PROCEEDINGS OF SCIENCE

Search for continuous phase transitions in 5D pure SU(2) lattice gauge theory

José Matos[a,*]

[a] *Centro de Física das Universidades do Porto, University of Porto, 4169-007, Porto, Portugal*

E-mail: up201506054@edu.fc.up.pt

The Renormalization Group (RG) is one of the central and modern techniques in quantum field theory. Indeed, quantum field theories can be understood as flows between fixed points of the RG flow, which represent Conformal Field Theories (CFT's). Hence, the search and classification of yet unknown non-trivial CFT's is a legitimate endeavor. Analytical considerations point to the existence of such a fixed point in pure SU(2) Yang-Mills fields in 5D. This issue has already been addressed, although inconclusively. We search for this putative fixed point using lattice Monte Carlo methods. An extended discussion can be found in [1].

The 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021 Zoom/Gather@Massachusetts Institute of Technology

*Speaker

https://pos.sissa.it/

PoS(LATTICE2021)311

1. Introduction

A widely used framework to classify the set of consistent field theories is the Renormalization Group (RG), see [2] for a review. In this framework, there is a special set of theories signaled by the fix points of the the RG flow. We investigate the existence of such a fixed point in the 5-dimensional pure SU(2) Yang-Mills theory. The existence of a non-trivial fixed point in this model was first proposed in[3], in the context of the ε-expansion in $4 + \varepsilon$ and subsequently studied numerically in [4]. If this putative fixed point exists, it would be one of the few examples of non-trivial conformal field theory (CFT's) in $D > 4$.

This model is particularly interesting due to the non-renormalizability of the theory in 5 dimensions, which excludes it from being studied perturbatively, similarly to the non-linear sigma model which is non-renormalizable in 3 dimensions. The latter, however, has a non-trivial fixed point [2, 5–7]. The existence of non-trivial fixed points in non-renormalizable theories is also the main idea behind the "asymptotic safety" program, see [8] for a review.

The study of this non-perturbative effects requires the use of non-perturbative methods. We elected to use Monte-Carlo methods to study a lattice version of the theory, in which fixed points of the RG flow are identified with continuous phase transitions. Based on the results from the ε-expansion [3], we expect the critical surface to have co-dimension one, which means that we should have to tune a single parameter (or in the unlikely situation that one of the parameters flows parallel to the critical surface, two) to hit the critical surface. Previous results [4, 9] rule out this possibility, as they only identified first order phase transitions.

We will follow a similar procedure to that presented on [4, 9], but we will take advantage of the freedom to chose different lattice actions which the same naive continuum limit but with different projections into the continuum operators. In particular, we will focus on lattice actions with two free parameters in order to increase our chances of hitting the critical surface. This work is presented in detail in [1].

2. Lattice Actions and Monte Carlo

We used two simple extensions of the usual Wilson action, introduced in [2], since it preserves local gauge invariance. This is achieved at the expense of introducing link variables $U_\mu^{\mathfrak{R}}$, belonging to any representation of $SU(2)$. In the pure version (i.e. without fermions) the only gauge invariant objects are traces of ordered products of the link variables around closed loops. These loops are usually referred to as Wilson loops. The simplest Wilson loop is a rectangular loop contained in a plane defined by two directions μ, ν and with dimension I and J, respectively. We will denote these loops as $\Box^R_{\mu\nu,I\times J}$. In this work we will only consider isotropic systems such that the directional indexes μ and ν can be dropped. With these conventions, the standard Wilson action takes the form

$$S = \sum_{\Box} \frac{\beta}{2} \operatorname{Tr}\left(1 - \Box^f_{1\times 1}\right),$$

where the sum is performed over all distinct 1×1 Wilson loops and the link variables are in the fundamental representation.

We will study two different variations of this action. The first, already studied in [4], considers an extra term where the trace of the link variables is taken in the adjoint representation[1]

$$S^{f,a}_{1\times1} = \sum_{\square_{1\times1}} \frac{\beta_f}{2} \operatorname{Tr}\left(1 - \square^f_{1\times1}\right) + \frac{\beta_a}{3} \operatorname{Tr}\left(1 - \square^a_{1\times1}\right). \tag{1}$$

We selected his extension as a starting point, since in [4] the authors suggest that a continuous phase transition may exist in this model for negative enough values of β_a. The second extension uses loops of different sizes in the fundamental representation

$$S^{f}_{1\times1,2\times2} = \sum_{\square_{1\times1}} \frac{\beta_1}{2} \operatorname{Tr}\left(1 - \square^f_{1\times1}\right) + \sum_{\square_{2\times2}} \frac{\beta_2}{2} \operatorname{Tr}\left(1 - \square^f_{2\times2}\right), \tag{2}$$

and it was chosen to explore a different region of the coupling space.

