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Quantum Counter-Terms for Lattice Field Theory on Curved Manifolds

Evan K. Owen,[a,*] **Casey E. Berger,**[b] **Richard C. Brower,**[a] **George T. Fleming,**[c] **Andrew D. Gasbarro**[d] **and Timothy G. Raben**[e]

[a] *Boston University, Boston, MA 02215, USA*
[b] *Smith College, Northampton, MA 01063, USA*
[c] *Yale University, Sloane Laboratory, New Haven, CT 06520, USA*
[d] *AEC Institute for Theoretical Physics, Universität Bern, 3012 Bern, Switzerland*
[e] *Michigan State University, East Lansing, MI 48824, USA*
E-mail: ekowen@bu.edu, cberger@smith.edu, brower@bu.edu, george.fleming@yale.edu, andrewgasbarro@gmail.com, rabentim@msu.edu

We present the necessity of counter-terms for Quantum Finite Element (QFE) simulations of ϕ^4 theory on non-trivial simplicial manifolds with semi-regular lattice spacing. By computing the local cut-off dependence of UV divergent diagrams we found that the symmetries of the continuum theory are restored for ϕ^4 theory on the manifolds S^2 and $S^2 \times \mathbb{R}$ in the weak coupling regime [1, 2]. Here we consider the construction of non-perturbative local counter-terms in an attempt to approach the strong coupling Wilson-Fisher IR fixed point.

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*Speaker

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1. Introduction

In 2012, Brower, Fleming and Neuberger [3, 4] explored lattice radial quantization of the $d = 3$ Ising spin model on a cylindrical geometry whose cross-section was a discretized icosahedron. They observed critical scaling but the computed critical exponents did not agree with the results of the continuum $d = 3$ Ising conformal field theory (CFT). The conclusion was that the inability of the discretization scheme based on Ising spins to recover full rotational invariance in the continuum limit prevented the recovery of full conformal invariance as well, leading to the discrepancy.

In subsequent work [1, 2, 5], Brower, Fleming and collaborators replaced the discretized icosehedron with a discretized 2-sphere S^2 and replaced the Ising spins with a real scalar field ϕ. The critical ϕ^4 theory is known to be in the same universality class as the Ising model in $d = 2, 3$. In this framework, a discretized classical action for ϕ^4 theory on S^2 can be constructed using the Finite Element Method (FEM) and was shown to converge to the continuum classical action, including full rotational invariance, as the lattice spacing goes to zero [6]. Unfortunately, recovering continuum symmetries of the classical action was insufficient for the recovery of full conformal invariance of the quantum critical theory. This is clearly demonstrated in Figure 1 where an attempt to locate the critical surface using the 4th-order Binder cumulant U_4 fails and the spherical plot on the right reveals the reason: The non-uniform discretization of the sphere leads to position-dependent lattice artifacts that must be renormalized in the interacting theory.

In the Quantum Finite Element (QFE) framework, it was conjectured for sufficiently small bare coupling that a position-dependent counterterm computed in bare lattice perturbation theory and added to the classical action would suffice to restore the full conformal invariance of the critical theory in the continuum limit [7, 8]. This conjecture has been demonstrated numerically for critical ϕ^4 theory on S^2 [2] and $\mathbb{R} \times S^2$ [1]. Despite the apparent success of this approach, it remains unclear why it works at all. The bare lattice coupling λ_0 is related to the dimensionful coupling of the continuum theory λ_{cont} via $\lambda_0 = a\lambda_{\text{cont}}$ as $a \to 0$ so that any finite bare lattice coupling corresponds to infinite coupling in the continuum theory. In [1], critical behavior was observed even in a region where λ_0 was small, but λ_{cont} was clearly in the non-perturbative regime. It is not known why continuum-like perturbative counter-terms work in a regime where continuum perturbation theory obviously fails.

Here, we study ϕ^4 theory on a discretized S^2 lattice and seek to modify the perturbative counter-terms in a manifestly non-perturbative way. We present a simple method of tuning the counter-terms so that critical behavior is observed in the non-perturbative region, but we find evidence that this method is not sufficient to restore full rotational symmetry in the continuum limit.

