Renormalization constants of quark bilinear operators in QCD with dynamical up, down, strange and charm quarks

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We present preliminary results of a calculation of the QCD renormalization constants (RCs) for quark bilinear operators, evaluated non-perturbatively on the lattice in the RI'-MOM scheme. The calculation is performed by using dedicated ensembles with N_f = 4 degenerate dynamical twisted mass (clover) fermions and the Iwasaki gauge action. A detailed analysis is reported, with emphasis on the control or subtraction of the hadronic contaminations occurring in the lattice estimators of RCs and a check of proper scaling with a^2 of the final results. Such a careful study of systematic errors is the counterpart of the high statistical precision reached by current calculations of RCs in the RI'-MOM scheme and is important in order to quote accurate results in phenomenological applications, such as the computation of quark masses.

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1. Introduction

In this contribution we report on the non-perturbative computation of the renormalization constants (RCs) of quark non-singlet bilinear operators in lattice QCD with Wilson-clover fermions using dedicated gauge ensembles with $N_f = 4$ degenerate dynamical quarks produced by the Extended Twisted Mass Collaboration (ETMC) [1]. We compute RCs in the RI'-MOM scheme [2] in which the bilinear operators $O_\Gamma = \overline{q}_1 \Gamma q_2$, with $\Gamma = \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu\nu}\}$ (referred to in the following as $\{S, P, V, A, T\}$, respectively), are renormalized by imposing the condition

$$Z_q^{-1} Z_\Gamma \operatorname{Tr}\left[\mathcal{V}_\Gamma(p) P_\Gamma\right]\big|_{p^2=\mu^2} = 1 \,, \tag{1}$$

where Z_Γ is the RC of the operator O_Γ, $\mathcal{V}_\Gamma(p)$ the amputated Green function of the operator between quark and antiquark states with momentum p, which is projected on its tree-level value with a suitable combination of Dirac matrices P_Γ, and Z_q is the quark field RC. In this work we adopt the definition of Z_q first proposed in Ref. [3]

$$Z_q = -\frac{i}{12N_p} \sum_\mu{}' \operatorname{Tr}\left[\frac{\gamma_\mu}{\tilde{p}_\mu} S_q^{-1}(p)\right]\Bigg|_{p^2=\mu^2} \,, \tag{2}$$

where $S_q(p) = \sum_x S_q(x,0)e^{-ipx}$ is the quark propagator with momentum p and the sum $\sum_\mu'$ is over the N_p non-zero components of the lattice momentum $a\tilde{p}_\mu \equiv \sin(ap_\mu)$. Throughout the present work we have used twisted mass Wilson-clover fermions in the twisted basis, after which, as usual, the RCs are named. In the pure gauge sector we use the Iwasaki action [4] and the clover term coefficient, c_{SW}, is set to its 1-loop [5] tadpole-boosted value (see Ref. [6] for details). The building block of the calculation of RCs is the lattice estimator of the vertex function $\mathcal{V}_\Gamma(p)$, which is constructed in terms of quark propagators with momentum p, and is computed for several lattice momenta, different values of the (valence and sea) quark masses and at three different lattice spacings denoted in the following as A, B and C (see Ref. [1] for details on the lattice setup). Then, we reduce lattice artifacts on $\mathcal{V}_\Gamma(p)$ by both applying a "democratic filter" on momenta to reduce hypercubic effects, and by subtracting the perturbative one-loop cut-off effects $O(g^2 a^\infty)$ [7]. Since RI'-MOM is a mass-independent scheme, an extrapolation of $\mathcal{V}_\Gamma(p)$ to the chiral limit ($m_q \to 0$) is needed to get the RCs. The determination of Z_P, in particular, requires more attention due to the chirally divergent behaviour of the vertex function $\mathcal{V}_P(p)$. In order to distinguish between p-dependent lattice artifacts and the natural dependence on $p^2 = \mu^2$ of scale-dependent operators, all RCs are evolved to a common reference scale μ_{ref} [8, 9]. The RC estimates are finally obtained at each lattice spacing by applying an appropriate fit procedure. In this work we will focus mainly on the fit ansatz and results, while more details will be presented in a forthcoming publication [10].