It is useful to define the lattice average of the trace of the Wilson loops

$$W^{\mathfrak{R}}_{I\times J} = 1 - \frac{\sum_{\square\times J} \operatorname{Tr}\left(\square^{\mathfrak{R}}_{I\times J}\right)}{N_{\text{loops}}\, d(\mathfrak{R})},$$

where N_{loops} is the number of loops of the given type and $d(\mathfrak{R})$ is the dimension of the representation. In this notation, the extended actions take a particularly simple form

$$S^{f,a}_{1\times1} = N_{\text{loops}} \left(\beta_f W^f_{1\times1} + \beta_a W^a_{1\times1}\right),$$
$$S^{f}_{1\times1,2\times2} = N_{\text{loops}} \left(\beta_1 W^f_{1\times1} + \beta_2 W^f_{2\times2}\right).$$

The standard Monte Carlo approach is based on generating large numbers of configurations to perform measurements of some average in the canonical ensemble. In this context, we identify a phase transition as a qualitative change in the dependence of this average in the coupling being changed (e.g. the order parameter of the transition should be zero for some values of β and non-zero for others). If we compute the average of the order parameter, this change is discontinuous for first order phase transition and it is continuous for a continuous (second order) phase transition.

For the models considered, this approach has two severe drawbacks. On one hand, there is no simple order parameter whose change from a non-zero value to zero can be used to identify a phase transition. On the other hand, from the previous works in [4], we need to distinguish very weak first order criticality from true second order criticality, such that we are in the regime where the noise of the data may be bigger than the changes in the parameters. We tackle these problems by introducing a generalized ensemble that allows us to directly estimate the first derivative of the entropy (i.e. the logarithm of the number of configurations with a given value of $W^f_{1\times1}$) with respect to $W^f_{1\times1}$, which enable us to identify the phase transition and its order. When the first derivative of the entropy is monotonic there are no phase transitions and when it is non-monotonic there is a first order phase transition. A second order phase transition occurs in the marginal case that separates the previous scenarios, when there is a point where the second derivative is zero.

[1]The numeric factors are added to normalize the traces as since the trace of the identity in the fundamental representation is 2 and in the adjoint representation is 3.

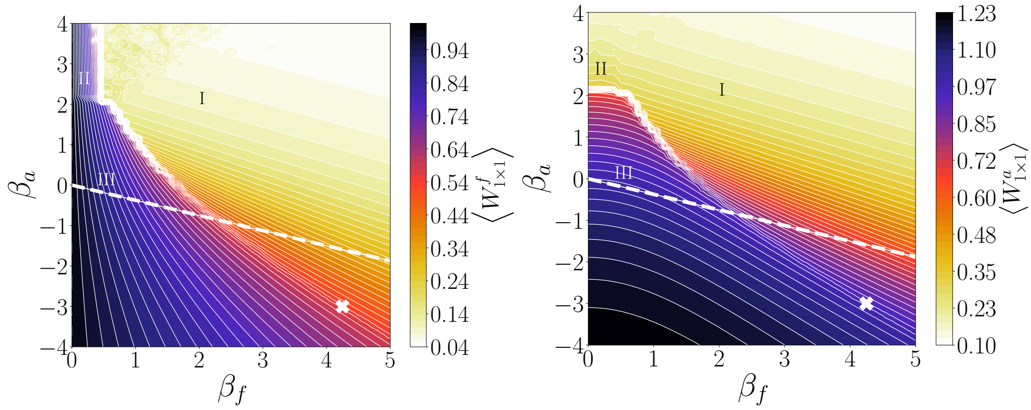

Figure 1: Survey of the parameter space, for $L = 4$. We show $\left\langle W^f_{1\times1}\right\rangle_{\beta_f,\beta_a}$ on the **left** and $\left\langle W^a_{1\times1}\right\rangle_{\beta_f,\beta_a}$ on the **right** . The white cross signals some of the parameters analysed in [4]. We identified three different "phases", labeled by the roman-numerals. The thin white lines are level curves. Discontinuities are identified when several white lines come together. The dashed white line distinguishes regions with different signs of the coupling in the naive continuum limit; below $1/g^2 < 0$ and above $1/g^2 > 0$. We used rectangular grid with a spacing of 0.1.

