# Finite temperature and $\delta$-regime in the 2-flavor Schwinger model

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The Schwinger model is often used as a testbed for conceptual and numerical approaches in lattice field theory. Still, some of its rich physical properties in anisotropic volumes have not yet been explored. For the multi-flavor finite temperature Schwinger model there is an approximate solution by Hosotani *et al.* based on bosonization. We perform lattice simulations and check the validity of this approximation in the case of two flavors. Next we exchange the rôle of the coordinates to enter the $\delta$-regime, and measure the dependence of the residual "pion" mass on the spatial size, at zero temperature. Our results show that universal features, which were derived by Leutwyler, Hasenfratz and Niedermayer referring to quasi-spontaneous symmetry breaking in $d > 2$, extend even to $d = 2$. This enables the computation of the Schwinger model counterpart of the pion decay constant $F_\pi$. It is consistent with an earlier determination by Harada *et al.* who considered the divergence of the axial current in a light-cone formulation, and with analytical results that we conjecture from 2d versions of the Witten–Veneziano formula and the Gell-Mann–Oakes–Renner relation, which suggest $F_\pi = 1/\sqrt{2\pi}$.

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## 1. The multi-flavor Schwinger model

The Schwinger model represents Quantum Electrodynamics — fermions coupled to an Abelian gauge field — in $d = 2$ space-time dimensions [1]. The original 1-flavor version was solved analytically, which revealed in particular an axial anomaly, whereas for $N_f > 1$ flavors the chiral condensate vanishes in the chiral limit. The multi-flavor version is still of interest, as we see from a number of contributions to this conference.

The Schwinger model shares qualitative features with QCD, in particular confinement and topology of the gauge field configurations. It does not capture, however, asymptotic freedom (the gauge coupling constant $g$ is indeed constant), nor spontaneous chiral symmetry breaking. Still, the spectrum contains $N_f - 1$ light bosons; at finite fermion mass and/or in finite volume, their behavior is similar to quasi-Nambu-Goldstone bosons. By analogy, and in agreement with the literature, we denote them as "pions".

In addition, the particle spectrum contains a heavier boson, which can be interpreted as the "photon", but — following another analogy — it is often denoted as the "$\eta$-meson". Since it is a flavor singlet, its 3-flavor QCD analogue is $\eta_1$, which is close to $\eta'$, but in the Schwinger model we also just call it $\eta$. In the chiral limit, its mass is given by [2]

$$m_\eta^2 = \frac{N_f g^2}{\pi} \tag{1}$$

(the coupling $g$ has mass dimension 1), while the pion is massless. At finite (degenerate) fermion mass $m$, no exact solution is known, but an approximate solution predicts the pion mass in infinite volume as [3]

$$m_\pi = 4e^{2\gamma} \sqrt{\frac{2}{\pi}} \, (m^2 g)^{1/3} = 2.1633 \ldots (m^2 g)^{1/3} \,, \tag{2}$$

where $\gamma = 0.577 \ldots$ is Euler's constant.

We present simulation results on regular, Euclidean lattices, with Wilson fermions and the plaquette gauge action, obtained with the Hybrid Monte Carlo algorithm. We are particularly interested in anisotropic volumes: first, we study this model at finite temperature, and compare the "meson" masses with theoretical predictions. A bosonization ansatz reduces the system to a quantum mechanical problem [3, 4], which we solve numerically.

By inverting the rôle of the coordinates, we access the $\delta$-regime, which is still unexplored in $d = 2$. We conjecture, and confirm, a residual pion mass $m_\pi^R \propto 1/L$ at $m = 0$. The proportionality constant provides a value for a parameter, which we denote — by analogy — as the "pion decay constant" $F_\pi$. It is dimensionless in $d = 2$, and its value is consistent with the Witten-Veneziano relation (if we identify $F_\pi = F_\eta$), and with the Gell-Mann–Oakes–Renner relation. It further agrees with a previous determination in the framework of a light-cone formulation, which refers to the divergence of the axial current [5].

