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Bosonization of Majorana modes in arbitrary geometries

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We briefly summarize the main results from [25] which extend the ones from [17, 18]. The bosonization method based on the properties of Clifford algebras is presented, together with the illustrating examples: system of Majorana fermions on honeycomb lattice and its relation to Kitaev's model, and the Hubbard model on rectangular lattices. The role of bundaries together with the presence of Majorana edge modes are discussed.

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1. Introduction and motivation

The search for dual descriptions of fermionic systems has a long-standing tradition. Several bosonization techniques were developed to resolve this issue and to describe femionic models in terms of certain bosonic variables. Interestingly, this was an incredibly active branch of both condensed matter as well as high energy physics. Different techniques and nomenclature used in these two fields were sometimes sources of misunderstandings and re-discovering certain results known before in the second of those branches of physics. One of the oldest and the most well recognised method is the Jordan-Wigner transformation [1]. Despite its simplicity in $(1 + 1)$-dimensional case, the straightforward generalization to higher dimensions leads to problems, mainly related to its non-local character due to the existence of the so-called *Jordan-Wigner string*. Several modifications [2–11] valid in spatial dimensions higher than one have been proposed. None of them were free of non-locality of a certain type. It was manifested either in the presence of non-local operators or by the existence of complicated constraints. The later were usually thought of as certain Gauss'-like laws for an underlying gauge theory. The proposed bosonization techniques contained, but were by no means limited to, schemes generalizing ideas originally applied to the Luttinger-Tomonaga model [12], both in the abelian as well as non-abelian case. One of the most famous non-abelian bosonization is Witten's proposal [13]. Yet another method is based on constructing an exact isomorphism of algebras describing the two systems - the original fermionic one and its bosonic counterpart. Both the bond algebra approach introduced in [14] as well as the ones studied in [3, 4] are of the later type. In [15] the proposal of bosonization, to which we refer to as the Gamma model, was introduced. It was partially analyzed in [16] and later on in [17, 18] we have demonstrated in a full glory its algebraic character, finally classifying this method as the one based on the existence of a certain isomorphism of algebras.

The notion of bosonization is quite closely related to the more general concept of dualities. Originating from the famous Kramers-Wannier dualities [19, 20], nowadays there exists the whole zoo of them, known as the so-called *web of dualities* [21–23]. The dual description usually involves higher gauge theories [3, 4, 24]. We have demonstrated in [18] that the Gamma model, in certain cases, also possesses such dual description in terms of higher gauge theory, and, moreover, constraints present in the model are closely related to modified (higher) Gauss' laws.

In this brief overview we summarize the main features of the Gamma model. We will argue that, under quite non-restrictive assumptions, one can always construct a Gamma model corresponding to a given fermionic system defined on a lattice with an arbitrary geometry. The original formulation [15–18] was limited to systems with a single fermion per lattice site, and, moreover, the allowed geometries of the lattice were restricted to ones with a site-independent even coordination number. We have shown that the existing constraints are equivalent to the pure gauge condition, and their modifications can be interpreted as coupling the fermionic system with an external background $\mathbb{Z}_2$ gauge field. Furthermore, the operators encoding these constraints are nothing else than the holonomies for these gauge models.