The idea behind our approach is replacing the canonical ensemble weight for a more adequate weight

$$e^{-\beta W N} \longrightarrow e^{-\left(\beta_1+\beta_2 \frac{W}{N}\right) W N},$$

where N is the total number of loops and W is the lattice average of the trace of some Wilson loop. The probability density for a given value of W in this generalized ensemble is

$$\rho\left(W\right) = e^{\mathbb{S}(W)-\left(\beta_1+\beta_2 \frac{W}{N}\right) W N}.$$

Notice that Monte Carlo simulations will generate an estimate for $\rho\left(W\right)$, such that we can always recover the entropy.

We can understand the advantage of this modification by recalling that Monte Carlo simulations usually sample the maximums of the probability distribution (Eq.2), and the number of local maximums of this distribution greatly impacts the quality of the simulations. In particular, when there are two local maximums, we usually observe hysteresis which can skew our measurements. It can be shown that, by choosing a value of β_2 sufficiently large, we can guarantee the existence of a single maximum, completely avoiding the hysteresis problem and sampling even the thermodynamically unstable region. With this method, we were able to obtain estimates of the entropy, for all values of the action around the critical region. This method is inspired by the usual multicanonical methods [10–13], in which the ensemble weight is chosen such that it cancels the entropy and the resulting histogram, obtained in the Monte Carlo simulations, is flat.

3. Results

In this section we will present the results obtained in this work. We start by exploring the fundamental + adjoint action Eq.1, with the aim of investigating the suggestion made in [4] about

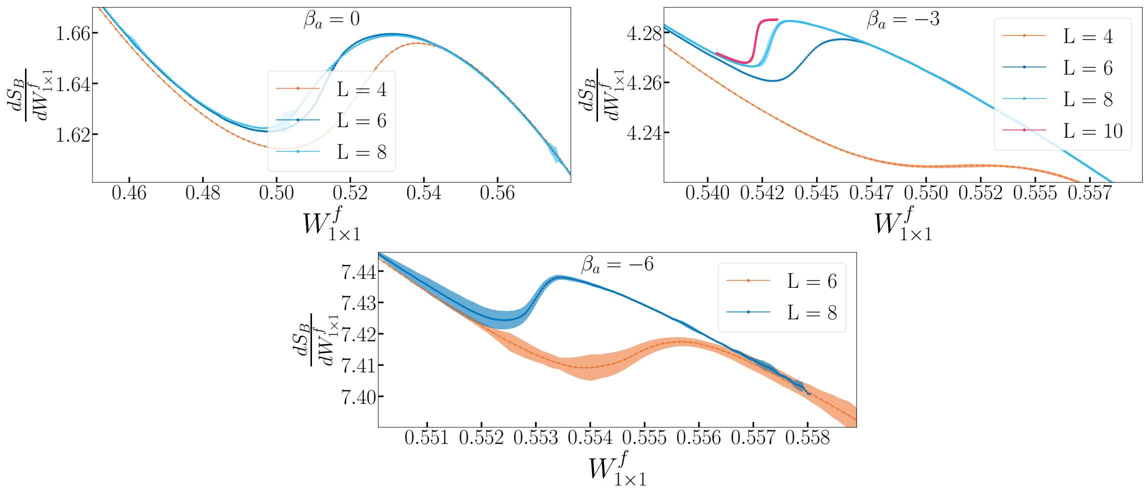

Figure 2: First derivative of the entropy with respect to $W^f_{1\times1}$, as a function of $W^f_{1\times1}$, for different lattice sizes and β_a. The colored region around the markers is an estimate of the error based on independent measurements.

the existence of second order criticality at $\beta_a \sim -6$. We surveyed the parameter space for small lattice sizes ($L = 4$) and identified the different phases of our model. Then, we measure the first derivative of the entropy around the region in which the first order phase transition becomes weaker, and perform finite size scaling to check if our results hold in the infinite limit size.

