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Lattice Observables for Parton Distributions

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In recent years there has been a momentous surge of interest in the determination of the parton content of nucleons using lattice QCD. Scalar field theory is a convenient toy model to understand the relation between Euclidean correlators and Parton Distribution Functions (PDFs). In this talk we review the factorization formulae that allow the extraction of PDFs from the pseudo-PDFs and quasi-PDFs computed on the lattice. The scalar theory provides a simplified setting where the main physical properties are easily analysed.

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1. Introduction

Following the seminal work in Ref. [1], there has been increasing interest in understanding the structure of nucleons from first-principle simulations of lattice QCD. Parton Distribution Functions (PDFs) are defined starting from the Fourier transform of a fermion bilinear along the light-cone,

$$f(x) = \int \frac{d\xi^-}{4\pi} \, e^{-ixP^+\xi^-} \, \langle P|\overline{\psi}(\xi^-)\gamma^+ U\left(\xi^-, 0\right) \psi(0)|P\rangle \,, \tag{1}$$

where $|P\rangle$ denotes a hadronic state with momentum $P^\mu = (P^0, 0, 0, P^z)$ and $U(\xi^-, 0)$ is a Wilson line connecting ξ^- to the origin, which ensures the gauge invariance of the bilocal operator. We use light-cone coordinates, so that $P^+ = \frac{P^0+P^z}{2}$, $\xi^- = \frac{\xi^0-\xi^z}{2}$ and γ^+ is defined in an analogous manner. The physical PDF is obtained from Eq. (1) after the operator is properly renormalized. As usual the renormalization programme requires that a particular scheme is specified and it has become conventional to use PDFs defined in the $\overline{\text{MS}}$ for phenomenology. We will discuss the issues related to the renormalization of the bilinears in detail below in the context of the scalar toy model.

Lattice QCD relies on the Euclidean formulation of field theories and therefore does not allow us to access the values of the correlators above along the light-cone. Following the prescription in Ref. [1], the observables that are obtained by Monte Carlo simulations are computed at space-like separations. In this talk we will review the definition of quasi-PDFs [1] and pseudo-PDFs [2], the Euclidean observables computed in lattice QCD, their renormalization and their relation to PDFs. Following early work by Collins [3], we are going to study these issues at one-loop in perturbation theory in scalar field theory, where all the relevant features can be studied while avoiding the complications of computing in QCD. The toy model allows us to highlight the important points without being bogged down in the technical details. The results summarised in this talk have been presented in Ref. [4], while we refer to the review talk at this Conference [5] and references therein.

In the scalar field theory the PDFs are defined starting from the matrix element of a two-point correlator between 'hadronic' states,

$$\mathcal{M}\left(\nu, z^2\right) = \langle P|\phi(z)\phi(0)|P\rangle \,, \tag{2}$$

where the hadronic state with momentum P is denoted by $|P\rangle$, and z is a space-time coordinate in Minkowski space. Note that because of the invariance under Lorentz transformations of the continuum theory, the matrix element can only depend on the Lorentz-invariant variables $\nu = p \cdot z$, which is usually referred to as 'Ioffe time', and z^2. The Ioffe-time distribution and pseudo distributions are obtained by computing the matrix element in Eq. (2) along the light-cone, choosing e.g. $z^\mu = (0, z_-, 0_\perp)$, and a spatial direction, $z^\mu = (0, 0, 0, z_3)$, respectively. [1] Taking the Fourier transform along a light-cone direction yields the light-cone PDFs, as in the QCD case shown in Eq. (1),

$$f(x) = \int \frac{dz^-}{4\pi} \, e^{-ixP^+z^-} \, \langle P|\phi(z^-)\psi(0)|P\rangle \,. \tag{3}$$

Note that in the scalar theory there is no need for the Wilson line and there is no spin structure, which simplifies the perturbative calculation while retaining all the important ingredients. Finally,

[1] Note that we have used light-code coordinates to specify z in the first case, while using the usual Cartesian axes for the second one.

the quasi-PDFs are defined by taking the Fourier transform along one spatial direction,

$$q(x) = \int \frac{dz^3}{4\pi}\, e^{-ixP^3z^3}\, \langle P|\phi(z^3)\psi(0)|P\rangle\,. \tag{4}$$

The main purpose of this note is to elucidate the relations between these different quantities.

2. Renormalization

As discussed above, the basic building block is the matrix element

$$\widehat{\mathcal{M}}\left(\nu, z^2\right) = \langle p|\phi(z)\phi(0)|p\rangle\,. \tag{5}$$

For the perturbative calculation we consider the partonic matrix element, denoted by the 'hat', and computed between one-particle partonic states $|p\rangle$, The tree-level and one-loop contributions are shown in Fig. (1).