2. Hadronic contaminations in RI-MOM RC estimators

A crucial aspect of this analysis is the careful study of hadronic contaminations in RI-MOM estimators of bilinear RCs. If not treated correctly, they can in fact spoil the determination of RCs. It is well known [2] that RC estimators in RI-MOM are affected by the contribution of hadronic states which are suppressed as $O(\Lambda_{\text{QCD}}^2/p^2)$. These have to be properly identified and removed

in order to only get the contribution from external quark states. The evaluation of the RC Z_P of the pseudoscalar density operator requires additional care, since hadronic effects in this case are also chirally divergent as $1/m_\pi^2$. The calculation of Z_P will be briefly described in Sec. 3 as it is extensively discussed in Ref. [1]. The origin of hadronic contaminations can be understood from the study of the large p^2 behaviour ($p^2 \gg \Lambda_{\rm QCD}^2$) of the 2-fermion Green function

$$G_\Gamma(p) = \int {\rm d}^4x\, {\rm d}^4y\, e^{-ip(x-y)} \langle q_1(x) O_\Gamma(0) \overline{q}_2(y) \rangle \sim S_q(p) \mathcal{V}_\Gamma(p) S_q(p) + \sum_H \frac{w_H^{(\Gamma)} \Lambda_{\rm QCD}^4}{(p^2 m_H)^2}\ . \quad (3)$$

The first term corresponds to the contribution from the quark states we want to isolate, while the second is the contribution from intermediate hadronic states. Here $w_H^{(\Gamma)}$ are dimensionless parameters that depend on the specific operator O_Γ and, possibly, on the mass of the intermediate hadron state m_H^2. The hadronic contributions in Eq. (3) arise from the time orderings where both the quark and antiquark fields are located at time distances either before or after the operator insertion. It follows that, after amputating $G_\Gamma(p)$ with inverse quark propagators ($S_q^{-1}(p) \sim \not{p}$) and projecting it on its tree-level value, the hadronic effects in $\mathcal{V}_\Gamma(p)$ (and hence in Z_Γ) will be suppressed as $O(\Lambda_{\rm QCD}^2)/p^2$ for all $\Gamma \neq \gamma_5$, and $O(\Lambda_{\rm QCD}^4)/(p^2 m_\pi^2)$ for $\Gamma = \gamma_5$. As a consequence, in our analysis we will check for the presence of $1/p^2$ contributions in the estimators of the RCs. We stress that the definition of $G_\Gamma(p)$ in Eq. (3) and the corresponding large p^2 behaviour on the RHS are peculiar of the RI-MOM scheme in which bilinear vertices have the exceptional kinematics $p_1^2 = p_2^2 = p^2$ and $q^2 = (p_1 - p_2)^2 = 0$. Therefore, other renormalization schemes like RI-SMOM [11] will show a different pattern for hadronic effects. In particular, when using non-exceptional momenta $p_1^2 = p_2^2 = q^2 = p^2$ chiral symmetry breaking and other infrared effects are expected to be more suppressed than in RI-MOM. An extended discussion on hadronic contaminations in the RI-MOM scheme is given in Ref. [1], and will be extended in Ref. [10], while a study of such effects in the RI-SMOM scheme will be part of future investigations.

3. Determination of Z_q, Z_P and Z_V in RI'-MOM

In this Section we describe the determination of the RCs of the quark field (Z_q), of the pseudoscalar density (Z_P), and of the (local) vector current (Z_V) obtained in the RI'-MOM scheme.

