

Determination of the continuous β function of SU(3) Yang-Mills theory

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In infinite volume the gradient flow transformation can be interpreted as a continuous real-space Wilsonian renormalization group (RG) transformation. This approach allows one to determine the continuous RG β function, an alternative to the finite-volume step-scaling function. Unlike step-scaling, where the lattice must provide the only scale, the continuous β function can be used even in the confining regime where dimensional transmutation generates a physical scale Λ_{QCD} . We investigate a pure gauge SU(3) Yang-Mills theory both in the deconfined and the confined phases and determine the continuous β function in both. Our investigation is based on simulations done with the tree-level Symanzik gauge action on lattice volumes up to 32^4 using both Wilson and Zeuthen gradient flow (GF) measurements. Our continuum GF β function exhibits considerably slower running than the universal 2-loop perturbative prediction, and at strong couplings it runs even slower than the 1-loop prediction.

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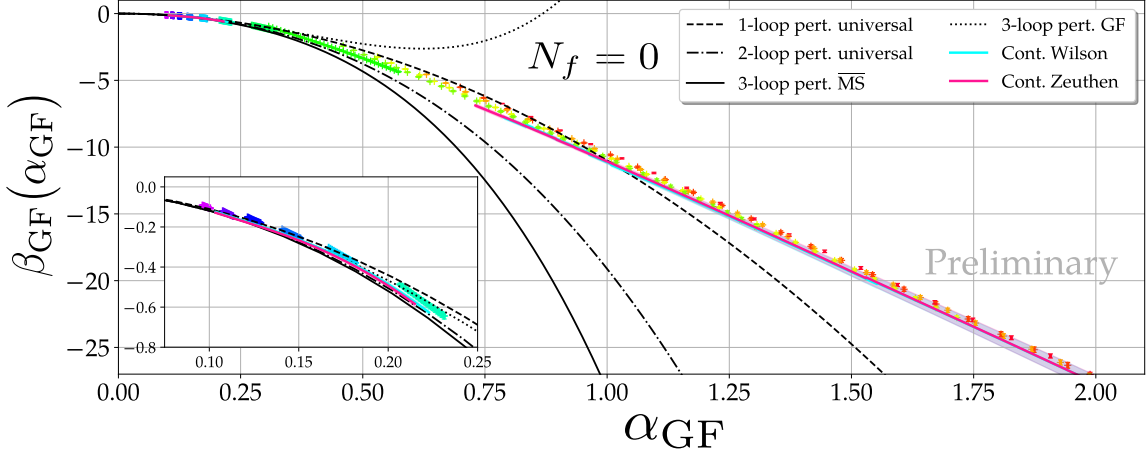


Figure 1: Comparison of the continuous β function in the continuum limit (magenta and cyan bands for Zeuthen flow and Wilson flow, respectively) against the 1- and 2-loop universal β functions and the 3-loop $\overline{\text{MS}}$ and gradient flow β functions (various black curves). The rainbow-colored data points correspond to “raw data” for the β function on 32^4 volumes from $\beta \equiv 6/g_0^2 = 8.50$ (magenta) down to $\beta = 4.30$ (red) using Zeuthen flow and the Symanzik operator (ZS) to determine $E(t)$. The insert on the lower left shows the details in the deconfined, small α_{GF} regime.

1. Introduction

The scale-dependent properties of a 4-dimensional Yang-Mills (YM) theory (with or without fermions) can be extracted from its renormalization group (RG) β function, defined as the logarithmic derivative of the renormalized coupling with respect to an energy scale. The gradient flow (GF) transformation [1–3] combined with Monte Carlo renormalization group (MCRG) [4] principles defines a complete RG scheme if the energy scale is set by the GF flow time as $\mu \propto 1/\sqrt{8t}$ for asymptotically large t . For a detailed discussion see Ref. [5]. The GF coupling defined in terms of the flowed energy density $E(t)$ [3, 6]

$$g_{\text{GF}}^2(t) \equiv \frac{128\pi^2}{3(N_c^2 - 1)} \langle t^2 E(t) \rangle \quad (1)$$

has both zero canonical and anomalous dimensions and therefore serves as a valid definition of a renormalized running coupling. The constants in Eq. (1) are chosen to match the $\overline{\text{MS}}$ coupling at tree level [3] and N_c denotes the number of colors. The corresponding GF β function in infinite volume is

$$\beta(g_{\text{GF}}^2) \equiv \mu^2 \frac{dg_{\text{GF}}^2}{d\mu^2} = -t \frac{dg_{\text{GF}}^2}{dt}. \quad (2)$$