Despite seemingly limited applications such construction covered relatively large number of physically interesting examples, e.g. toroidal geometries which are of main interest for lattice Monte Carlo simulations. In particular, relation with the famous Kitaev's toric code was demonstrated in [18]. Even in the case of a simple rectangular geometry the problem arises when we consider the boundaries of the lattice. The reason is that the coordination number for vertices on the boundary is smaller than the one in the bulk. This subtlety was already noticed in [18] when the simplest adjustment of our bosonization method on boundary was proposed. It followed from that considerations that the existence of the boundaries may lead to the presence of additional Majorana modes localized on them, and those modes could be connected by strings (constructed of out bosonic operators) that feature the braid relations, and therefore the possible link to well-established applications of Majorana modes for the quantum computation could be visible also from this construction. In [25] we have generalized the bosonization scheme based on the Gamma model into arbitrary geometries and for fermionic systems with arbitrary (not necessarily constant) number of Majorana fermions per lattice site. In contrast with the previous construction the Majorana modes do not necessarily form usual Dirac fermions, but rather are the fundamental variables. This in particular allows for the dual description of several spin liquid systems. In particular, the Kitaev's honeycomb model emerges as a result of such a bosonization scheme. The possibility of having larger number of Majorana modes per lattice site allows also for the consideration of models with more than one flavour of fermionic variables, e.g. labeled by the spin as it happens in case of the Hubbard model. We remark that similar techniques were also investigated in a different context in [26–28] and applied to certain higher spin models. More recently, similar findings were also obtained in [29, 30]. To some extent the last approach overlap with our last results from [25] and the previous ones from [17, 18]. The brief overview and comparison between these schemes can be found in [31].

The organization of this overview, summarizing the main findings from [25], is as follows. In section 2 we present the general construction, starting with the detailed characterization of fermionic theories in the subsection 2.1 followed by the formulation of their bosonized version in the subsection 2.2, and finally discussing the role of the constraints therein. We illustrate the general method on simple examples. We start with the discussion of the fermionic system defined on honeycomb lattice with one Majorana fermion per lattice site. The relation to the Kitaev's model is demonstrated. Next, the simplest example with more than one fermionic flavour - the Hubbard model - is discussed. The role of boundaries and the presence of additional Majorana modes is discussed in section 4. We conclude in section 5.

2. The bosonization scheme

We summarize here the general construction introduced in [25]. We refer the reader to [25] (and also to [18]) for the detailed proofs of all the results stated here. As it was already announced in the preceding section, the lattice geometry is allowed to be a generic finite graph. In particular, the coordination number may vary from site to site, and all types of boundaries are in principle allowed. This covers lattices usually considered in both high energy and condensed matter physics, but also opens the possibility to analyze more exotic systems.

2.1 Fermionic theories - the general framework

On the aforementioned lattices we consider fermionic systems whose fundamental degrees of freedom are Majorana fermions. We do not assume that they have to be paired in a way to form a usual Dirac fermions, but rather they themselves are the degrees of freedom here. In particular,

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models with single Majorana variable per lattice site are one class of systems of interest. We allow, in general, for arbitrary number of Majorana fermions per lattice site, and, moreover, this number is allowed to be site-dependent. This, in particular, will allow later for considering fermionic systems with extra edge modes. Let $\psi_\alpha(x)$ denote the Majorana fermion of type α, located on site x. Here $\alpha = 0, \ldots, n(x)$ is the index enumerating flavours of Majorana fermions located on site x. Therefore, the total number of Majorana variables is given by $n = \sum_x (n(x)+1)$. Since the graph was finite, i.e. had only finite number of vertices (and edges), this quantity is a well-defined positive integer. The product of all Majorana operators forms a fermionic parity,

$$(-1)^F = i^{\frac{n}{2}} \prod_x \prod_{\alpha=0}^{n(x)} \psi_\alpha(x). \tag{1}$$

We would like to consider fermionic models with the fermionic parity being a conserved quantity. This is the only assumption we make on the fermionic Hamiltonian describing the system. To achieve this goal it is enough to restrict ourselves to cases with the total number of Majorana fermions, n, being an even integer. Indeed, then $(-1)^F$ anticommutes with every $\psi_\bullet(\cdot)$.