Quark field renormalization   The simplest quantity to determine is Z_q, as it only depends on the inverse quark propagator and hence does not suffer from hadronic contaminations. Moreover, the dependence of Z_q on the (valence and sea) quark mass is tiny, thus justifying the use of a constant fit ansatz in the chiral extrapolation. The p^2-dependence of the data after the evolution to a common reference scale of $\mu_{\rm ref}^2 = 13\,{\rm GeV}^2$ is shown in Fig. 1 for the three lattice spacings. The final value of Z_q is then obtained using two different methods. The first method (M1) consists in fitting the $Z_q(\mu_{\rm ref}^2)$ data linearly in p^2 in the range $p^2 \in [10 - 16]\,{\rm GeV}^2$ with the aim of removing $O(a^2 p^2)$ cut-off effects, while in the second method (M2) the same data are fitted to a constant [3, 12]. In principle, M2 is by construction much less sensitive than M1 to small residual higher order perturbative effects or hadronic contaminations but leaves some $O(a^2)$ artifacts in the RC estimates. In the absence of such systematics, the two methods should yield compatible results for renormalized matrix elements in the continuum limit. This can be tested, for example, by studying the scaling of
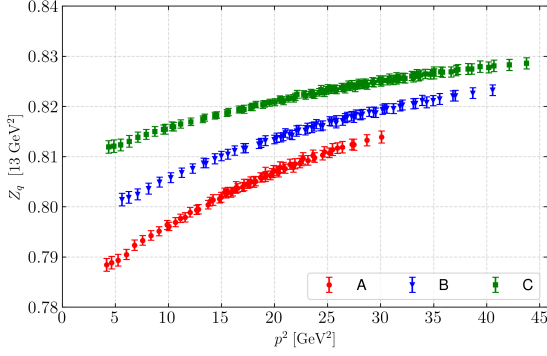
Figure 1: Dependence of $Z_q[13\,\text{GeV}^2]$ on the momentum p^2 (in physical units) for the three lattice spacings $a \simeq \{0.0938, 0.0807, 0.0690\}$ fm, labelled as A, B and C, respectively.
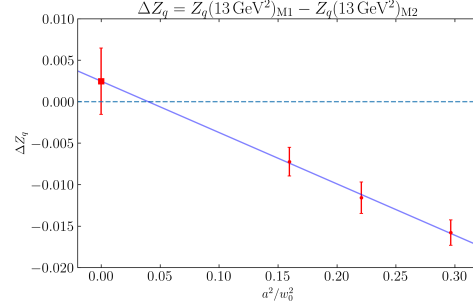
Figure 2: Scaling of the difference $\Delta Z_q[\text{M1}] - Z_q[\text{M2}]$ with $(a/w_0)^2$ evaluated at $\mu_{\text{ref}}^2 = 13\,\text{GeV}^2$.

the difference $\Delta Z_\Gamma = Z_\Gamma[\text{M1}] - Z_\Gamma[\text{M2}]$ with a^2. The value of ΔZ_q, as expected, results to be well compatible with zero in the limit $a^2 \to 0$, as shown in Fig. 2.

The pseudoscalar density   As mentioned in Sec. 2, the calculation of Z_P is more delicate due to the pion contamination showing up as a pole $1/m_\pi^2$ in the lattice estimator of the vertex function $\mathcal{V}_P$. In our analysis we adopted a partially quenched (PQ) setup[1]. Therefore, exploiting that $[m_\pi^2]_{\text{val}} \sim \mu_{\text{val}}$ at LO in PQ chiral perturbation theory [13] we can extrapolate the vertex lattice estimator to $\mu_{\text{val}} = 0$ at fixed p^2 with the following fit ansatz

$$v_P(p, \mu_{\text{val}}, \mu_{\text{sea}}) = \mathcal{V}_P(p, \mu_{\text{sea}}) + \frac{K'}{p^2} + \frac{K}{p^2}\frac{1}{\mu_{\text{val}}} + \frac{K''}{p^2}\mu_{\text{val}} + \dots \tag{4}$$