If the YM theory contains fermions, the fermion mass would be a relevant parameter. Unless it is set to zero, its running affects g_{GF}^2 . Equation (2) therefore defines the standard RG β function only in the chiral limit. Attempts to determine the RG β function using lattice simulations are usually performed with $am_f = 0$ in small volumes in the deconfined regime, where simulations with massless fermions are feasible. However, in the confined regime finite mass simulations might be necessary. In that case the chiral $am_f \rightarrow 0$ limit has to be taken first [7].

Similarly, the volume is a relevant parameter. If it is kept finite in physical units, the Callan-Symanzik RG equation contains a term proportional to Ldg^2/dL . The volume dependence of g_{GF}^2 is non-trivial, and absorbing Ldg^2/dL would modify the properties of the β function. The finite volume correction is present even if the flow time is tied to the volume, i.e. choosing $\sqrt{8t} = cL$. In step-scaling studies this issue is avoided by scaling both the flow time and the volume at the same time. For scale change s the GF step-scaling β function is [6]

$$\beta_{c,s}(u) = \frac{g_{\text{GF}}^2(s^2t, sL; g_0^2) - g_{\text{GF}}^2(t, L; g_0^2)}{\log(s^2)} \Bigg|_{u=g_{\text{GF}}^2(t, L; g_0^2)} \quad (3)$$

To avoid introducing an unknown volume-dependent function when using Eq. (2), the first step in obtaining the continuous β function is to perform an extrapolation to the $L/a \rightarrow \infty$ limit. We perform the infinite volume extrapolation at fixed lattice flow time t/a^2 and fixed bare coupling $\beta = 6/g_0^2$. Thus the resulting β function corresponds to the $c = \sqrt{8t}/L = 0$ GF renormalization scheme [6, 8, 9]¹. The continuum physics along the renormalized trajectory (RT) may be extracted in the second step, the ‘‘infinite lattice flow time’’ extrapolation ($a^2/t \rightarrow 0$) at fixed g_{GF}^2 . By fixing the renormalized coupling we effectively fix the flow time t . Taking the lattice flow time $t/a^2 \rightarrow \infty$ at fixed t is equivalent to the usual $a \rightarrow 0$ continuum limit and automatically tunes the bare coupling g_0^2 to zero. In summary, the continuous β function analysis proceeds via the following steps:

1. Infinite volume extrapolation at fixed lattice flow time t/a^2 and bare coupling $\beta = 6/g_0^2$.
2. Continuum limit extrapolation ($a^2/t \rightarrow 0$) at fixed g_{GF}^2 .

The step-scaling approach to calculating the RG β functions on the lattice [6, 10–16] requires that the lattice size L provides the only energy scale in the system. In contrast the continuous β function [7–9] requires an infinite volume extrapolation but permits the existence of a physical energy scale that might be generated by the underlying dynamics of the system. An example of such a scenario is the 4-dimensional pure gauge $SU(3)$ YM system. Here, dimensional transmutation gives rise to an external energy scale Λ_{QCD} that characterizes infrared behavior of the system.