Since $(-1)^F$ is assumed to be the conserved quantity, we have to consider only even fermionic operators. They form an algebra $\mathcal{A}$, which can be explicitly described in terms of certain generators and relations satisfied by them. The most convenient choice for generators is by taking as them operators $S(e) = \psi_0(x)\psi_0(y)$ for every edge e with starting point x and target point y, and $T_\alpha(x) = \psi_0(x)\psi_0(x)$ for any $\alpha \neq 0$ and any site x. In order to describe in a compact manner the set of relations that these operators are subject to, we introduce formal $\mathbb{Z}_2$-linear combinations of vertices and the pairing between them as a $\mathbb{Z}_2$-bilinear operation $(\cdot,\cdot)$ satisfying $(x, y) = \delta_{x,y}$, for any two vertices x, y. The boundary of an edge e from x to y can be written as $\partial e = x + y$. Having said that, the set of relations satisfied by the aforementioned generators is:

1. $S(e)^2 = T_\alpha(x) = -1$, $S(e)^\dagger = -S(e)$ and $T_\alpha(x)^\dagger = -T_\alpha(x)$, for any x and any $\alpha \neq 0$,
2. $S(e)S(e') = (-1)^{(\partial e, \partial e')} S(e')S(e)$, for any two edges e, e',
3. $S(e)T_\alpha(x) = (-1)^{(x,\partial e)} T_\alpha(x)S(e)$, for any edge e and any vertex x,
4. $T_\alpha(x)T_\beta(y) = (-1)^{(x,y)(1-\delta_{\alpha,\beta})} T_\beta(y)T_\alpha(x)$, for any two vertices x, y, and any α, β,
5. $S(e_1) \ldots S(e_m) = 1$, for any loop $e_1 \ldots e_m$ (i.e. with $e_{m+1} = e_1$).

We have proven in [18] that all the above relations are independent, and there is no further independent relation between generators of $\mathcal{A}$. Therefore, the above set gives a complete description of the algebra $\mathcal{A}$. In order to find an equivalent, bosonic, formulation of a given fermionic model it is therefore enough to construct, out of the bosonic variables, the same number of generators as in the fermionic case, and to prove that they satisfy all the above relations. This is an essence of the bosonization scheme we discuss here.

2.2 "The Gamma model" as a bosonized version of the fermionic theory

The bosonization proposed in [25] and being a generalization of the one analyzed in the full glory in [18], is based on associating certain generators of the Clifford algebra to the edges and vertices of the lattice on which the fermionic system is defined. More precisely, with each lattice site x we associate the Clifford algebra generated by elements

$$\{\Gamma(x,e)|e - \text{edge incident to } x\} \cup \left\{\Gamma'_\alpha(x)\right\}_{\alpha=0}^{n(x)}. \tag{2}$$

This is illustrated, in case of the square lattice, in Fig. 1. We assume that Gamma matrices

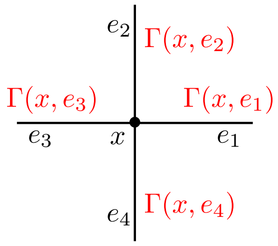

Figure 1: Operators $\Gamma(x,e)$ associated to edges having site x as a common endpoint.

corresponding to distinct vertices commutes with each other. The only anticommutations are present when there is endpoint-sharing. In this sense this is a (hard-core) bosonic system. Out of these variables we define $\widehat{S}(e) = i\Gamma(x,e)\Gamma(y,e)$, for an edge e connecting x with y, and $\widehat{T}_\alpha(x) = i\Gamma'(x)$, for any site x and nonzero α. As it was argued in [25] those operators satisfy all the relations describing the algebra $\mathcal{A}$ except the loop ones. We then introduce operators $W(\ell) = \widehat{S}(e_1)\ldots\widehat{S}(e_m)$ for any loop ℓ consisting with edges $e_1,\ldots,e_m$. Instead of considering the whole Hilbert space we restrict ourselves only to its subspace $\mathcal{H}_0$ determined by the conditions $W(\ell)|\phi\rangle = |\phi\rangle$ for any $|\phi\rangle \in \mathcal{H}_0$. These constraints play an important role. Their modification, $W(\ell)|\phi\rangle = \omega(\ell)|\phi\rangle$, with $\omega(\ell) = \pm 1$, has a natural interpretation in terms of a $\mathbb{Z}_2$-gauge theory. More precisely, it was argued in [18, 25] that $\omega(\ell)$ can be thought of as the holonomy along ℓ, and the fermionic system, corresponding to such a constraint on the bosonic side, is nothing else than the original one with the fermions minimally coupled to the external $\mathbb{Z}_2$-gauge field for the fermionic parity symmetry. The detailed discussion of constraints in the case of toroidal geometry was also performed in [17]. In this case the two "Polyakov" lines corresponding to the two generators of the fundamental group of the torus play an important role. We refer to [17, 18] for a detailed discussion of the constraints for such geometries.