where $\mathcal{V}_P(p, \mu_{\text{sea}})$ is our target vertex, the K-coefficients contain in general lattice artifacts and the ellipsis stands for terms suppressed by higher powers of $1/p^2$ as $p^2 \to \infty$. In practice, we noticed that the linear dependence on μ_{val} is mostly due to an $O(a^2)$ effect and therefore K'' was not included in the fit, thus absorbing this cut-off effect in the extrapolated value at $\mu_{\text{val}} = 0$. Since at fixed p^2 the target vertex $\mathcal{V}_P$ and the finite hadronic contamination K'/p^2 cannot be disentangled, after the chiral extrapolation it is necessary to check the data against both p^2-dependent lattice artifacts **and** $O(a^0)$ residual hadronic effects scaling as $1/p^2$. As described in Ref. [1], we did this in the case of $Z_P(\mu_{\text{ref}}^2)$ both by fitting directly the $1/p^2$ pole (in a range of p^2 around $p^2 = \mu_{\text{ref}}^2$) and by studying the a^2 behaviour of the non-perturbative step scaling function $\Sigma_P(\mu_a^2, \mu_b^2) = Z_P(\mu_a^2)/Z_P(\mu_b^2)$. In the first case we obtained a result compatible with zero for the fit parameter of the $1/p^2$ pole, while in the second case we got perfect agreement of numerical data extrapolated linearly in a^2 with the continuum perturbative result of Σ_P (computed at N^{3}LO). Both findings suggest that residual systematic uncertainties on Z_P are negligible within our small statistical errors. The final results for Z_P are obtained by applying the methods M1 and M2 at two different reference scales and using different p^2-ranges for the fits. For further details on the calculation of Z_P we refer to Ref. [1].

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[1]At all β values we compute propagators with 9 different values of the valence masses μ_{val} (from ~ 8 to $\sim$16 times larger than the average up-down quark mass) for each of the four sea mass μ_{sea} used.

The local vector current We now turn to compute the RC for the local vector current, Z_V. In this case, unlike the determination of Z_P, the chiral extrapolation is easier due to the absence of chirally divergent effects and thanks to the tiny linear dependence on μ_{val}, that allows us to safely neglect the linear term in the chiral fit ansatz. However, also in this case one has to take into account the contribution of residual $1/p^2$ hadronic effects, which turn out to be significant for Z_V even at relatively high values of p^2. In Fig. 3 we show the p^2-dependence of Z_V at three different values of the lattice spacing $a \simeq \{0.0938, 0.0807, 0.0690\}$ fm. The impact of the $\sim 1/p^2$ hadronic effects can be seen very clearly by studying the scaling of $\Delta Z_V = Z_V[\text{M1}] - Z_V[\text{M2}]$ with a^2 (the methods M1 and M2 being applied in the range $p^2 \in [18\text{-}22]$ GeV^2 where the curves in Fig. 3 look pretty linear in p^2 at all three β's) and by comparing the non-perturbative estimate of the step scaling function $\Sigma_V(\mu_a^2, \mu_b^2)$ with its exact value $\Sigma_V = 1$. Both tests fail dramatically: from Fig. 4 we see that the quantity ΔZ_V does not scale linearly to zero with $(a/w_0)^2$, while in Fig. 5 the same behaviour is shown for the difference of the lattice step scaling function $\Sigma_V(21.5 \text{ GeV}^2, 14.3 \text{ GeV}^2)$ from unit value. This means that the methods M1 and M2 are not adequate in this case to extract the value of Z_V. Therefore, we introduce a new method M3, consisting in a fit of the data on large p^2 ranges with the following fit ansatz

$$\zeta_V(p^2) = Z_V + d_2^{(V)} \cdot (a^2 p^2) + \frac{\epsilon_V}{p^2}. \tag{5}$$

Here $\zeta_V(p^2)$ is the lattice estimator of the RC (corresponding to the curves in Fig. 3), while Z_V is our target RC, which is free from leading cut-off effects and hadronic contaminations. The method M3 is applied in the ranges $p^2 \in [4\text{-}20]$ GeV^2, $p^2 \in [4\text{-}25]$ GeV^2 and $p^2 \in [4\text{-}30]$ GeV^2 for the lattice spacings A, B and C, respectively. The choice of different upper bounds is motivated by the fact that higher order cut-off effects start contributing at different values of physical p^2 depending on the lattice spacing. We also checked that the fit is not sensitive to small variations of the lower bound. The results of the fit parameters obtained with the M3 method are reported in Table 1, where the second uncertainty quoted for Z_V is a conservative systematic error.
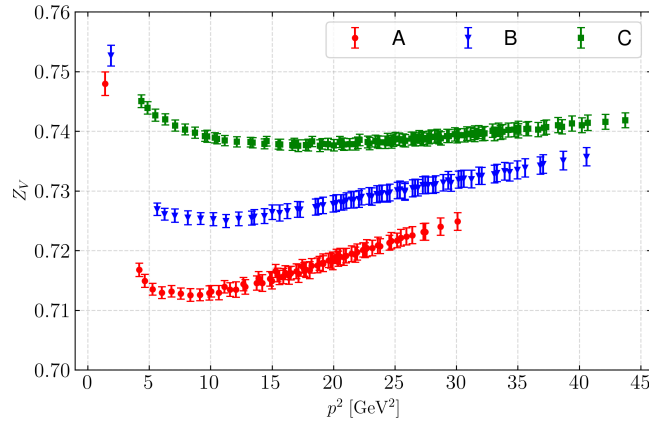

