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Finite volume corrections to forward Compton scattering off the nucleon

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We calculate the spin-averaged amplitude for doubly virtual forward Compton scattering off nucleons in the framework of manifestly Lorentz-invariant baryon chiral perturbation theory at complete one-loop order $O(p^4)$. The calculations are carried out both in the infinite and in a finite volume. The obtained results allow for a detailed estimation of the finite-volume corrections to the amplitude which can be extracted on the lattice using the background field technique.

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1. Introduction

The study of the Compton amplitude has gained much attention in recent years. It plays a central role in the analysis of many fundamental problems such as, for example, the evaluation of the Lamb shift in muonic hydrogen [1], or the calculation of the proton-neutron mass difference [2–9]. Hence, the calculation of this amplitude on the lattice would definitely contribute to the solution of the above problems. However, lattice results are always plagued by finite-volume artifacts which may be sizable in some cases. In order to carry out a precise extraction of the Compton amplitude, these finite-volume corrections should be reliably estimated and removed from the lattice data.

Different approaches have been proposed so far for the extraction of the Compton amplitude on the lattice. In this contribution, we shall discuss an approach based on the background field method [10–16]. The calculations are done in Baryon Chiral Perturbation Theory, up-to-and-including $O(p^4)$, where p is a small momentum/mass. Our study is focused on the calculation of the so-called subtraction function, which is related to the Compton amplitude in a particular kinematics. First, the forward doubly virtual Compton scattering amplitude off nucleons is evaluated, and the behavior of the subtraction function at small values of the photon momentum is discussed. Furthermore, the full set of the finite-volume corrections to the subtraction function is evaluated up-to-and-including $O(p^4)$. It is shown that, despite the poorly known low-energy constants at this order, the finite-volume artifacts can be evaluated quite accurately and do not preclude one from an accurate measurement of the subtraction function on the lattice.

2. Infinite volume Comptom scattering

2.1 Doubly-virtual Compton scattering in forward direction in the infinite volume

In this contribution we follow the notations of Ref. [5]. We define the unpolarized forward Comptom scattering amplitude as an average over the nucleon spins:

$$T^{\mu\nu}(p,q) = \frac{i}{4}\sum_s \int d^4x\, e^{iq\cdot x}\langle p,s|Tj^\mu(x)j^\nu(0)|p,s\rangle\,, \tag{1}$$

where (p,s) are the four-momenta and spin projections of incoming and outgoing nucleon, respectively, and q is the momentum of the (virtual) photons in the initial and final state, respectively. Further, j^μ denotes the electromagnetic current.

Using Lorentz-invariance, current conservation and parity, this amplitude can be expressed through two invariant amplitudes:

$$\begin{aligned}
T^{\mu\nu}(p,q) &= T_1(\nu,q^2)K_1^{\mu\nu} + T_2(\nu,q^2)K_2^{\mu\nu}\,,\\
K_1^{\mu\nu} &= q^\mu q^\nu - g^{\mu\nu}q^2\,,\\
K_2^{\mu\nu} &= \frac{1}{m^2}\left\{(p^\mu q^\nu + p^\nu q^\mu)\,p\cdot q - g^{\mu\nu}(p\cdot q)^2 - p^\mu p^\nu q^2\right\}\,,
\end{aligned} \tag{2}$$

where $\nu = p\cdot q/m$ and m is the nucleon mass.

Further, the invariant amplitudes can be split into the elastic and inelastic parts. Here we would like to mention that the definition, which is used in the following, unambiguously follows from the

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requirement that the elastic amplitude vanishes in the limit $\nu \to \infty$, for fixed q^2, and thus obeys an unsubtracted dispersion relation in the variable ν. Under this requirement, the elastic part is given by:

$$\begin{aligned} T_1^{\mathrm{el}}(\nu,q^2) &= \frac{4m^2q^2\left\{G_E^2(q^2)-G_M^2(q^2)\right\}}{(4m^2\nu^2-q^4)(4m^2-q^2)}\,, \\ T_2^{\mathrm{el}}(\nu,q^2) &= -\frac{4m^2\left\{4m^2G_E^2(q^2)-q^2G_M^2(q^2)\right\}}{(4m^2\nu^2-q^4)(4m^2-q^2)}\,, \end{aligned} \tag{3}$$

where G_E, G_M denote the electric and magnetic (Sachs) form factors of the nucleon.