2. Background

The discretized ϕ^4 action used in our simulations is

$$S = \frac{1}{2} \sum_{\langle xy \rangle} \frac{V_{xy}}{\ell_{xy}^2} \left(\phi_x - \phi_y\right)^2 + \sum_x \sqrt{g_x} \left(\frac{1}{2} m^2 \phi_x^2 + \lambda_0 \phi_x^4\right)$$

where the site- and link-dependent coefficients are geometrical factors determined using the finite element method as described in detail in [2]. We discretize S^2 using a refined icosahedron as shown in Figure 2. The continuum limit is approached by increasing the refinement L.

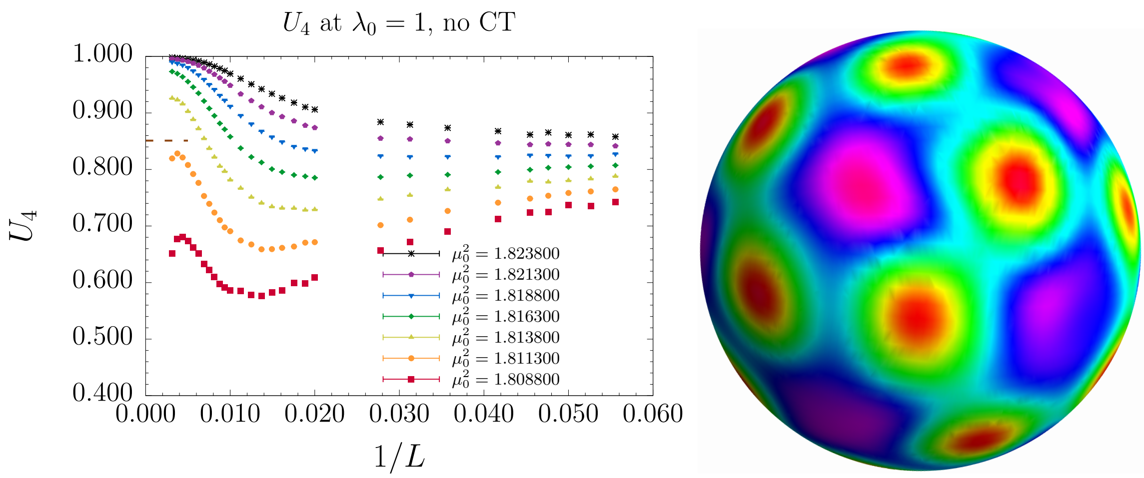

Figure 1: On the left, the fourth-order Binder cumulant does not show critical scaling to the known continuum result as the UV cutoff $\propto 1/L \to 0$ for the unrenormalized action. On the right, a spherical plot of the local deviation of the magnetization $\phi^2(x)$ from the average over the sphere $\overline{\phi^2}$ in the pseudocritical region showing a mixed phase structure with clear icosahedral symmetry. Reprinted from [2].

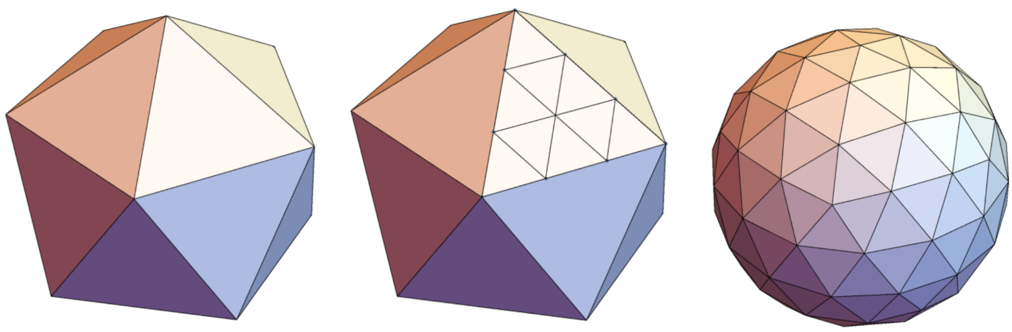

Figure 2: Refined icosahedron used to discretize S^2, shown for a refinement of $L = 3$. Each edge of the icosahedron is subdivided into L segments of equal length, which are then used to form a uniform triangular lattice on each face. The resulting points are then projected onto a unit sphere.