Figure 3: Dependence of Z_V on the momentum p^2 (in physical units) for the three lattice spacings $a \simeq \{0.0938, 0.0807, 0.0690\}$ fm, labelled as A, B and C, respectively.
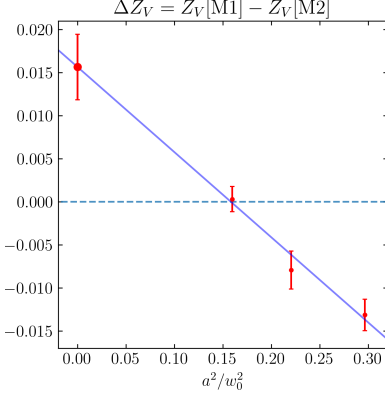
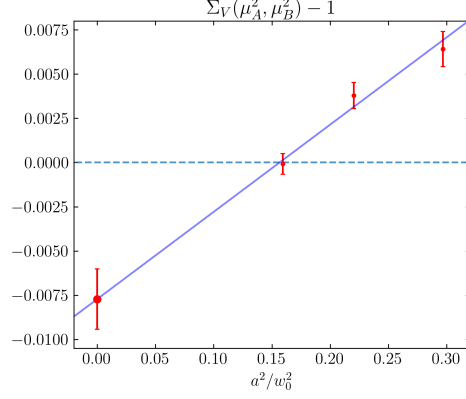**Figure 4:** Scaling of the difference $\Delta Z_V = Z_V [\text{M1}] - Z_V [\text{M2}]$ with $(a/w_0)^2$.

Figure 5: Scaling of the function $\Sigma_V (21.5 \, \text{GeV}^2, 14.3 \, \text{GeV}^2) - 1$ with $(a/w_0)^2$.

β	Z_V	$d_2^{(V)}$	ϵ_V [GeV^2]
1.726	0.6989 (15)(10)	0.0038 (3)	0.057 (6)
1.778	0.7148 (22)(10)	0.0032 (4)	0.051 (11)
1.836	0.7306 (13)(10)	0.0018 (4)	0.061 (6)

Table 1: Results of the fit on $\zeta_V (p^2)$ using the M3 fit ansatz of Eq. (5).

β	Z_V [WTI]
1.726	0.6960 (7)
1.778	0.7131 (6)
1.836	0.7310 (5)

Table 2: Results of Z_V, in the chiral limit, obtained from the WTI in Eq. (6).

The RC Z_V can be obtained also in alternative and completely independent ways from the study of hadronic correlators involving the vector current. In particular, we can extract Z_V from the PCAC Ward-Takahashi identity (WTI) for twisted mass fermions, namely

$$Z_V \sum_{\mathbf{x}} \langle \tilde{\partial}_0 V_0(t, \mathbf{x}) P^\dagger(0) \rangle^{(\chi)} = (\mu_1 + \mu_2) \sum_{\mathbf{x}} \langle P(t, \mathbf{x}) P^\dagger(0) \rangle^{(\chi)}, \tag{6}$$