The last remaining ingredient of the bosonization discussed here is related to the fermionic parity operator. In order to finalize this construction the following operators were defined:

$$\gamma(x) \sim \prod_{e:\, x-\text{endpoint of } e} \Gamma(x,e) \prod_{\alpha\neq 0} \Gamma'_\alpha(x), \tag{3}$$

with the phase chosen so that it squares to one. Consider now the quantity $N(x) = \deg(x) + n(x)$, where $\deg(x)$ is the number of edges having x as an endpoint. As it was discussed in [25] the parity of $N(x)$ determines whether $\gamma(x)$ commutes or anticommutes with all Gamma matrices generating the Clifford algebra at site x. For simplicity, we assume in this overview that this number is even, and we refer the reader to [25] for a more detailed discussion of the role of this quantity in the generic case. In the models with even $N(x)$ we end up with the constraint between the geometric structure of the lattice and the number of Majorana modes (per lattice site) allowed in the system. More precisely, in such a case the relation $\deg(x) + n(x) = 0$, modulo 2, has to be fulfilled. Moreover, we are free to chose one of the sectors defined by the conditions $\gamma(x) = \pm 1$. In other words, the main result of [25] is that the constrained bosonic system is completely equivalent to the sector of the fermionic one. One of the two possible values of $(-1)^F$ characterizes this choice. This also explicitly shows the role of restricting our attention to fermionic systems described by Hamiltonians preserving the fermionic parity.

3. Illustrating examples

Here we briefly discuss two simplest examples, used in [25] to illustrate the general construction. First of them is the system with only one Majorana fermion ψ per lattice site. The simplest, nontrivial, geometry for which such a construction is possible is the honeycomb lattice, which is a trivalent graph. This system is totally different than ones analyzed previously in [17, 18]. Indeed, in this case Majorana fermions are themselves fundamental variables and they are not just a convenient way to parameterize complex Dirac fermions. This shows that the proposed bosonization scheme can be applied to a single Majorana mode as well. The Gamma matrices, suitable for this model, are just the usual Pauli ones, $\sigma_X, \sigma_Y, \sigma_Z$, assigned to the three types of edges as depicted in Fig. 2. The loop constraints are generated by the ones associated to elementary plaquettes which on the

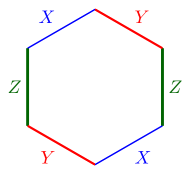

Figure 2: The assignment of Pauli matrices to the elementary plaquette of the honeycomb lattice.

Hilbert space $\mathcal{H}_0$ take value -1. One possible choice from the zoo of systems that could be defined on such lattice and can be bosonized using our scheme, is the one described by the Hamiltonian $H = i \sum_{I\in\{X,Y,Z\}} \sum_{\substack{\langle xy\rangle \\ \text{type I edges}}} J_I \psi(x)\psi(y)$, where J_I with $I \in \{X, Y, Z\}$ are constants characterising the model. We have shown in [25] that its bosonized version corresponds to the famous Kitaev's model restricted to the sector corresponding to the value -1 of the plaquette operators. In this sense our scheme defines an inverse of the Kitaev's fermionization procedure. The discussed example is the simplest spin liquid model and our techniques, since defined for Majorana modes from the very beginning, allow also to analyze more sophisticated spin liquids, e.g. the ones proposed in [32].