3.1 Fundamental + Adjoint Action

In this extension, we identified several lines of first order criticality. Although we could identify a curve over which first order criticality becomes weaker, we show that the existence of second order criticality is very unlikely. As mentioned, we started our search by broadly exploring the phase diagram for small lattice sizes. The results are shown in Fig.1 and suggest the existence of 3 distinct phases with 3 curves of phase transitions separating them. We studied each of these curves and found first order criticality in all of them. However, in the boundary between phase I and phase III, we observed a weakening of critical behavior as we decrease β_a. This is the curve previously identified in [4]. Besides the results presented in this report, we also performed measurements of the "Creutz ratios" [14] that suggest this line separates a confined phase from a deconfined phase.

These results motivated a more complete study of this region. We studied in detail the phase transition over this line by measuring first derivative of the entropy in several points and studying how some of its characteristics change over this curve. The results for $\beta_a = 0$, $\beta_3 = -3$, and $\beta_a = -6$ are presented in Fig.2. The two main features of this measurements are: 1) The width of the thermodynamically unstable region decreases when β_a decreases, suggesting that for negative enough β_a, the first order phase transition may vanish; 2) Increasing the system size not only does not increase the width of the thermodynamically unstable region, but it also makes it more unstable (the slope of the second derivative in the thermodynamically unstable region increases). We verified this behavior up to $\beta_a = -10$ and $L = 8$.

A second order phase transition requires that both the width of the thermodynamically unstable region and the maximum of the second derivative to go to zero. Assuming the tendencies observed

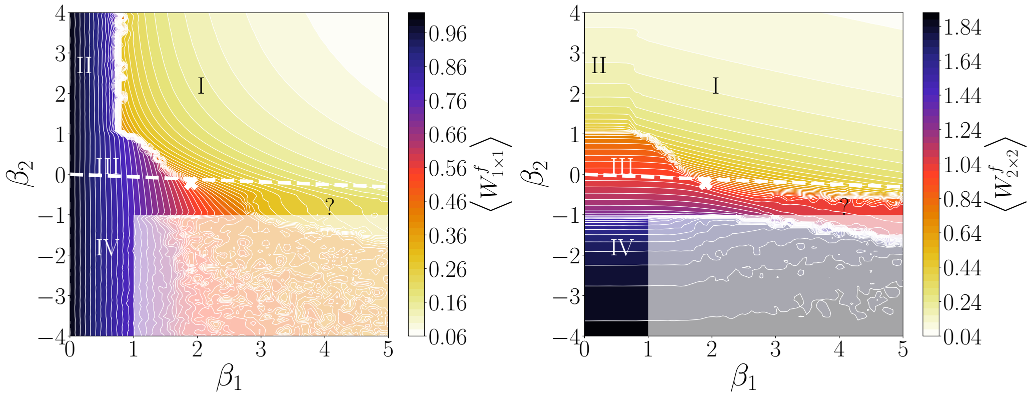

Figure 3: Survey of the parameter space, for $L = 4$. The image on the **left** hand-side shows $\left\langle W^f_{1\times1} \right\rangle_{\beta_1,\beta_2}$ and the one on the **right** hand-side $\left\langle W^f_{2\times2} \right\rangle_{\beta_1,\beta_2}$. The white cross marks the point we selected for subsequent analysis. We clearly identified four different phases, labeled by roman-numerals. The thin white lines are level curves. Discontinuities are identified when several white lines come together. The dashed white line distinguishes regions with different signs of the coupling, for the naive continuum limit ($1/g^2 < 0$ below the line, $1/g^2 > 0$ above the line). There might be a new phase in the region around the question mark (?).

hold outside the range of system sizes and couplings measured, we conclude that it is very unlikely that a second order phase transition is presented in this extension.

3.2 Variable Size Wilson Loops

The extension with square Wilson loops of size 1 and 2, given by Eq.2, was selected with the goal of surveying a different region of the coupling space. We performed a similar analysis as previously, in which we start by surveying the parameter space to identify the relevant regions and then we measure the first and second derivative of the entropy over the regions of interest. The results are shown in Fig.3. Similarly to the previous extension we can identify several phases, and there is a single boundary over which we may find a second order phase transition. It is important to point out that the bottom left corner is washed out since we observed frustration and were not able to properly perform any measurements.

We performed measurements at several points in the boundary between region I and III, but we will focus on the results for $\beta_2 = -0.22$ (signaled by the white cross in Fig.3), shown in Fig.4. Although the observed behavior is similar to the fundamental + adjoint, there is a significant difference. The first order phase transition actually disappears over this line, and for $\beta_2 = -0.22$ we no longer observe first order criticality. In the bottom plot of Fig.4 we can see that the maximum of the second derivative is always smaller than zero, excluding first and second order criticality.

