusive results are from [20].

The puzzle and the slight tension between the exclusive and inclusive values remain, as the uncertainties are not yet small enough to say anything conclusive. Interestingly the latest calculation by the JLQCD collaboration that uses fully relativistic action for the b quark agrees with both the exclusive determinations (by HFLAV and FLAG) and the inclusive determination by HFLAV. The ongoing lattice QCD calculations will help here, as well as the use of more processes ($B_s \to K \ell \nu$ and $B_c \to D \ell \nu$) when the experimental data becomes available. Work is also being done to calculate the inclusive semileptonic decays on the lattice — see [23, 24].

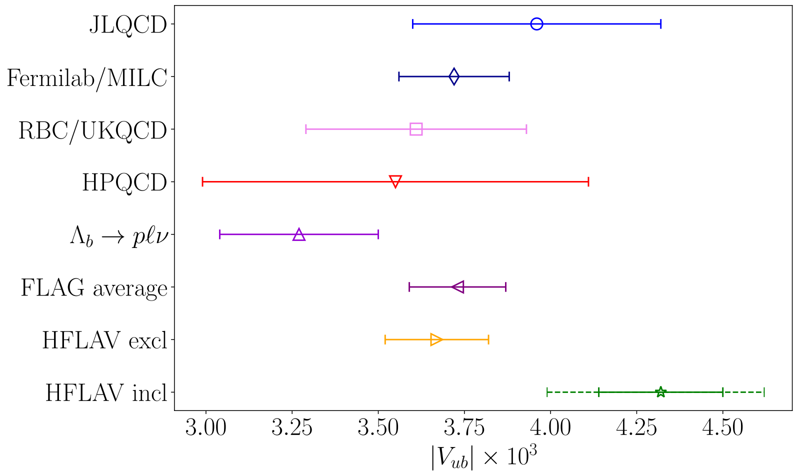

Figure 5: Comparison of lattice QCD calculations with exclusive and inclusive determinations by HFLAV and FLAG. The results are from the following publications: the Fermilab Lattice and MILC collaborations [5] (from 2015), the RBC/UKQCD Collaboration [3] (from 2015), the JLQCD Collaboration [17] (from 2021) and the HPQCD Collaboration [22] (from 2007, using a nonrelativistic (NRQCD) action for the b quark). Note that these are all from calculations of the $B \to \pi\ell\nu$ process. The value labelled $\Lambda_b \to p\ell\nu$ is from [19], a lattice QCD calculation of the form factors of the Λ_b to p process. The FLAG average is from the 2019 report [21], and the HFLAV exclusive and inclusive results are from [20]. The inclusive data point is from their GGOU analysis, with a second (dashed) error bar to show the spread of values from using different frameworks.

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