In these proceedings we aim to demonstrate the properties of the continuous β function in the 4-dimensional pure gauge $SU(3)$ YM system. This system has been studied extensively with step-scaling up to $g_{\text{GF}}^2/4\pi \sim 1$ [17]. We consider volumes and bare couplings both in the deconfined (small volume, weak coupling) and in the confining (large volume, strong coupling) regimes. Figure 1 presents our preliminary results. The rainbow colored data points show the predicted $\beta_{\text{GF}}(\alpha_{\text{GF}})$ as a function of the running coupling $\alpha_{\text{GF}} \equiv g_{\text{GF}}^2/4\pi$ at several bare coupling values on 32^4 lattice volumes using Zeuthen flow and the Symanzik operator (ZS)². In addition the continuum limit extrapolated results both for Zeuthen (magenta line) and Wilson (cyan line) flow are shown, together with the 1- and 2-loop universal predictions and the 3-loop prediction within the GF [18] and $\overline{\text{MS}}$ schemes. A striking feature of Fig. 1 is the smooth curve mapped out by the rainbow colored data. These data points represent ‘‘raw’’ numerical results using Eqs. (1) and (2). They have not been altered in any way, in particular they are not extrapolated to the continuum limit. Nonetheless,

¹An alternative approach to the infinite volume extrapolation is to fix g_{GF}^2 instead of the bare coupling β (cf. Sec. 2).

²We will define the various gradient flows, operators and other simulation details in Sect. 2

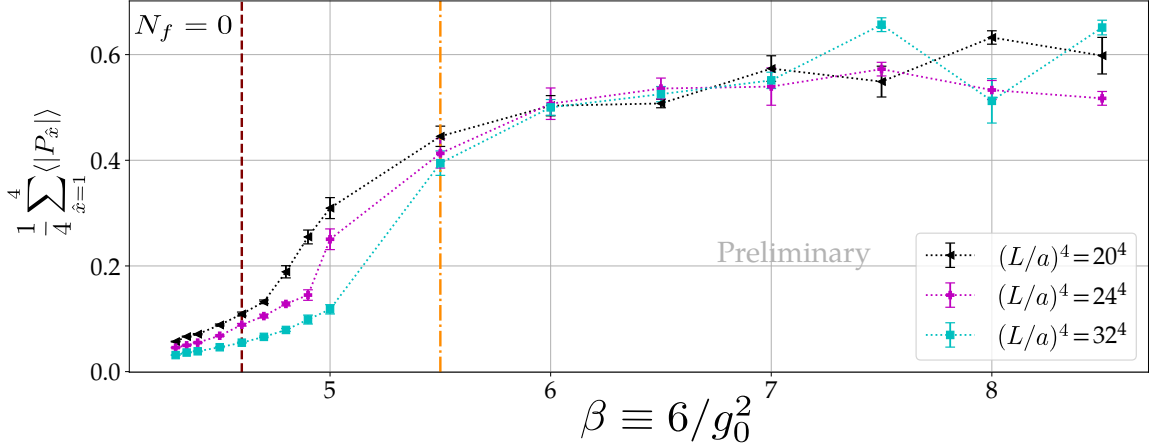


Figure 2: The magnitude of the Polyakov loop as a function of the bare coupling β for our three volumes. Symbols with the same color represent data at the same volume and are connected with a dotted line to guide the eye. Below the left dashed vertical line ($\beta \leq 4.6$) all our volumes are confined while above $\beta \geq 5.5$ (right dash-dotted vertical line) all are deconfined.

these raw data points match very closely the infinite volume continuum limit extrapolated curve, indicating that the chosen action-flow-operator combination has very small cutoff effects. This is expected in the weak coupling perturbative regime since Zeuthen flow is $O(a^2)$ improved [19, 20]. Therefore the combination of Symanzik gauge action, Zeuthen flow and Symanzik operator has no tree-level $O(a^2)$ correction. However, surprisingly, this feature persists even up to $\alpha_{\text{GF}} \lesssim 2.0$, well into the confining regime. We also observe that the β function follows the 3-loop GF prediction up to about $\alpha_{\text{GF}} \lesssim 0.3$, where the GF prediction peels off. The numerical result follows closely the 1-loop universal curve up to $\alpha_{\text{GF}} \lesssim 1.0$, but predicts an even slower running beyond. In the next Section we describe our analysis in greater detail and explain how the continuum limit result shown in Fig. 1 is obtained.