The inelastic invariant amplitudes are defined as $T_i^{\mathrm{inel}} = T_i - T_i^{\mathrm{el}}$, with $i = 1, 2$. The amplitudes T_i^{inel} obey dispersion relations in the variable ν:

$$\begin{aligned} T_1^{\mathrm{inel}}(\nu,q^2) &= T_1^{\mathrm{inel}}(\nu_0,q^2)+2(\nu^2-\nu_0^2)\int_{\nu_{\mathrm{th}}}^{\infty}\frac{\nu' d\nu'\, V_1(\nu',q^2)}{(\nu'^2-\nu_0^2)(\nu'^2-\nu^2-i\varepsilon)}\,, \\ T_2^{\mathrm{inel}}(\nu,q^2) &= 2\int_{\nu_{\mathrm{th}}}^{\infty}\frac{\nu' d\nu'\, V_2(\nu',q^2)}{\nu'^2-\nu^2-i\varepsilon}\,. \end{aligned} \tag{4}$$

Here, one has already taken into account the fact that, according to Regge theory, the dispersion relations for T_1^{inel}, T_2^{inel} require one subtraction and no subtractions, respectively. The lower integration limit is equal to $\nu_{\mathrm{th}} = (W_{\mathrm{th}}^2 - m^2 - q^2)/(2m)$, with $W_{\mathrm{th}} = m + M_\pi$, where M_π is the pion mass. The quantities V_1, V_2 denote the absorptive parts of T_1^{inel}, T_2^{inel}. They can be expressed through the experimentally observed total (transverse, longitudinal) electroproduction cross sections $\sigma_T(\nu, q^2)$, $\sigma_L(\nu, q^2)$.

The choice of the subtraction point ν_0 is arbitrary. The quantity $S_1^{\mathrm{inel}}(q^2) = T_1^{\mathrm{inel}}(0, q^2)$ is usually referred to as the subtraction function. Analogously, one can define the full subtraction function that includes the elastic part as well: $S_1(q^2) = S_1^{\mathrm{el}}(q^2) + S_1^{\mathrm{inel}}(q^2) = T_1(0, q^2)$. At $q^2 = 0$ the inelastic part of the subtraction function is given by

$$S_1^{\mathrm{inel}}(0) = -\frac{\kappa^2}{4m^2} - \frac{m}{\alpha}\beta_M\,, \tag{5}$$

where κ, β_M denote the anomalous magnetic moment and the magnetic polarizability of the nucleon, respectively, and $\alpha \simeq 1/137$ is the electromagnetic fine-structure constant.

Recently, a different subtraction function was introduced in Refs. [8, 9]. The subtraction point has been chosen at $\nu_0 = iQ/2$, where $Q = \sqrt{-q^2}$. The new subtraction function is expressed through the amplitude $\bar{T}$:

$$\bar{S}(q^2) = \bar{T}^{\mathrm{inel}}(\nu_0, q^2)\,, \quad \bar{S}(0) = -\frac{\kappa^2}{4m^2} + \frac{m}{2\alpha}(\alpha_E - \beta_M)\,. \tag{6}$$

The two subtraction functions are closely related to each other. Namely, the difference $S_1^{\mathrm{inel}}(q^2) - \bar{S}(q^2)$ is given through a convergent integral over the experimentally measured electroproduction cross sections. Hence, it suffices to calculate one of these subtraction functions. Since the choice $\nu_0 = 0$, in contrast to $\nu_0 = iQ/2$, can be implemented on the lattice in a straightforward manner [15, 16], we stick to this choice.