The next step is to try to understand the behavior of this system when we increase the lattice size. Although the maximum of the second derivative seems to increase with the lattice size, it is not clear if it will reach zero, as required for a second order phase transition. In order the clarify this point we plotted the maximum of the second derivative of the entropy as a function of the inverse lattice size for different values of β_2 around $\beta_2 = -0.22$. The results, presented in Fig. 5, show that increasing the lattice size and decreasing β_2 have opposite effects in the maximum of

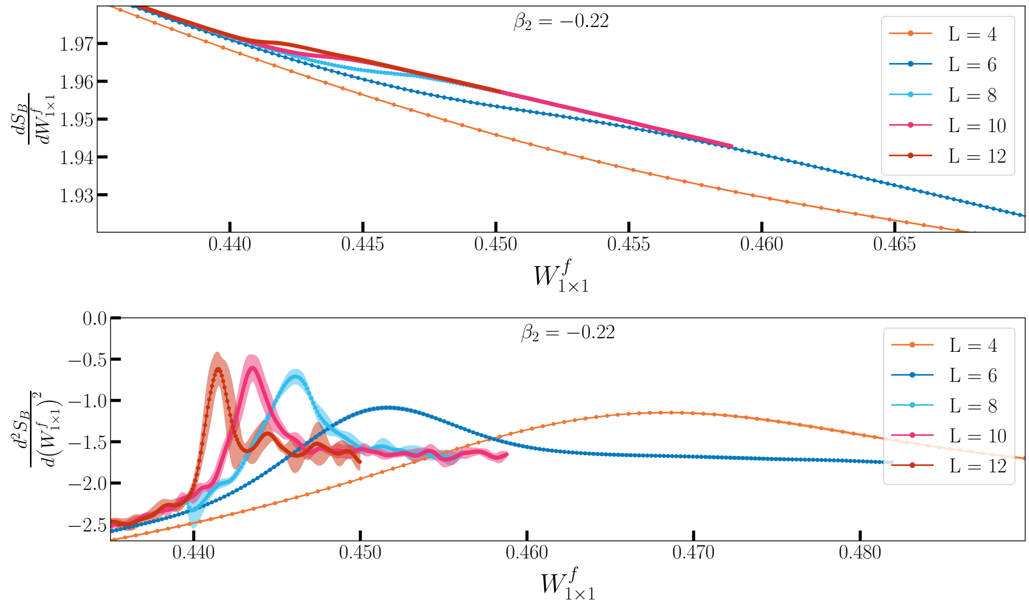

Figure 4: First (**top**) and second derivative (**bottom**) of the entropy as a function of $W^f_{1\times 1}$, for different lattice sizes and $\beta_2 = -0.22$. The colored region around the markers is an estimate of the error based on independent measurements.

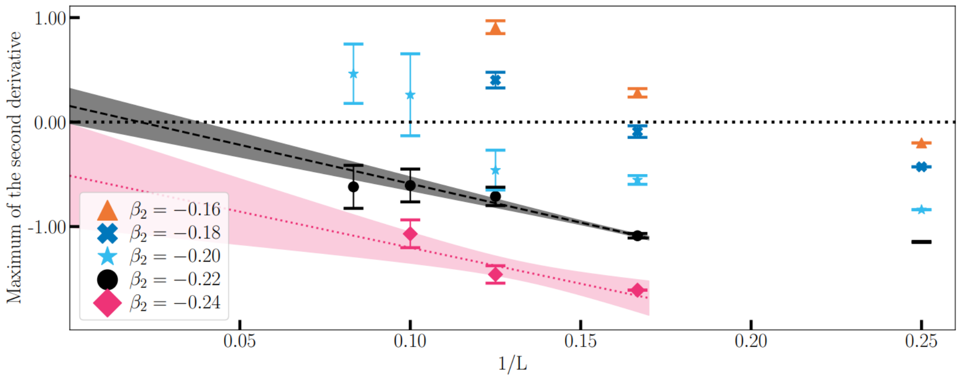

Figure 5: Dependence of the maximum of the second derivative of the entropy as a function of $1/L$, for different β_2 (colors). The error bars are estimate from independent simulations.

the second derivative. This allow us to speculate that there might be a special value of β_2 such that, when we increase the lattice size, the maximum of the second derivative goes to zero. To emphasize this point, we performed a linear fit (dashed lines) for $\beta_2 = -0.22$ and $\beta_2 = -0.24$ and estimated its uncertainty (shaded region). This fit was made to guide the eye and not as a quantitative extrapolation of the infinite lattice size limit, as we cannot quantitatively study this limit.