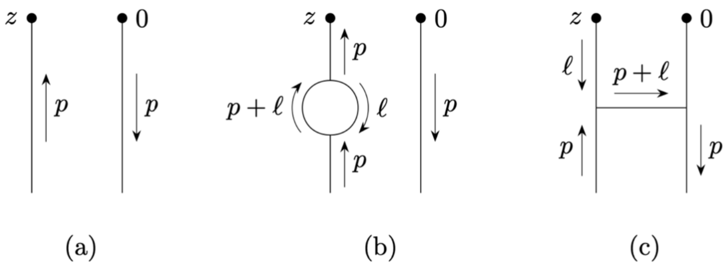

Figure 1: Tree-level, (a), and one-loop, (b) and (c), contributions to the partonic matrix element.

The tree-level contribution is readily evaluated to yield

$$\hat{\mathcal{M}}^{(0)} = \exp\left(-i\nu\right)\,. \tag{6}$$

The one-loop contributions can be expressed in terms of $\hat{\mathcal{M}}^{(0)}$:

$$\widehat{\mathcal{M}}_b\left(\nu, z^2\right) = \frac{\alpha}{6}\left(\frac{1}{\epsilon} + \log\frac{m^2}{\mu^2} + b\right)\widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right)\,, \tag{7}$$

$$\widehat{\mathcal{M}}_c\left(\nu, z^2\right) \propto \int_0^1 d\xi\,(1-\xi)\,K(z^2, M^2)\,\widehat{\mathcal{M}}^{(0)}\left(\xi\nu, 0\right)\,, \tag{8}$$

$$K(z^2, M^2) = \frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T} T^{3-D/2} e^{-TM^2}\, e^{-z_E^2/(4T)}\,, \tag{9}$$

$$z_E^2 = -z^2\,, \quad M^2 = m^2\left(1 - \xi + \xi^2\right)\,. \tag{10}$$

The contribution from $\widehat{\mathcal{M}}_b$ is the usual multiplicative renormalization of the field ϕ.

On the light-cone More interestingly, $\widehat{\mathcal{M}}_c$ has a divergent kernel $K(0, M^2)$ on the light-cone. It is clear from the explicit expression in Eq. (8) that $\widehat{\mathcal{M}}\left(\nu, z^2\right)$ is renormalized by convolution with a kernel. Specifically, in the $\overline{\mathrm{MS}}$ scheme,

$$\widehat{\mathcal{M}}_R\left(\nu, \mu^2\right) = \int_0^1 dy\, \mathcal{K}_R\left(y\right) \widehat{\mathcal{M}}\left(y\nu, 0\right) , \tag{11}$$

with the renormalization kernel:

$$\mathcal{K}_R\left(y\right) = \left[1 - \frac{\alpha}{6}\frac{1}{\epsilon}\right] \delta\left(1-y\right) - \alpha\, \frac{1}{\epsilon}\left(1-y\right) . \tag{12}$$

The renormalized light-cone matrix element can be written as

$$\begin{aligned}\widehat{\mathcal{M}}_R\left(\nu; \mu^2\right) &= \left[1 + \frac{\alpha}{6}\left(\log\frac{m^2}{\mu^2} + b\right)\right] \widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right) \\ &+ \alpha \int_0^1 dx\, \left(1-x\right)\, \log\frac{\mu^2}{m^2\left(1-x+x^2\right)} \widehat{\mathcal{M}}^{(0)}\left(x\nu, 0\right) .\end{aligned} \tag{13}$$

Note that the expression is finite and depends on the scale μ^2 that appears naturally in the process of renormalizing the matrix element. We have therefore added an explicit dependence on μ^2 in the arguments of $\widehat{\mathcal{M}}_R$. At the same time we have also removed the dependence on z^2 since this is constant and equal to zero on the light-cone.

Spatial separation The case where z is a space-like vector is simpler, since the non-vanishing value of z^2 provides a regulator for the kernel in Eq. (9), which becomes a Bessel function:

$$K\left(-z_3^2, M^2\right) = \frac{1}{(4\pi)^3} \int_0^\infty \frac{dT}{T}\, e^{-T M^2} e^{-\frac{z_3^2}{4T}} = \frac{1}{(4\pi)^3}\, 2K_0\left(M z_3\right) , \tag{14}$$

so that the renormalized space-like correlator is given by

$$\begin{aligned}\widehat{\mathcal{M}}_R\left(\nu, z_3^2; \mu^2\right) &= \left[1 + \frac{\alpha}{6}\left(\log\frac{m^2}{\mu^2} + b\right)\right] \widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right) \\ &+ \alpha \int_0^1 dx\, \left(1-x\right)\, 2K_0\left(M z_3\right) \widehat{\mathcal{M}}^{(0)}\left(x\nu, 0\right) .\end{aligned} \tag{15}$$

3. Factorization

The expressions for the renormalized correlators are the starting point to derive factorization theorems for the pseudo-PDFs. Analogous results for the quasi-PDFs are obtained by taking the Fourier transform of the space-like correlators to momentum space.