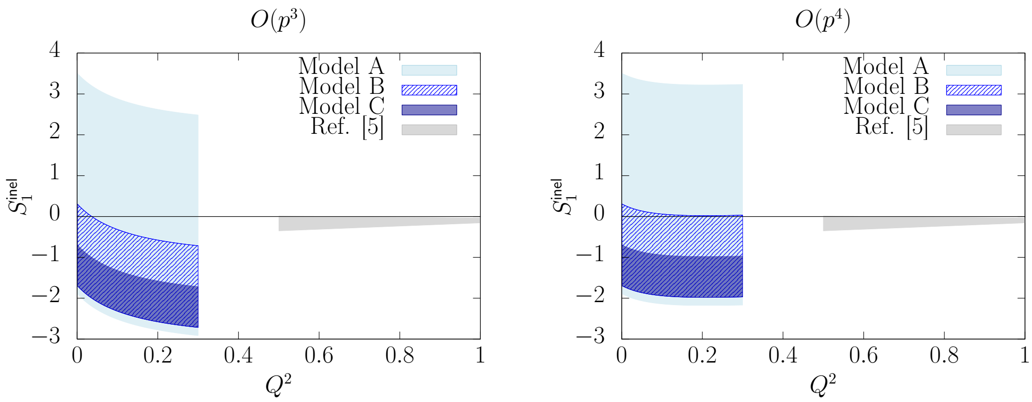

Figure 1: The subtraction function $S_1^{\text{inel}}(q^2)$ for proton minus neutron, at order p^3 and p^4 in the left and right panel, respectively. Here, $Q^2 = -q^2$. The light blue, dashed and dark blue bands show the results of Models A, B and C, respectively. The result of Ref. [5], which is obtained with the use of the Reggeon dominance hypothesis, is shown by the gray band. GeV units are used everywhere.

To compute our results, we make use of three sets of polarizabilities, referred to as model A [17], model B [5] and model C [18] corresponding to the purely experimental input, the Reggeon dominance hypothesis and the combination of the lattice results with experimental data, respectively. Below, we evaluate the difference of the subtraction functions for the proton and the neutron[1] using the input from the three distinct models:

$$\begin{aligned}
&\text{model A:} \qquad S_1^{\text{inel}}(0) = (0.8 \pm 2.7)\,\text{GeV}^{-2}\,, \qquad \bar{S}(0) = (-0.2 \pm 2.6)\,\text{GeV}^{-2}\,,\\
&\text{model B:} \qquad S_1^{\text{inel}}(0) = (-0.7 \pm 1.0)\,\text{GeV}^{-2}\,, \qquad \bar{S}(0) = (-1.7 \pm 0.8)\,\text{GeV}^{-2}\,,\\
&\text{model C:} \qquad S_1^{\text{inel}}(0) = (-1.2 \pm 0.5)\,\text{GeV}^{-2}\,, \qquad \bar{S}(0) = (-2.2 \pm 0.6)\,\text{GeV}^{-2}\,.
\end{aligned} \tag{7}$$

2.2 Infinite volume results

To calculate $S_1^{\text{inel}}(q^2)$ and $\bar{S}(q^2)$ independently, we made use of the leading-order effective Lagrangian of pions interacting with external sources [19] together with the order four effective Lagrangian given in Ref. [20].

Note from Fig. 1 and Fig. 2 that the experimental input, used in corresponding papers, leads to very large uncertainty at small values of Q^2, which comes mainly from the resonance region above the Δ-resonance. On the other hand, the results of Chiral Perturbation Theory become generally unreliable at higher values of Q^2. Thus, combining both calculations, one can get a coherent picture of the Q^2-dependence of the subtraction function in a wide interval and check the consistency of the Reggeon dominance hypothesis.

Notice, however, that the uncertainty in the different models considered in the present work lead to very large error bars in these plots. This concerns, especially, the results of model A, which completely overlap with the results of the models B and C, thus bringing no independent constraints.

[1] In order to ease the notations, we do not attach the superscript $p-n$ to the subtraction functions, corresponding to the difference proton minus neutron. From the context it is always clear, which subtraction function is meant.