As mentioned in the introduction, this model does not appear to have a stable critical surface in the continuum limit. A second issue, likely related to the first issue, is that spherical symmetry is not restored in the continuum limit.

3. Weak Coupling Regime

At weak coupling, the theory can be renormalized to remove UV divergent behavior by adding perturbative mass counter-terms to the action. In $d = 2$, the 1-loop diagram is logarithmically UV divergent and all higher order diagrams are finite. Performing the perturbative expansion in λ_0, renormalization requires the addition of local mass counter-terms. To first order in the bare coupling, and after subtracting off a position-independent logarithmically divergent piece, the mass counter-terms have the form

$$\delta m_x^2 = 6Q\lambda_0 \log\left(\sqrt{g_x}\right) \tag{1}$$

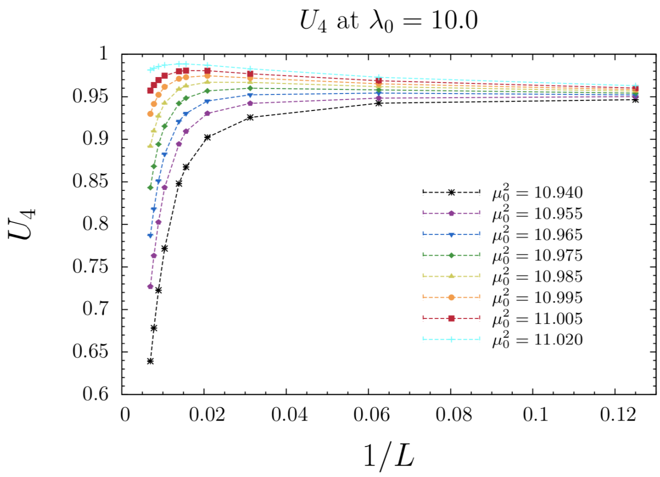

Figure 3: At $\lambda_0 = 10$ with perturbative counter-terms, the Binder cumulant U_4 does not approach the continuum value $U_4^* \simeq 0.851\ldots$ as the lattice spacing is taken to zero. Here and in Figure 1, μ_0^2 is related to the bare mass by $m^2 = -2\mu_0^2$. Reprinted from [10].

where $Q = \sqrt{3}/8\pi$ and $\sqrt{g_x}$ is the normalized area associated with site x. The new lattice action is

$$S \to S + \sum_x \sqrt{g_x}\delta m_x^2 \phi_x^2 \tag{2}$$

As noted in the introduction, it has been shown numerically that this renormalization effectively stabilizes the critical surface and restores spherical symmetry in the perturbative regime ($\lambda_0 \leq 1$) [2].

For larger values of λ_0 however, we find that this renormalization does not stabilize the critical surface. Specifically, we expect the 4th order Binder cumulant U_4 to approach its exact continuum value $U_4^* \simeq 0.851\ldots$ [2, 9] in the continuum limit, where U_4 is defined as

$$U_4 = \frac{3}{2}\left(1 - \frac{\langle m^4\rangle}{3\langle m^2\rangle^2}\right) \tag{3}$$

and m is the magnetization

$$m = \sum_x \sqrt{g_x}\phi_x \tag{4}$$

At $\lambda_0 = 10$ we find that an obstruction begins to appear at a refinement of $L = 64$ that causes U_4 to drop to zero at fixed bare mass. This behavior is shown in Figure 3.

4. Strong Coupling Regime

We conjecture that it may be possible to stabilize the critical surface at strong coupling by modifying the perturbative counter-terms in a manifestly non-perturbative way. As a first attempt, we simply multiply δm^2 from the perturbative expansion by a tunable coefficient $C(\lambda_0)$, i.e.