## 2. The masses $m_\pi$ and $m_\eta$ at finite temperature

In the 1990s, Hetrick, Hosotani and Iso discussed the bosonization of the multi-flavor Schwinger model [3, 4]. Here we particularly refer to a system of non-linear differential equations given in

Refs. [7], which represent a quantum mechanical description of the model at finite temperature. These are Schrödinger-type equations, which imply the values of $m_\pi$, $m_\eta$ and the chiral condensate $\Sigma$, as functions of the degenerate fermion mass $m$. We solved them numerically, as an eigenvalue problem, by three numerical methods. They lead to consistent results, which stabilize for increasing matrix size. Figure 1 shows these results for $m_\pi$ and $m_\eta$, as functions of $m$, for $N_f = 2$ flavors.[1] The pion mass is compared to the approximation of eq. (2), which predicts a larger (smaller) $m_\pi$ at small (moderate) $m$.

As a test, we measured $m_\pi$ and $m_\eta$ by simulations on a $L_t \times L = 10 \times 64$ lattice. In this case, the (renormalized) fermion mass is measured by referring to the PCAC relation. Simulations at various values of $\beta \equiv 1/g^2$ show that the lattice artifacts are small at $\beta = 4$.[2] The notorious problems close to the chiral limit prevent reliable results at $m \lesssim 0.02$. In the range of $0.02 \lesssim m \lesssim 0.05$ the bosonization prediction is compatible with the simulation results. At larger fermion mass, however, this approximation significantly overestimates both $m_\pi$ and $m_\eta$. On the other hand, around $m \approx 0.2$ formula (2) for $m_\pi$ is in agreement with the simulation results.
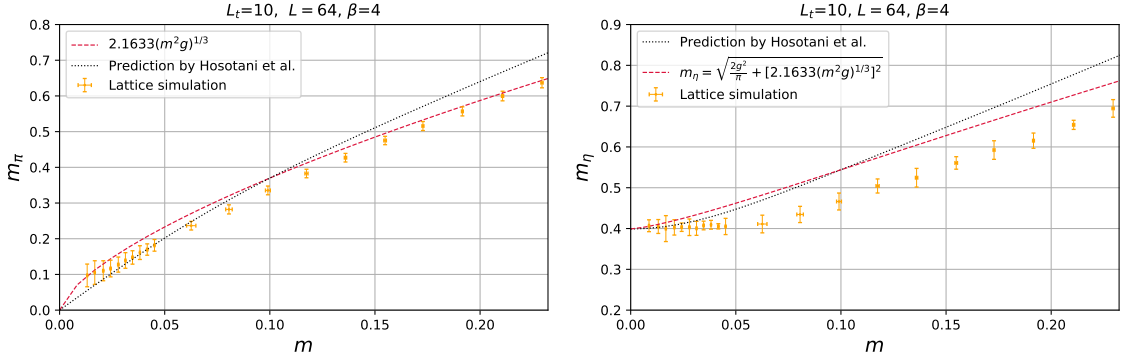


**Figure 1:** The masses $m_\pi$ and $m_\eta$ as functions of fermion mass $m$, at finite temperature, obtained from the approximation (2) of Ref. [3], from a bosonization method [7], and from lattice simulations.

The formulae of Refs. [7] also include the case of an arbitrary vacuum angle $\theta$, which could be of interest to probe simulation methods which try to overcome the sign problem. However, here we see that these formulae are only reliable at small $m$, where the simulations are confronted with additional difficulties.

## 3. Residual pion mass in the δ-regime

Chiral perturbation theory, as a systematic effective low-energy theory for QCD, distinguishes the regimes of large space-time volume ($p$-regime), small space-time volume ($\epsilon$-regime) and small spatial volume but a large extent in (Euclidean) time, $L \ll L_t$ ($\delta$-regime); the length scale is set by the inverse pion mass.

Here we address the δ-regime, which is least explored, and where finite-size effects entail a residual pion mass $m_\pi^R$ even in the chiral limit. It was introduced by Leutwyler [8], who approximated the quasi-1d system by quantum mechanics, such that $m_\pi^R$ corresponds to the mass gap of a quantum

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[1]For the chiral condensate, obtained from bosonization, we refer to Ref. [6].

[2]We are using lattice units. For a general lattice spacing $a$, this relation takes the form $\beta \equiv 1/(ag^2)$.

rotor,

$$m_\pi^{\rm R} = \frac{N_\pi}{2\Theta} , \quad \Theta \simeq F_\pi^2 L^3 . \tag{3}$$

$N_\pi$ is the number of pions, and $\Theta$ is the moment of inertia, which is given here to leading order, in $d = 4$ [8].