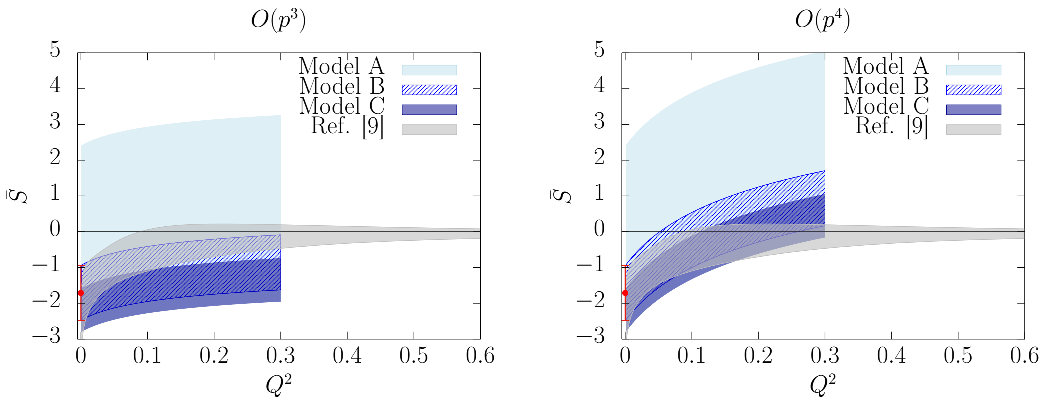

Figure 2: Results for the subtraction function $\bar{S}(q^2)$ for proton minus neutron. The notations are the same as in Fig. 1. The gray band shows the result of Ref. [9], obtained with the use of the Reggeon dominance. GeV units are used everywhere. The data point at the origin shows the prediction of the $\bar{S}(0)$ in model B with Reggeon dominance, see Eq. (7).

What is more important in our opinion, is that the amplitudes calculated in the effective field theory show a smooth behavior in the vicinity of $Q^2 = 0$, stated differently, no rapid variations are observed. Note also that the convergence is quite poor, and the picture changes significantly when going from $O(p^3)$ to $O(p^4)$. Still, we do not observe any apparent disagreement to the Reggeon dominance.

3. Finite volume vorrections

3.1 Extraction of the subtraction function on the lattice

The diagrams that contribute to the Compton amplitude are the same in the infinite and in a finite volume. The only difference consists in replacing the three-dimensional integrals with the sums over discrete lattice momenta (we take that the effects related to a finite size of a lattice in the temporal direction are already taken into account during the measurement of the energy levels). Assuming periodic boundary conditions, in the loop integrals one has to replace:

$$\int \frac{d^4k}{(2\pi)^4 i} \to \int_V \frac{d^4k}{(2\pi)^4 i} \equiv \int \frac{dk_0}{2\pi i} \frac{1}{L^3} \sum_{\mathbf{k}}, \qquad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}, \quad \mathbf{n} = \mathbb{Z}^3 . \tag{8}$$

Here, we consider the nucleon placed in a periodic magnetic field on the lattice with a spatial size L (the temporal size of the lattice is assumed to be much larger and is effectively set to infinity). The configuration of the magnetic field is chosen as:

$$\mathbf{B} = \big(0, 0, -eB\cos(\omega \mathbf{n}\mathbf{x})\big), \qquad \mathbf{n} = (0,1,0), \tag{9}$$

where e denotes the proton charge. In Ref. [16] it has been shown that the spin-averaged level shift in the magnetic field with a given configuration is given by:

$$\delta E = -\frac{1}{4m}\left(\frac{eB}{\omega}\right)^2 T_L^{11}(p,q) + O(B^3), \qquad p^\mu = (m, \mathbf{0}), \quad q^\mu = (0,0,\omega,0). \tag{10}$$