2. Simulation Details

Our study is based on simulations performed using the tree-level improved Symanzik (Lüscher-Weisz) gauge action [21, 22] and three lattice volumes of size $(L/a)^4 = 20^4$, 24^4 and 32^4 . The analysis in the confined phase is based on five bare gauge couplings ($\beta = 6/g_0^2 = 4.3, 4.35, 4.4, 4.5, 4.6$). In the deconfined phase, we use seven bare gauge couplings (5.5, 6.0, 6.5, 7.0, 7.5, 8.0, 8.5). The division of the simulations to confining/deconfining regimes is based on the Polyakov loop, an order parameter in pure gauge systems. In Fig. 2 we show the magnitude of the Polyakov loop $P_{\hat{x}}$ (averaged over all four directions \hat{x} of the lattice) as a function of the bare coupling β for our three volumes. Because simulations are performed in a finite box, we do not observe a *true* phase transition. However, there is a clear drop in the magnitude of the Polyakov loop as a function of β on each volume. Scatter plots of $P_{\hat{x}}$ at each gauge configuration in the Argand plane indicate that all of our volumes with $\beta \leq 4.6$ (at or below the dashed maroon vertical line) are confined, while all volumes with $\beta \geq 5.5$ (at or above the dash-dotted orange vertical line) are deconfined.

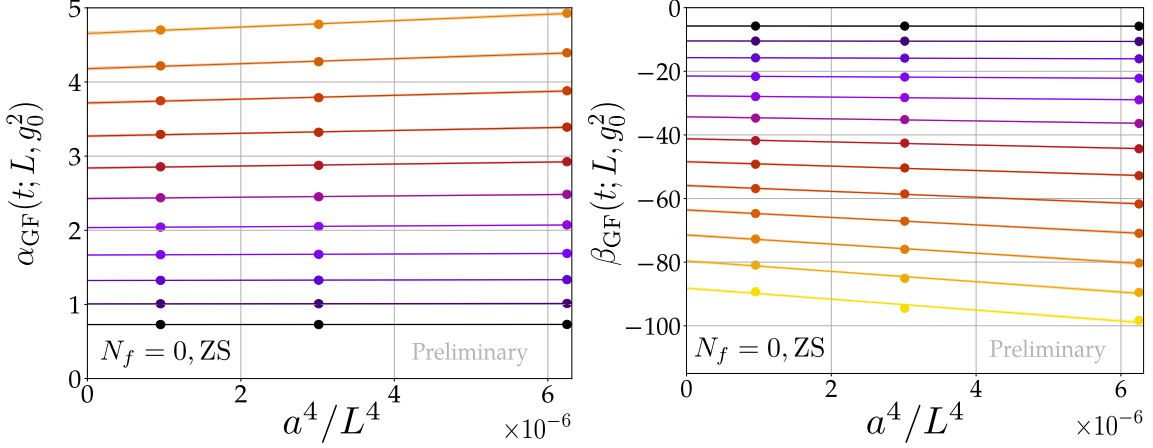


Figure 3: Example of the infinite volume extrapolation for $\alpha_{\text{GF}}(t; L, g_0^2)$ (left) and $\beta_{\text{GF}}(t; L, g_0^2)$ (right) at $\beta \equiv 6/g_0^2 = 4.3$ for the ZS combination. Different colors correspond to different flow times, with the flow time starting at $t/a^2 = 1.0$ (black) and ending at $t/a^2 = 7.72$ (yellow). The contributing lattice volumes are $(L/a)^4 = 20^4, 24^4$ and 32^4 .

We express the GF coupling as $\alpha_{\text{GF}} \equiv g_{\text{GF}}^2/4\pi$ and use the definition [6]

$$\alpha_{\text{GF}}(t; L, g_0^2) = \frac{32\pi}{3(N_c^2 - 1)} \frac{1}{1 + \delta(t; L)} \langle t^2 E(t) \rangle, \quad (4)$$

where the factor $1 + \delta(t; L)$ corrects for gauge zero modes that appear when imposing periodic boundary conditions in a finite box [6]. We take the logarithmic derivative of $g_{\text{GF}}^2 = 4\pi\alpha_{\text{GF}}$ with respect to the lattice flow time in Eq. (2) using a 5-point stencil.