$$\delta m^2 \to C(\lambda_0)\delta m^2 \tag{5}$$

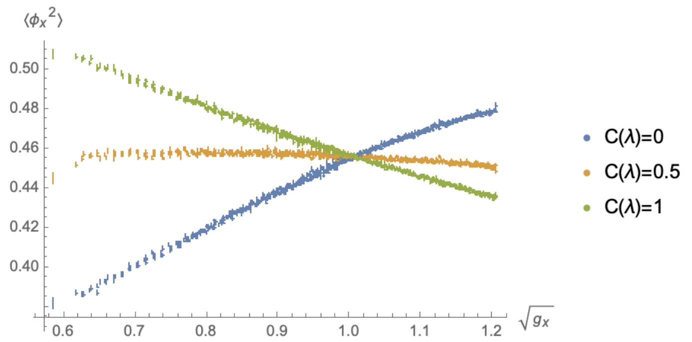

Figure 4: Measurement of the operator $\langle \phi_x^2 \rangle$ at different positions on the discretized S^2 lattice with a bare coupling of $\lambda_0 = 10$ and a refinement of $L = 96$. Smaller values of $\sqrt{g_x}$ are points near the icosahedral vertices, and larger values are near the centers of the icosahedral faces. The data becomes approximately uniform across the sphere when the coefficient $C(\lambda_0 = 10)$ is tuned to about 0.5.

To see the effect of this change, we measure ϕ_x^2 at different sites around the sphere, parametrized by the normalized area associated with each site $\sqrt{g_x}$. Spherical symmetry requires local operators like ϕ_x^2 to be uniform on the sphere, so any non-uniformity of this measurement gives an explicit indication of spherical symmetry breaking. In Figure 4, we show results for three different choices of $C(\lambda_0)$. The measurement is clearly non-uniform in the case without counter-terms as well as the case with unmodified perturbative counter-terms. For the intermediate case, we have found that it is possible to tune the coefficient so that the measurement is approximately uniform everywhere on the sphere.

Because we are only tuning one parameter, there is still some residual position-dependence in the measurement that can be seen as a slight curvature of the data. We therefore expect that this simple approach will not be sufficient to ensure a full restoration of spherical symmetry in the continuum limit. Despite this, we again attempt to approach the continuum limit with our tuned coefficient, keeping the coefficient fixed as the refinement increases. As shown in Figure 5, we can now identify a critical mass for which the Binder cumulant remains constant as the lattice spacing goes to zero. In addition, the Binder cumulant seems to be approaching the correct continuum value.

We repeated this process at several different values of λ_0 to see if this parametrization of the counter-terms is effective at even stronger coupling. We note that the tuning was performed "by eye" using plots similar to Figure 4, rather than by using a numerical method. Determining a more precise tuning method is the subject of future work. We find that the tuned values approximately follow the form

$$C(\lambda_0) = e^{-Q\lambda_0} \tag{6}$$

where $Q = \sqrt{3}/8\pi$ is the same geometric factor from the perturbative counterterm. With this modification, the counter-terms become

$$\delta m_x^2(\lambda_0) = 6Q\lambda_0 e^{-Q\lambda_0} \log\left(\sqrt{g_x}\right) \tag{7}$$

This behavior, shown in Figure 6, has a maximum around $\lambda_0 \simeq 14.5$. This implies that the counter-terms have the largest effect at this coupling, which potentially implies a faster approach

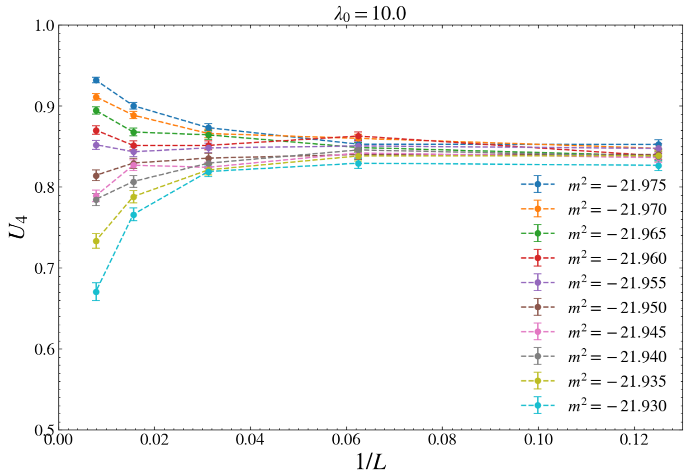

Figure 5: After tuning the perturbative counter-terms to make the operator $\langle \phi_x^2 \rangle$ approximately uniform on the sphere, a mass can be found such that the Binder cumulant U_4 remains constant with increasing refinement and appears to approach the correct continuum value $U_4^* \simeq 0.851\ldots$