In the framework of O($N$) models, with $N_\pi = N - 1$, Hasenfratz and Niedermayer generalized this formula with respect to the space-time dimension $d > 2$, and computed $\Theta$ to next-to-leading order [9],

$$\Theta = F_\pi^2 L^{d-1} \left[ 1 + \frac{N_\pi - 1}{2\pi F_\pi^2 L^{d-2}} \left( \frac{d-1}{d-2} + \cdots \right) \right] . \tag{4}$$

We see that $F_\pi$ has the mass dimension $d/2 - 1$. The restriction to $d > 2$ avoids a possible singularity in the last term, in agreement with the concept of would-be Nambu-Goldstone bosons in infinite volume.

There are only few lattice QCD studies in the $\delta$-regime. In the next-to-next-to-leading order, sub-leading low-energy constants appear [10], and the comparison with QCD simulation results led in particular to a reasonable value of the controversial constant $l_3$ [11]. The transition to the $p-$ and $\epsilon$-regime is investigated in Ref. [12].

In our case, the next-to-leading order term has the prefactor $N_\pi - 1 = 0$. We dismiss it, despite the denominator $d - 2$, so we conjecture for the 2-flavor Schwinger model

$$m_\pi^{\rm R} \simeq \frac{1}{2F_\pi^2 L} . \tag{5}$$

In order to test this conjecture, we performed simulations on lattices with spatial size $L \ll L_t = 64$, at $\beta = 3, 4$ and $5$. The value of $m_\pi^{\rm R}$ is obtained by a chirally extrapolated plateau; two examples are illustrated in Figure 2.
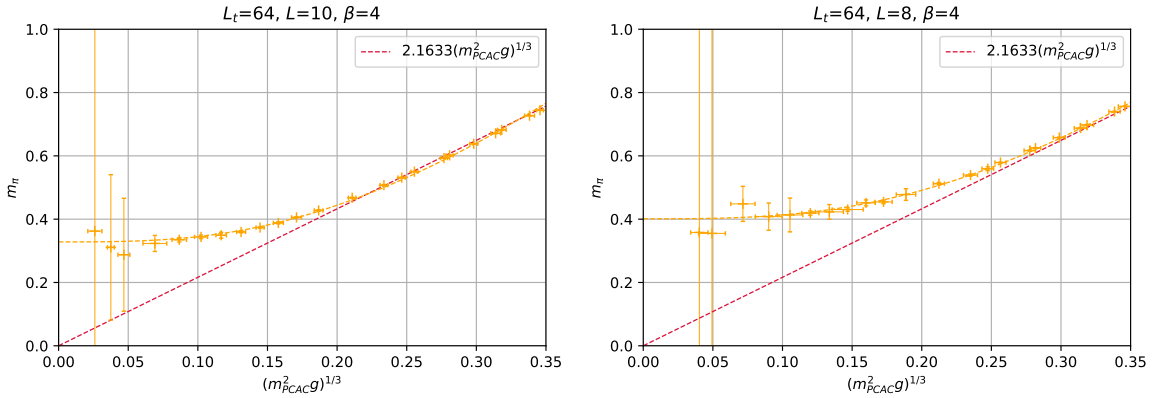


**Figure 2:** Illustration of the residual pion mass plateaux in spatial sizes $L = 10$ and $L = 8$, at $\beta = 4$.

The plots in Figure 3 show an example for the PCAC fermion mass depending on the hopping parameter $\kappa$, and a multitude of results for $m_\pi^{\rm R}$ at fixed $\beta$, but different $L$. We observe good agreement with the conjectured proportionality $m_\pi^{\rm R} \propto 1/L$.