Our gauge configurations are generated using GRID [23, 24] and we perform gradient flow measurements using QLUA [25, 26]. We consider three operators, Wilson plaquette (W), Symanzik (S) and clover (C) to calculate the energy density $E(t)$ using both Wilson flow (W) and Zeuthen flow (Z). From now on we use the shorthand notation ‘flow-operator’ to refer to a given combination, e.g. ‘ZS’ refers to Zeuthen flow and Symanzik operator.

2.1 Infinite volume extrapolation

The infinite volume extrapolation predicts β_{GF} as a function of α_{GF} at fixed lattice flow time t/a^2 as $L^2/t \rightarrow \infty$. This can be achieved by interpolating the finite volume data $\alpha_{\text{GF}}(t; L, g_0^2)$ and $\beta_{\text{GF}}(t; L, g_0^2)$ in the bare coupling and taking the infinite volume limit at fixed α_{GF} and t/a^2 [8, 9]. The interpolation is necessary in slowly running systems, but can be avoided if the running coupling evolves fast and predictions from different bare couplings show significant overlap. Here we apply the infinite volume extrapolation to *both* $\alpha_{\text{GF}}(t; L, g_0^2)$ and $\beta_{\text{GF}}(t; L, g_0^2)$ at fixed bare coupling g_0^2 and lattice flow time t/a^2 . Such a procedure is well-justified, as long as the leading t/L^2 finite volume corrections are small even for the smallest volume used in the extrapolation. Alternatively, one may first extrapolate α_{GF} in Eq. (4) to the infinite volume limit, then plug the result into Eq. (2). We find that both approaches agree within error over a reasonable range of t/a^2 . However, the

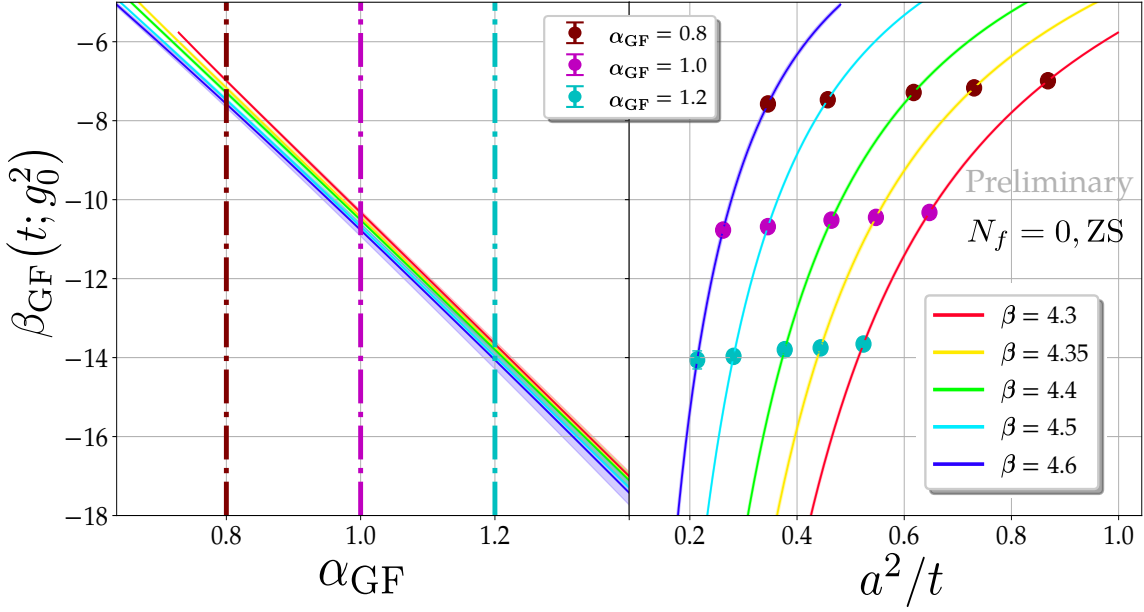


Figure 4: Example of the continuum extrapolation for Zeuthen flow. On the left panel we show the infinite volume extrapolated $\beta_{\text{GF}}(t; g_0^2)$ as a function of α_{GF} with the Symanzik operator. At sample values of α_{GF} (0.8 (maroon), 1.0 (magenta) and 1.2 (cyan)) we identify $(\beta_{\text{GF}}, t/a^2)$ pairs at available bare couplings $\beta \equiv 6/g_0^2$. On the right panel we show these pairs.

former method tends to be numerically more stable at larger lattice flow times t/a^2 that are well below the threshold for non-negligible t/L^2 effects.