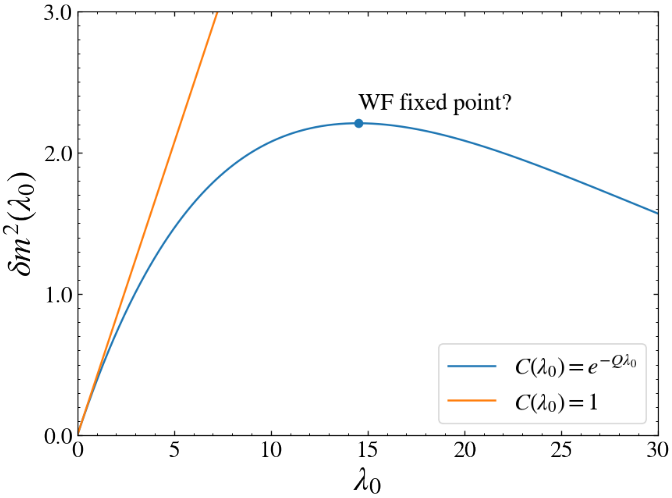

Figure 6: Conjectured form of the mass counter-term obtained by tuning the perturbative counter-term so that the operator $\langle \phi_x^2 \rangle$ is approximately uniform on the sphere.

to the continuum limit as the lattice spacing decreases. We are therefore tempted to guess that this value of the bare coupling might correspond to the Wilson-Fisher fixed point, though we have no additional evidence to prove this conjecture at this time.

The form for the counter-terms at strong coupling in equation 7, though pleasing in its simplicity, seems to be insufficient to ensure a stable critical surface at all values of λ_0, as we will now show. In particular, it may be that the reappearance of the factor Q is merely a coincidence. We have shown that this method greatly improves the stability of the critical surface at $\lambda_0 = 10$. However, at $\lambda_0 = 20$ we find that the obstruction still exists at a refinement above about $L = 96$, as shown in Figure 7. It could be the case that additional tuning of $C(\lambda_0)$ would further improve the results at stronger coupling, or perhaps additional counter-terms for other terms in the action are required.

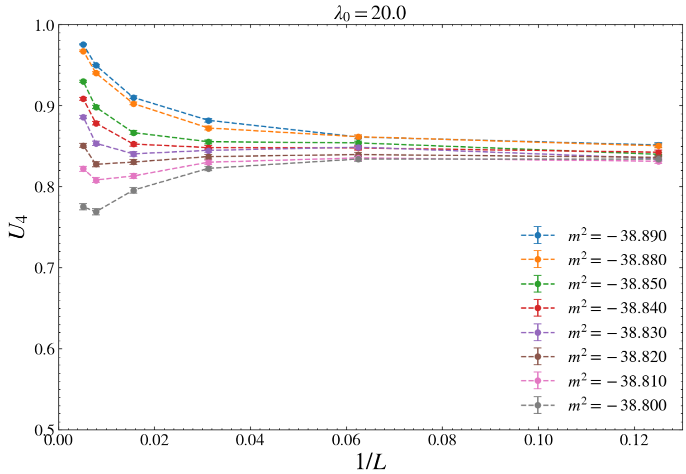

Figure 7: Same as Figure 5 but at $\lambda_0 = 20$, using the counterterm coefficient given in equation 6. The Binder cumulant does not remain constant for any value of the mass as the lattice spacing is taken to zero.

5. Conclusion

We have presented a method for modifying perturbative counter-terms for ϕ^4 theory on a discretized S^2 lattice which reduces residual breaking of spherical symmetry in the strong coupling regime. We have also shown that the modified counter-terms result in a stable critical surface at higher refinement when compared to either using perturbative counter-terms without modification or not using counter-terms at all. We have also found indications that there is a value of the bare coupling at which the effect of the counter-terms is strongest, possibly indicating that this coupling corresponds to the Wilson-Fisher fixed point of the theory.

We note that the simple method we used to modify the perturbative counter-terms was only an improvement. It did not completely restore spherical symmetry, nor did it lead to a well-defined critical surface at all values of the bare coupling. It is possible that a more complex generalization of this method could fully restore spherical symmetry and stabilize the critical surface in the continuum limit. Future work is required to constrain the possible form of such a generalization.

6. Acknowledgements

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