This property allows us to proceed and extract the "pion decay constant" according to eq. (5). The fits at fixed $\beta$ lead to the $F_\pi$-values in Table 1. The results at $\beta = 3, 4$ and $5$ agree to percent
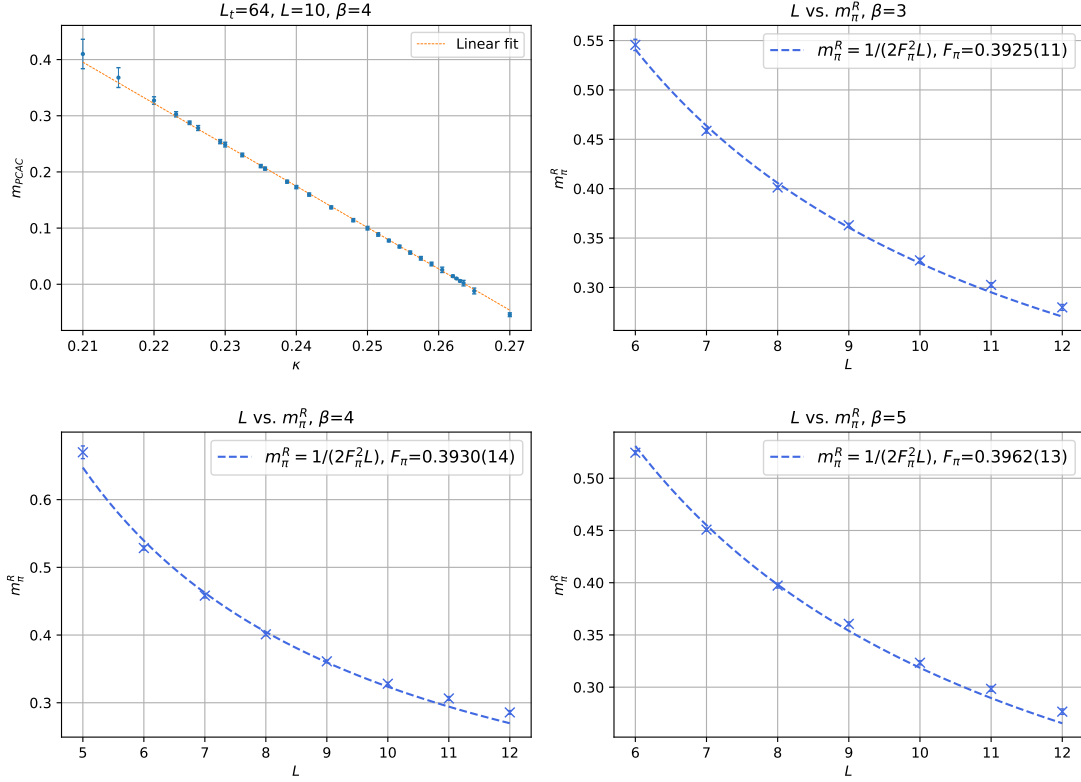
**Figure 3:** Top, left: Fermion mass $m_{\text{PCAC}}$ as a function of the hopping parameter $\kappa$, in one example. Generally we observe an approximately linear behavior. Rest: Residual pion mass $m_\pi^{\text{R}}$ in the $\delta$-regime, as a function of the spatial size $L$, at fixed $\beta \equiv 1/g^2$. The simulation results are consistent with the hypothesis $m_\pi^{\text{R}} \propto 1/L$. In each case, a 1-parameter fit to relation (5) provides the value of $F_\pi$ in Table 1.

| $\beta \equiv 1/g^2$ | 3 | 4 | 5 |
|---|---|---|---|
| $F_\pi$ | 0.3925(11) | 0.3930(14) | 0.3962(13) |

**Table 1:** Results for $F_\pi$, obtained by fits to eq. (5), at three values of $\beta$.

level, but for increasing $\beta$ (*i.e.* suppressed lattice artifacts) we observe a slight trend up — we will come back to it.