Motivated by the finite-volume scaling behavior of the β function in the deconfined phase [8, 9] we choose our extrapolating function to be linear in a^4/L^4 . We have also explored a number of alternative fitting functions, including polynomial dependencies on a/L , exponential forms and “resummed” functional forms with a generic $k_1/(1+k_2(a/L)^\alpha)$ ($\alpha = 2$ or 4) structure. In all cases, fits to $a_1 + a_2(a/L)^4$ and $b_1/(1+b_2(a/L)^4)$ exhibit good p -values over a wide range of lattice flow times. The final results for both functional forms agree within errors and in these proceedings we present results only for linear fits to $(a/L)^4$. Investigations of other possible fitting functions are still underway.

Figure 3 shows an example of the infinite volume extrapolation at $\beta = 4.3$ using the ZS flow-operator combination. The small slopes for both α_{GF} and β_{GF} over the entire range of lattice flow times considered confirms that finite-volume effects are small in the confined phase. Small finite-volume effects allow the use of larger lattice flow times t/a^2 . In turn, simulations in the confined phase cover more of the renormalized trajectory. Finite volume effects in the deconfined phase, where the lattice size provides the only infrared cutoff, are expected to be larger. We observe that finite volume effects become significant at much smaller t/L^2 values in the deconfined regime than in the confining regime. The larger finite-volume corrections in the deconfined regime constrain the flow time values that can be used in the continuum extrapolation.