The next example will illustrate the possibility of bosonizing systems with more than one flavour of the usual fermions. One of the simplest systems with this feature is the Hubbard model defined on the square lattice. In this case we have an extra index, the spin, labeling fermionic variables. We can decompose them into four Majorana fermions (by analogy of decomposing a complex number into its real and imaginary part) and proceed by finding the bosonized version of the Hamiltonian together with the form of the plaquette constraints. Moreover, one can also find a bosonized versions of the generators of symmetries (e.g. the generators of the $SU(2)$ group in case of the Hubbard model) possessed by the fermionic system, showing explicitly that our bosonization scheme preserves all the symmetries of the original system, and moreover, on-site symmetries in the fermionic models are mapped to on-site ones on the bosonic side.

4. The role of the boundary

The question posed in [18] that was the motivation for generalizing the construction in [25] was related to the role of the boundary of finite lattices. Here we summarize those results. For concreteness, we consider rectangular lattice of size $L_x \times L_y$, and systems with two Majorana fermions ψ_0, ψ_1 at each lattice site. One can identify two distinguished directions, $i = 1, 2$, on the lattice, and at the edges having site x as an endpoint put four Gamma matrices $\Gamma_{\pm i}(x)$ (similarly as in Fig. 1 with $\Gamma(x, e_1) = \Gamma_1(x)$, $\Gamma(x, e_4) = \Gamma_{-2}(x)$ etc.). The operator $\Gamma_1'(x)$ is not independent and can be expressed as a product all other Gammas located at edges neighbouring x. The plaquette constraints are identical with the ones for the Hubbard model. The bosonization prescription in the bulk is a straightforward application of the general scheme. However, the situation on the boundary is slightly different. We illustrate this by considering the northern boundary, where at each site x_i there is a *missing* Gamma matrix in the direction 1. To solve this issue, in [25] we have introduced spurious Majorana fermions $\chi_N(x_i)$ and proceed with the bosonization by identifying $i\Gamma_1(x_i)$ with $\psi_0(x_i)\chi_N(x_i)$. Moreover, we have also observed that our bosonization yields the identification (up

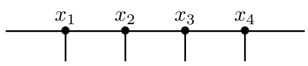

Figure 3: The northern boundary of the $L_x \times L_y$ lattice

to a controllable phase) of $\psi_0(1, b)\psi_0(L_x, b)$ with $\Gamma_{-1}(1, b)B_{-1,1}(b)\Gamma_1(L_x, b)$, where $B_{-1,1}(b)$ is a product of all $\Gamma_{-1}(a, b)\Gamma_1(a, b)$ along the path connecting points $(1, b)$ and (L_x, b). (Here we are using the natural notation $x = (a, b)$ for the points at rectangular lattice.) As a result, the bosonic system is equivalent to the fermionic one but with additional Majorana edge modes. The intriguing strings connecting those modes are also present, as expected from other well-established properties of Majorana modes and their relations to braiding and quantum computation.

5. Conclusions and outlook

In this brief overview we have summarized the main findings in [25] extending some of the results from [17, 18]. The presented construction, in particular, solves the problem related to the role of the boundary. Moreover, it provides a computational method to bosonize systems with Majorana fermions as fundamental variables. This leads to a prospering technique that may be applied to several, even quite exotic, spin liquid models. We have illustrated this scheme on the simplest model of this type - the Kitaev's honeycomb - as well as models with more than one flavour of usual fermions - the Hubbard model. The allowance for site-dependence in our construction leads also to possible applications to more complex geometries. e.g. for quasi-crystals or different types of junctions, where the on-site boundary-like Majorana modes may play a crucial role. We postpone the detailed discussion of those application for the future research.

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