## 4. The 2d Witten–Veneziano formula

In the large-$N_c$ limit of QCD, at finite 't Hooft coupling $g_s\sqrt{N_c}$, the 3-flavor chiral symmetry breaking has the structure $\text{U}(3) \otimes \text{U}(3) \to \text{U}(3)$. This implies 9 Nambu-Goldstone bosons, which include — in addition to the meson octet built of $\pi$, $K$ and $\eta$ — the $\eta'$-meson. If one considers $1/N_c$-corrections, the latter picks up a mass, which (with massless quarks $u$, $d$, $s$) is given by the Witten–Veneziano formula [13], $m_{\eta'}^2 F_{\eta'}^2 = 2N_f \chi_t^{\text{q}}$, where $\chi_t^{\text{q}}$ is the quenched topological susceptibility: to this order, quark loops do not contribute, and $F_{\eta'} = F_\pi$. According to lattice simulation results for $\chi_t^{\text{q}}$, the fact that the $\eta'$-meson is so heavy in Nature (heavier than a nucleon,
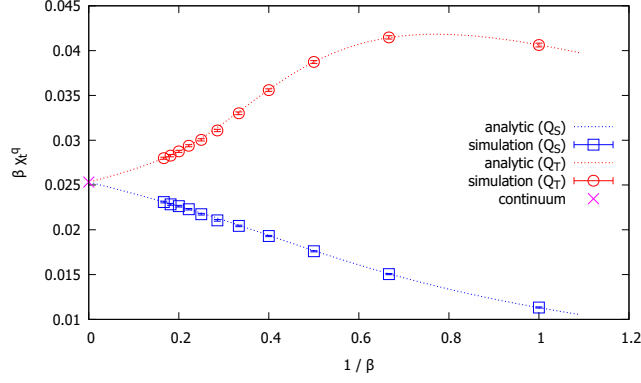
**Figure 4:** The quenched topological susceptibility, for two different lattice formulations of the topological charge, based on the numerical evaluation of implicit expressions, and on simulations.

and therefore not interpretable as a quasi-Nambu-Goldstone boson) can indeed be understood along these lines, as a topological effect.

According to Ref. [14], the conceptual basis of the Witten–Veneziano relation is more solid in the multi-flavor Schwinger model. In the chiral limit it reads

$$m_\eta^2 = \frac{2N_f}{F_\eta^2} \chi_t^q \ . \tag{6}$$

In this case there is no need to speculate (in QCD one assumes $N_c = 3$ to behave similarly to large $N_c$). On the other hand, we do not have any obvious justification for setting $F_\pi = F_\eta$, but we are going to consider this scenario nevertheless.

Ref. [15] computed the topological susceptibility in 2d U(1) pure gauge theory in the continuum, and infinite volume,

$$\beta \chi_t^q = \beta \lim_{V \to \infty} \frac{\langle Q^2 \rangle}{V} = \frac{1}{4\pi^2} \qquad (Q \ : \ \text{topological charge}) \ . \tag{7}$$

This value is consistent with the continuum limit of lattice results for $\chi_t^q$. In particular, Ref. [16] considered the (non-integer) topological lattice charge $Q_S = \frac{1}{2\pi} \sum_P \sin(\theta_P)$, where $\theta_P$ is the plaquette variable, and derived the exact expression $\beta \chi_t^q = I_1(\beta)/[4\pi^2 I_0(\beta)]$.

If we refer to the usual lattice definition $Q_T = \frac{1}{2\pi} \sum_P \theta_P \in \mathbb{Z}$, there is no closed expression for $\beta \chi_t^q$, but it can be evaluated numerically to arbitrary precision [17]. Figure 4 shows both analytic expressions as functions of $1/\beta$. As a consistency check we compare them to simulation results, which accurately agree, and we also see that the continuum limit coincides in both cases with eq. (7).

Thus eq. (7) is confirmed, and along with eqs. (1) and (6) we obtain, in the chiral limit,

$$F_\eta^2 = \frac{2N_f}{m_\eta^2} \chi_t^q = 2N_f \frac{\pi\beta}{N_f} \frac{1}{4\pi^2\beta} = \frac{1}{2\pi} \ , \tag{8}$$

If we assume $F_\pi = F_\eta$, as in large-$N_c$ QCD, we obtain $F_\pi = 1/\sqrt{2\pi} = 0.3989\ldots$, which is close to the value of $F_\pi$ that we obtained in the δ-regime, given in Table 1 — in particular the continuum limit seems perfectly compatible.

Finally we are now going to amplify our perspective and consider $F_\pi$ in the 2-flavor Schwinger model obtained by various formulations.