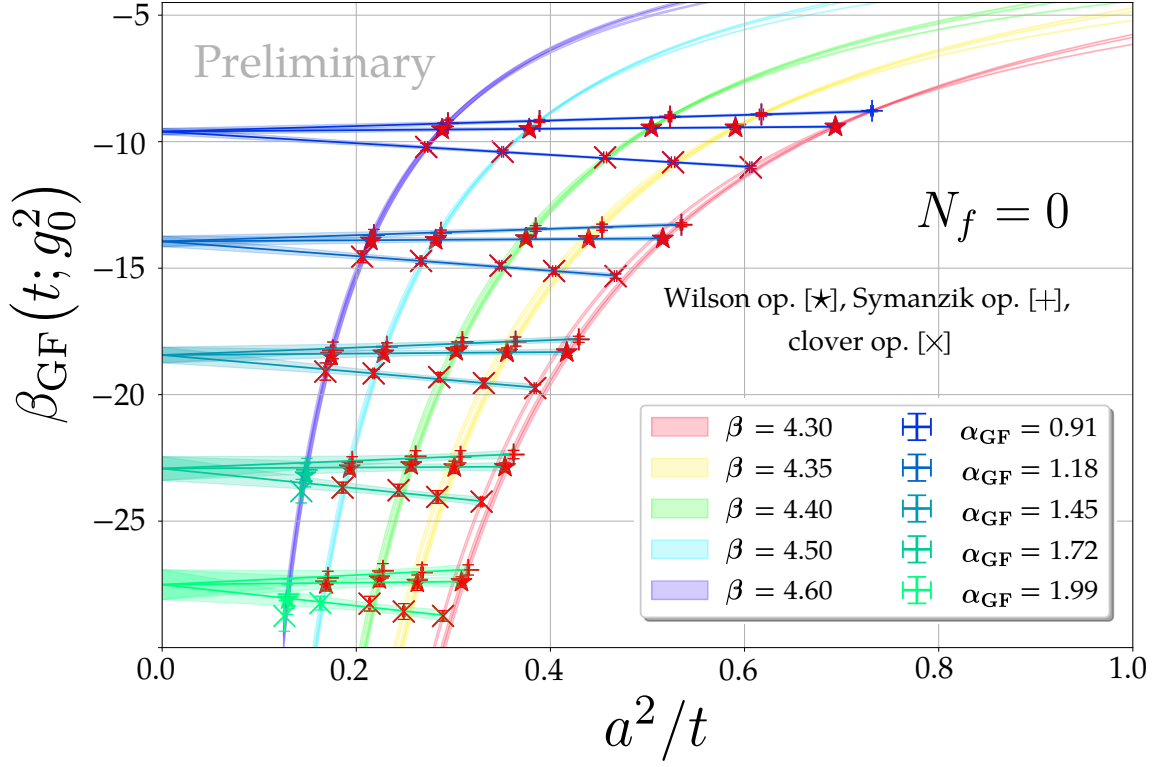


Figure 5: Results of a simultaneous linear fit to all three operators. The straight lines (blue to green) correspond to the extrapolation (with statistical errors only) at different fixed α_{GF} . Red symbols indicate data points that were used to obtain each fit over the lattice flow time range $t/a^2 \in [1.42, 6.0]$.

2.2 Continuum extrapolation

We determine the β function along the renormalized trajectory by removing the lattice cutoff (taking the continuum limit) at a given GF coupling α_{GF} . In-between renormalization group fixed points (RGFP) the mapping $t \mapsto \alpha_{\text{GF}}(t)$ is bijective. Therefore, fixing α_{GF} determines the dimensionful flow time t uniquely. One can then take the continuum limit at a given α_{GF} simply by extrapolating $\beta_{\text{GF}}(t; g_0^2)$ to $a^2/t \rightarrow 0$.

The left panel of Fig. 4 demonstrates this procedure for Zeuthen flow and Symanzik operator. On the left side, we fix α_{GF} at 0.8 (maroon), 1.0 (magenta) and 1.2 (cyan). These fixed values of the coupling will intersect the β functions determined with different bare gauge couplings (i.e., different lattice cutoffs), and we can ask at what value of t/a^2 does each α_{GF} intersect $\beta_{\text{GF}}(t; g_0^2)$. By collecting each of these t/a^2 values, we arrive at the right panel, where we plot $\beta_{\text{GF}}(t; g_0^2)$ as a function of a^2/t . The different colored symbols (maroon, magenta and cyan) represent the data points at which each fixed α_{GF} intersects $\beta_{\text{GF}}(t; g_0^2)$. We see that $\beta_{\text{GF}}(t; g_0^2)$ is roughly linear in a^2/t , i.e. cutoff effects scale with a^2 .

In practice, we obtain very good results from a simultaneous linear fit to all three operators at fixed α_{GF} , as illustrated in Fig. 5. The data are highly correlated and we use an SVD cut to tame the correlation matrix. The range of t/a^2 values entering the continuum extrapolation fit must be chosen with care. Lattice artifacts may contaminate small flow times. At large flow times residual

finite volume effects can enter due to an imperfect infinite volume extrapolation. In the confined phase, we find that our continuum extrapolation of $\beta_{\text{GF}}(t; g_0^2)$ is consistent with a linear dependence over $t/a^2 \in [1.42, 6.0]$. Removing the first or last bare gauge couplings that contribute to the continuum limit does not significantly impact the central value, but, depending on the number of data points used, could affect the errors.

Despite each flow having different cutoff effects, different flows should give the same continuum limit. Therefore, as a consistency check, we compare predictions of different gradient flows against each other to ensure they agree. This is indeed the case, as indicated by the overlap of the cyan and magenta curves in Fig. 1.

3. Conclusions and Further Prospects

We have calculated the continuous β function in the deconfined and confined phases of a pure gauge Yang-Mills system using the gradient flowed gauge coupling to determine the renormalized running coupling. The advantage of the continuous β function over the more traditional step-scaling function is that it can be used even in the confining phase where dimensional transmutation introduces a physical energy scale. In addition we demonstrate that large volume simulations in a fully $\mathcal{O}(a^2)$ improved setup have the potential to provide a good approximation of the renormalized trajectory of a 4-dimensional renormalized Yang-Mills system.