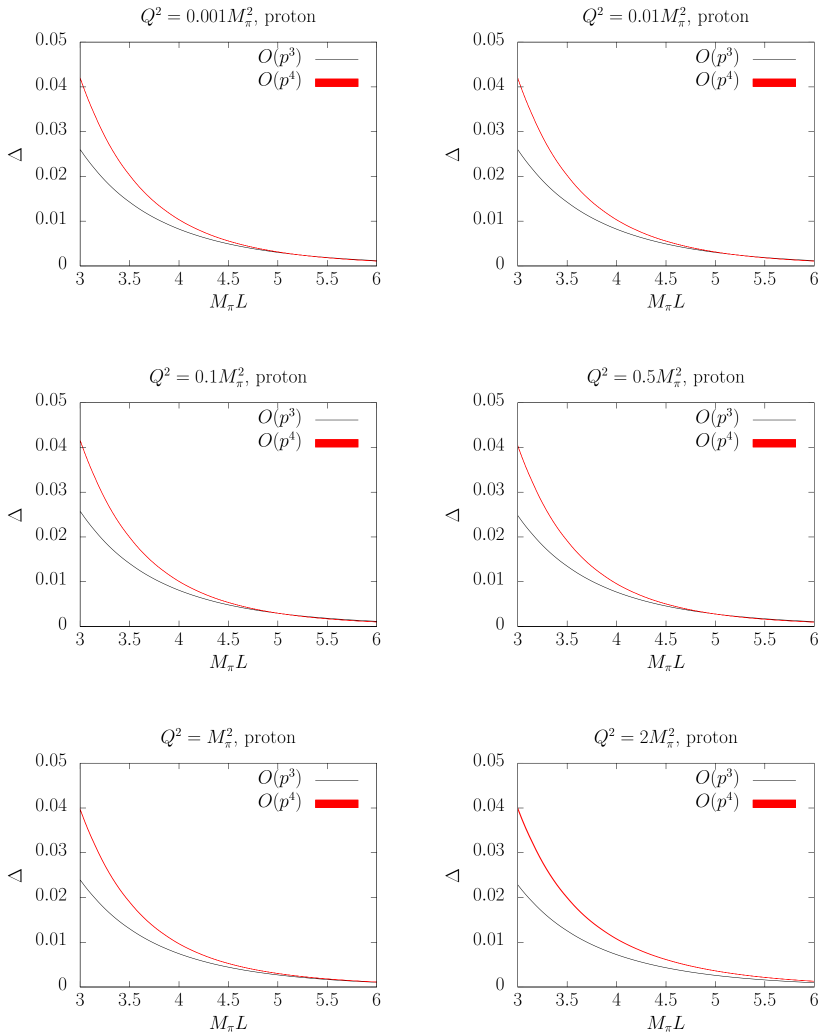

Figure 3: The finite-volume effect in the proton amplitude versus the dimensionless variable $M_\pi L$. The uncertainty in the knowledge of the LECs does not translate into a large uncertainty in the final results, the width of the red band is barely visible by eye.

Note that here $T_L^{11}(p,q)$ denotes the 11-component of the full Compton scattering amplitude in a finite volume (in other words, $T_L^{11}(p,q)$ includes both inelastic and elastic parts). Further, $q^2 = -\omega^2$. Hence, placing a nucleon in the periodic magnetic field enables one to extract the amplitude at non-zero (albeit discrete) values of $q^2 < 0$. The other variable is $\nu = p \cdot q/m = 0$ in the given kinematics.

3.2 Finite volume results

In this subsection, we present the finite-volume corrections to the 11-component of the Compton tensor, which enters the expression of the energy shift in the periodic background field. The quantity,