We conclude that our data is compatible with the existence of second order criticality for some value of β_2 over the boundary between region I and III identified in Fig.3. However, as we cannot perform a quantitative study of the infinite lattice size limit, we cannot verify the existence of this fixed point.

PoS(LATTICE2021)311

4. Conclusion

In this worked we looked for the putative non-trivial CFT in 5-dimensional pure SU(2) Yang-Mills. This search, motivated by the ε-expansion [3], was performed using an extension of the lattice action proposed by [2], where we added to the action terms either in the adjoint representation (fundamental + adjoint action) or for square Wilson loops of side 2. We used Monte Carlo methods in which we replaced the usual canonical ensemble by a generalized ensemble, allowing us to distinguish very weak first order phase transitions from second order phase transitions.

We examined the suggestion made in [4], that for negative enough values of β_a we might find a second order phase transition. While this holds true for small lattices ($L = 4$), first order criticality is always recovered for larger lattice sizes ($L \geq 6$), up to at least $\beta_a = -10$. Our analysis shows that, although the critical region becomes narrower along the boundary between phases I and III, it is unlikely that first order criticality disappears.

In the second extension, we observed a behavior compatible with the existence of a second order phase transition. However, as we are very limited in the lattice sizes, we could not perform a quantitative extrapolation of the large lattice size limit, which would be required to prove the existence of the putative fixed point. Larger simulations, possible with more sophisticated algorithms, will be required to definitely answer this question.

References

[1] A. Florio, J.a.M.V.P. Lopes, J. Matos and J.a. Penedones, *Searching for continuous phase transitions in 5D SU(2) lattice gauge theory*, 2103.15242.

[2] K.G. Wilson and J.B. Kogut, *The Renormalization group and the epsilon expansion*, *Phys. Rept.* **12** (1974) 75.

[3] M.E. Peskin, *Critical point behavior of the Wilson loop*, *Phys. Lett. B* **94** (1980) 161.

[4] H. Kawai, M. Nio and Y. Okamoto, *On existence of nonrenormalizable field theory: Pure SU(2) lattice gauge theory in five-dimensions*, *Prog. Theor. Phys.* **88** (1992) 341.

[5] A.M. Polyakov, *Interaction of Goldstone Particles in Two-Dimensions. Applications to Ferromagnets and Massive Yang-Mills Fields*, *Phys. Lett. B* **59** (1975) 79.

[6] W.A. Bardeen, B.W. Lee and R.E. Shrock, *Phase Transition in the Nonlinear σ Model in $2 + \epsilon$ Dimensional Continuum*, *Phys. Rev. D* **14** (1976) 985.

[7] I.Y. Arefeva, E.R. Nissimov and S.J. Pacheva, *Bphzl Renormalization of 1/N Expansion and Critical Behavior of the Three-dimensional Chiral Field*, *Commun. Math. Phys.* **71** (1980) 213.

[8] A. Eichhorn, *An asymptotically safe guide to quantum gravity and matter*, *Front. Astron. Space Sci.* **5** (2019) 47 [1810.07615].

[9] M. Creutz, *Confinement and the Critical Dimensionality of Space-Time*, *Phys. Rev. Lett.* **43** (1979) 553.

PoS(LATTICE2021)311

[10] J. Viana Lopes, M.D. Costa, J.M.B. Lopes dos Santos and R. Toral, *Optimized multicanonical simulations: A proposal based on classical fluctuation theory*, *Phys. Rev. E* **74** (2006) 046702.

[11] B.A. Berg and T. Neuhaus, *Multicanonical ensemble: A New approach to simulate first order phase transitions*, *Phys. Rev. Lett.* **68** (1992) 9 [hep-lat/9202004].

[12] J. Lee, *New monte carlo algorithm: Entropic sampling*, *Phys. Rev. Lett.* **71** (1993) 211.

[13] A.M. Ferrenberg and R.H. Swendsen, *Optimized monte carlo data analysis*, *Phys. Rev. Lett.* **63** (1989) 1195.

[14] M. Creutz, *Asymptotic Freedom Scales*, *Phys. Rev. Lett.* **45** (1980) 313.