In the future, we aim to further explore alternative approaches to extrapolating to the infinite volume limit. Moreover, we would like to predict the continuous β function in the transition region between deconfined and confined regimes. We aim to explore the use of our setup to extract the Λ -parameter and will also compare our findings to alternative approaches (see e.g. [27, 28]).

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References

- [1] R. Narayanan and H. Neuberger, [JHEP](#) **03** (2006) 064 [[hep-th/0601210](#)].
- [2] M. Lüscher, [Commun.Math.Phys.](#) **293** (2010) 899 [[0907.5491](#)].
- [3] M. Lüscher, [JHEP](#) **08** (2010) 071 [[1006.4518](#)].

- [4] R. H. Swendsen, [Phys. Rev. Lett.](#) **42** (1979) 859.
- [5] A. Carosso, A. Hasenfratz and E. T. Neil, [Phys. Rev. Lett.](#) **121** (2018) 201601 [1806.01385].
- [6] Z. Fodor, K. Holland, J. Kuti et al., [JHEP](#) **11** (2012) 007 [1208.1051].
- [7] Z. Fodor, K. Holland, J. Kuti et al., [EPJ Web Conf.](#) **175** (2018) 08027 [1711.04833].
- [8] A. Hasenfratz and O. Witzel, [Phys. Rev. D](#) **101** (2020) 034514 [1910.06408].
- [9] A. Hasenfratz and O. Witzel, [PoS LATTICE2019](#) (2019) 094 [1911.11531].
- [10] M. Lüscher, P. Weisz and U. Wolff, [Nucl. Phys.](#) **B359** (1991) 221.
- [11] Z. Fodor, K. Holland, J. Kuti et al., [PoS LATTICE2012](#) (2012) 050 [1211.3247].
- [12] P. Fritzsche and A. Ramos, [JHEP](#) **10** (2013) 008 [1301.4388].
- [13] Z. Fodor, K. Holland, J. Kuti et al., [JHEP](#) **09** (2014) 018 [1406.0827].
- [14] A. Hasenfratz, C. Rebbi and O. Witzel, [Phys. Lett. B](#) **798** (2019) 134937 [1710.11578].
- [15] A. Hasenfratz, C. Rebbi and O. Witzel, [PoS LATTICE2018](#) (2019) 306 [1810.05176].
- [16] A. Hasenfratz, C. Rebbi and O. Witzel, [Phys. Rev. D](#) **100** (2019) 114508 [1909.05842].
- [17] M. Dalla Brida and A. Ramos, [Eur. Phys. J. C](#) **79** (2019) 720 [1905.05147].
- [18] R. V. Harlander and T. Neumann, [JHEP](#) **06** (2016) 161 [1606.03756].
- [19] S. Sint and A. Ramos, [PoS LATTICE2014](#) (2015) 329 [1411.6706].
- [20] A. Ramos and S. Sint, [Eur. Phys. J.](#) **C76** (2016) 15 [1508.05552].
- [21] M. Lüscher and P. Weisz, [Commun. Math. Phys.](#) **97** (1985) 59.
- [22] M. Lüscher and P. Weisz, [Phys. Lett.](#) **158B** (1985) 250.
- [23] P. Boyle, A. Yamaguchi, G. Cossu et al., [PoS LATTICE2015](#) (2015) 023 [1512.03487].
- [24] P. Boyle, G. Cossu, A. Portelli et al., <https://github.com/paboyle/Grid>, 2015.
- [25] A. Pochinsky, [PoS LATTICE2008](#) (2008) 040.
- [26] A. Pochinsky et al., <https://usqcd.lns.mit.edu/w/index.php/QLUA>, 2008.
- [27] L. Del Debbio and A. Ramos, 2101.04762.
- [28] O. Francesconi, M. Panero and D. Preti, 2108.13444.
- [29] J. Anderson, P. J. Burns, D. Milroy et al., [Proceedings of PEARC17](#) **8** (2017) 1.