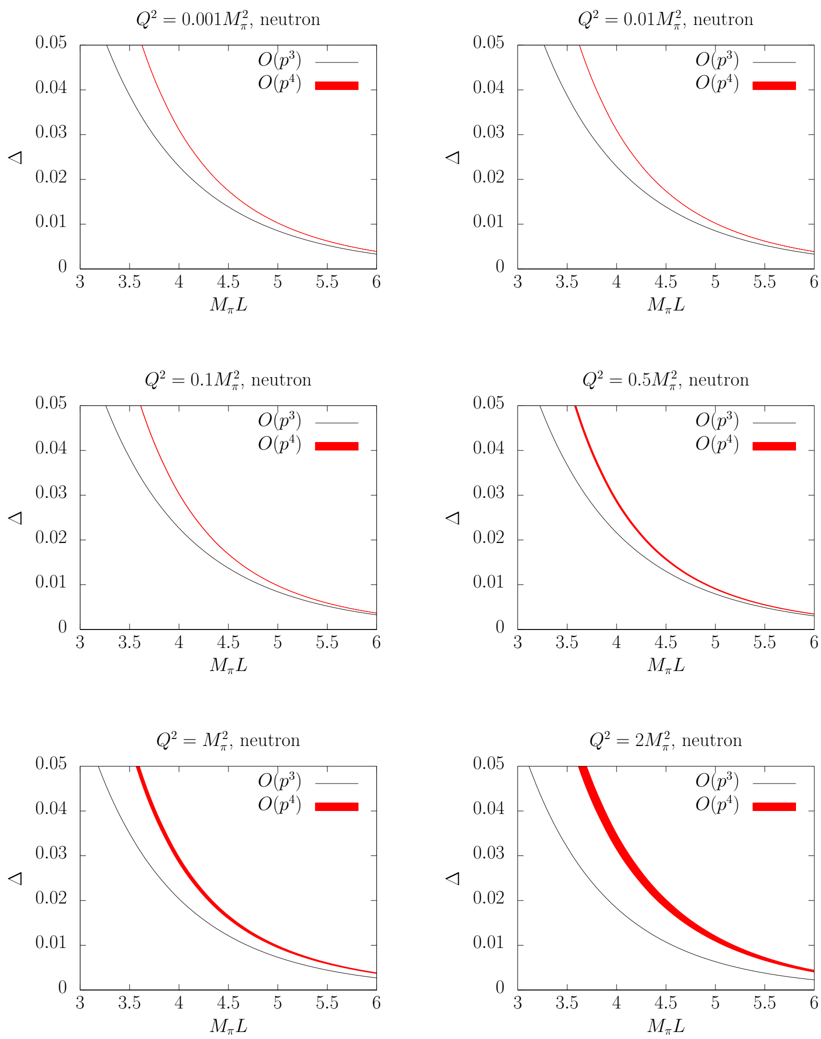

Figure 4: The same as in Fig. 3 for the neutron.

which is calculated here, is given by:

$$\Delta = \frac{T_L^{11}(p,q) - T^{11}(p,q)}{T^{11}(p,q)} \,. \tag{11}$$

We calculated this quantity for the physical value of the pion mass and for several different values of q^2, separately for the proton and the neutron.

The finite-volume corrections to the Compton amplitude are shown in Figs. 3 and 4 for the following values of the variable Q^2:

$$Q^2 = 0.001M_\pi^2\,,\ 0.01M_\pi^2\,,\ 0.1M_\pi^2\,,\ 0.5M_\pi^2\,,\ M_\pi^2\,,\ 2M_\pi^2\,. \tag{12}$$

These figures contain our main result. It is seen that, both for the proton and the neutron, the finite-volume corrections are encouragingly small for $M_\pi L \geq 4$. Moreover, the uncertainty caused by the poor knowledge of the $O(p^4)$ LECs is indeed moderate in the final results.

4. Summary

Using Baryon Chiral Perturbation Theory and the EOMS renormalization scheme of Refs. [21, 22], we have evaluated the spin-averaged amplitude of the doubly virtual Compton scattering off nucleons up to $O(p^4)$ in the infinite volume. The main result of this work is the calculation of the finite-volume corrections to the Compton scattering amplitude. These results are interesting, first and foremost, in view of the perspective of a measurement of the subtraction function on the lattice using periodic external fields. On the other hand, what is extracted from the nucleon mass shift, measured on the lattice, is a particular component of the second-rank Compton tensor T_L^{11}, which is a well-defined quantity and which tends to $S_1(q^2)$ (modulo an overall kinematic factor) in the infinite-volume limit.

Our results show that the exponentially vanishing finite-volume corrections to the quantity T_L^{11} amount up to 2-3% or less at $M_\pi L \simeq 4$ both for the proton and the neutron. This means that one can extract the infinite-volume subtraction function with a good accuracy already using reasonably large lattices. We also note that the convergence of our results is rather good, and the poor knowledge of the $O(p^4)$ LECs does not pose a real obstacle as the resulting uncertainty is very small.

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