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## 5. The "pion decay constant" in the Schwinger model: an overview

In QCD, the pion decay constant $F_\pi$ appears in a variety of relations, for instance

(a)   $\langle 0|J_\mu^5(0)|\pi(p)\rangle = \mathrm{i}p_\mu F_\pi$
(b)   $\langle 0|\partial_\mu J_\mu^5(0)|\pi(p)\rangle = m_\pi^2 F_\pi$
(c)   Coefficient to the leading term of $m_\pi^\mathrm{R}(L)$ in the $\delta$-regime
(d)   Witten–Veneziano formula
(e)   Gell-Mann–Oakes–Renner relation.

This list is incomplete, of course, one might add *e.g.* the rôle as a leading Low Energy Constant in the pion effective Lagrangian, the Goldstone-Wilczek current in the effective low-energy theory for the neutral pion decay, or the coefficient of the axial current correlation function in the $\epsilon$-regime (for lattice studies, see *e.g.* Refs. [18]), but in the following we are only going to refer to the relations (a) to (e).

Here the meaning of $F_\pi$ is always the same, but this is not obvious anymore when we refer to one of these relations to introduce — by analogy — a "pion decay constant" in the 2-flavor Schwinger model (although that "pion" does not decay). To the best of our knowledge, the only previous study of this kind was performed in Ref. [5], which referred to relation (b). Working with a light-cone formulation (at $m > 0$), Harada, Sugihara and Taniguchi obtained

$$F_\pi(m) = 0.394518(4) + 0.040(1)m/g \ . \tag{9}$$

In Section 3 we referred to property (c), and from the fits to $m_\pi^\mathrm{R}(L)$ we obtained the values in Table 1, which agree to two digits. Section 4 refers to relation (d), and if we add the hypothesis $F_\pi = F_\eta$, we arrive at $F_\pi = 1/\sqrt{2\pi}$.

Let us finally consider (e), the Gell-Mann–Oakes–Renner relation in the Schwinger model [19]

$$F_\pi^2(m) = \frac{2m\Sigma}{m_\pi^2} \ , \tag{10}$$

where $\Sigma = -\langle\bar\psi\psi\rangle$ is the chiral condensate. Ref. [3] derives explicit small-$m$ formulae for $\Sigma$ and $m_\pi$. In a large volume (and at vacuum angle $\theta = 0$), the latter is consistent with eqs. (1) and (2). Inserting both into the Gell-Mann–Oakes–Renner relation (10) exactly confirms the result that we conjectured in Section 4,

$$\Sigma = \frac{1}{\pi}\left(\frac{e^{4\gamma}m\,m_\eta^2}{4}\right)^{1/3} , \quad m_\pi = \left(4e^{2\gamma}m^2 m_\eta\right)^{1/3} \quad \Rightarrow \quad F_\pi = \frac{1}{\sqrt{2\pi}} \ . \tag{11}$$

In this form, $F_\pi$ does not depend on $m$, nor on $m_\eta$, and therefore neither on the coupling $g$.

Actually Ref. [3] distinguishes (in its eqs. (36) and (38)) three different regimes, depending on mass and size. In eq. (11) we reproduced the formula for $\Sigma$ and $m_\pi$ which are valid if $m\sqrt{m_\eta}L^{3/2} \gg 1$, $m_\pi L \gg 1$ and $m_\eta \gg m_\pi$. Interestingly, when we insert in eq. (10) the formulae in any of the two other regimes, the result for $F_\pi$ is exactly the same.

We conclude that relations (b), (c), (d) and (e) all lead to values for the "pion decay constant" which are consistent with $F_\pi = 1/\sqrt{2\pi}$, which looks highly satisfactory.

We close with two open questions:

- The consideration in Section 4, which refers to property (d), suggests the relation $F_\pi = F_\eta$ in the chiral limit. In fact, Ref. [14] also predicts $F_\eta = 1/\sqrt{2\pi}$ in the chiral limit of the 2-flavor Schwinger model, but its relation to $F_\pi$ remains to be understood.

- Relation (a) is often considered the standard way to define $F_\pi$ in QCD. If we try to employ its analogue to define $F_\pi$ in the Schwinger model, it seems to imply $F_\pi(m = 0) = 0$,[3] since the pions are sterile, *i.e.* free, if we are strictly in the chiral limit (this is how a contradiction with the Mermin-Wagner-Hohenberg-Coleman theorem is evaded [20]). In light of the results presented here, also that aspect remains to be understood.

## Acknowledgments

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