Position space Combining Eqs. (13) and (15), we obtain a relation between the light-cone and the spatial renormalized matrix elements. As we already mentioned in the Introduction, the renormalized light-cone matrix element is related to the light-cone PDFs; inverting the Fourier transform

$$\widehat{\mathcal{M}}_R\left(\nu, \mu^2\right) = \int_{-1}^1 d\xi\, e^{i\xi\nu}\, \widehat{f}_R\left(\xi, \mu^2\right) , \tag{16}$$

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we obtain for the space-like correlator

$$\widehat{\mathcal{M}}_R\left(\nu, -z_3^2; \mu^2\right) = \int_{-1}^{1} d\xi\, \tilde{C}\left(\xi\nu, m z_3, \frac{\mu^2}{m^2}\right) \hat{f}_R\left(\xi, \mu^2\right) . \tag{17}$$

The detailed expression for $\tilde{C}\left(\xi\nu, m z_3, \frac{\mu^2}{m^2}\right)$ can be worked out from the equations above, but its detailed form is not so important. What is important, though, is that C contains IR divergences when the scalar field becomes massless. In order to have a proper factorization theorem, we want the coefficient function $\tilde{C}$ to be IR safe. This is achieved in the limit $Mz^3 \ll 1$, where we have

$$\widehat{\mathcal{M}}_R\left(\nu, -z_3^2;\, \mu^2\right) = \int_{-1}^{1} d\xi\, \tilde{C}\left(\xi\nu, \mu^2 z_3^2\right) \hat{f}_R\left(\xi, \mu^2\right) + O(m^2 z_3^2)\,, \tag{18}$$

$$\tilde{C}\left(\xi\nu, \mu^2 z_3^2\right) = e^{i\xi\nu} - \alpha \int_0^1 dx\ (1-x)\log\left(\mu^2 z_3^2\, \frac{e^{2\gamma_E}}{4}\right) e^{ix\xi\nu} . \tag{19}$$

Expanding the Bessel function in this limit, cancels exactly the logarithmic divergence in the renormalized expression for $\widehat{\mathcal{M}}_R\left(\nu; \mu^2\right)$, and yields a factorization formula, Eq. (18), akin to the one that relates the Deep Inelastic Scattering (DIS) structure functions to the PDFs. Three interesting observations are in order:

- the Wilson coefficient $\tilde{C}\left(\xi\nu, \mu^2 z_3^2\right)$ is IR finite;
- there are power corrections to the factorization formula: in the case of position-space observables the corrections are ordered in powers of z_3, i.e. the corrections to the factorized expression are larger at larger distances;
- the dependence on z^2 only appears at $O(\alpha)$, this is Bjorken scaling in position space.

Momentum space Analogous factorization formulae are obtained for the quasi-PDF by performing the Fourier transform

$$q_R\left(y, \mu^2, P_3^2\right) = \frac{P_3}{2\pi} \int_{-\infty}^{\infty} dz_3\, e^{-iyP_3 z_3}\, \widehat{\mathcal{M}}_R\left(P_3 z_3, -z_3^2\right) , \tag{20}$$

and expanding the for small values of $M^2/(\xi^2 P_3^2)$. Once again the mass dependence cancels and the resulting Wilson coefficient is IR finite:

$$q_R\left(y, \mu^2, P_3^2\right) = \int_{-1}^{1} \frac{d\xi}{|\xi|}\, C\left(\frac{y}{\xi}, \frac{m^2}{\xi^2 P_3^2}, \frac{\mu^2}{m^2}\right) f_R\left(\xi, \mu^2\right) \tag{21}$$

$$\lim_{\frac{M^2}{\xi^2 P_3^2}\to 0} C\left(\eta, \frac{M^2}{\xi^2 P_3^2}, \frac{\mu^2}{M^2}\right) = C\left(\eta, \frac{\mu^2}{\xi^2 P_3^2}\right) =$$

$$= \delta\left(1-\eta\right) + \alpha \begin{cases} (1-\eta)\log\frac{\eta}{\eta-1} + 1 \\ (1-\eta)\log\left[4\eta\left(1-\eta\right)\frac{\xi^2 P_3^2}{\mu^2}\right] + 2\eta - 1 \\ -(1-\eta)\log\frac{\eta}{\eta-1} - 1 \end{cases} \tag{22}$$

4. Conclusions

We have performed a one-loop calculation in the simple context of a scalar field theory, showing that space-like correlators are related to the light-cone PDFs through factorization formulae, just like for instance the structure functions of DIS. These space-like correlators can be obtained by analytical continuation of correlators computed by Monte Carlo simulations in Euclidean field theories, as discussed for instance in Ref. [6]. The lattice data stand exactly on the same footing as experimental results. They can be used to solve the inverse problems defined in Eqs. (18), (21), and (22). Ideally lattice data should not be used on their own, but rather they should be included in global analyses, where they can be treated on the same footing as any other experimental determination, following the methodology suggested in Refs. [7, 8]. Taming the systematic errors of lattice QCD simulations becomes of paramount importance in order to maximise the impact of lattice data in a global fit. While the lattice data struggle to compete with experimental data in the kinematical regions where the latter are abundant, it would be interesting to fine tune the kinematics of the lattice simulations to target the flavor combinations and the regions in x where the PDFs are less constrained by experiments.

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