where $\tilde{\partial}_0$ is the symmetric lattice time derivative, P is the pseudoscalar density and the suffix (χ) denotes that the matrix elements are computed using twisted mass fermions in the *twisted* basis, in which the renormalized (physical) axial current can be written as $(A_R)_\mu = Z_V A_\mu^{(q)} = -i Z_V V_\mu^{(\chi)}$. The results obtained with this method using the same $N_f = 4$ gauge ensembles are reported in Table 2. A remarkable advantage of this approach is that the correlators in Eq. (6) can be computed very precisely and Z_V can be extracted without any systematic effect other than $O(a^2)$ artifacts. The values of Z_V obtained from the WTI can then be used to check the reliability of the RI-MOM results in Tab. 1. The two determinations, if hadronic contaminations are under control, should in fact only differ by $O(a^2)$ discretization effects. The scaling of the difference $Z_V [\text{RI'}] - Z_V [\text{WTI}]$ is shown in Fig. 6. The larger errorbars on the data in the figure represent the statistical+systematic uncertainty propagated from the results in Table 1. We show the results of a constant fit and a linear fit to the data (using the larger errorbars), both yielding an intercept well compatible with zero. Notice that all three points at finite lattice spacing are compatible to zero within 1-2σ. The calculation of Z_V through the WTI is certainly preferable, since it allows one to reach a very high level of precision. However, we believe that the calculation of Z_V in the RI'-MOM scheme is valuable and
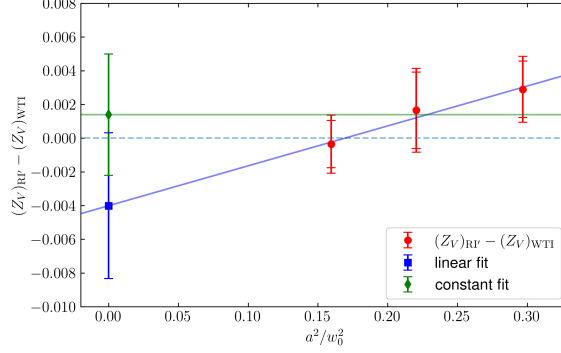
Figure 6: Scaling of the difference $(Z_V)_{\mathrm{RI'}} - (Z_V)_{\mathrm{WTI}}$ versus a^2/w_0^2. The larger errorbars on the data represent the statistical+systematic uncertainty propagated from the results in Table 1. A linear fit (blue) and a constant fit (green) to the data are also reported.

instructive, since the good agreement between the two approaches highlights the reliability of our analysis procedure and shows that the use of RI-MOM does not prevent precise determinations of RCs at the per mille level, once hadronic contaminations are properly treated.

4. Evaluation of the other renormalization constants

For the determination of the RCs of the axial current (Z_A) and of the scalar density (Z_S) we have employed a different approach, since our safety checks against hadronic contaminations failed in both cases and we did not have alternative determinations as precise as Z_V[WTI] to compare with. Therefore our approach here consists in an indirect evaluation based on a combination of results obtained in RI'-MOM and using alternative hadronic methods.

The axial current   We determine Z_A as the combination of (Z_A/Z_V)[RI'] obtained in the RI'-MOM scheme and Z_V[WTI] from Ward identities. In fact, the lattice estimator of the ratio Z_A/Z_V shows a suppression of the hadronic contaminations compared to Z_A alone. We then follow the same procedure described in Sec. 3 for Z_V and compute the ratio Z_A/Z_V in the RI'-MOM scheme applying the method M3 to the lattice estimator $\zeta_{A/V}(p^2)$ in the ranges $p^2 \in$ [4-20] GeV^2, $p^2 \in$ [4-25] GeV^2 and $p^2 \in$ [4-30] GeV^2 for the lattice spacings A, B and C, respectively. The results of the fit are reported in the second column of Table 3. As a safety check, these results can be compared with an alternative and independent determination of Z_A/Z_V that only makes use of hadronic matrix elements and exploits the properties of two variants of the twisted-mass fermion action [3]. In this case, the ratio Z_A/Z_V is extracted by expressing the renormalized physical matrix element of the axial current $\langle 0|(A_0)_R|\pi\rangle$ in terms of matrix elements of bare operators regularized using either the twisted mass (tm) or Osterwalder-Seiler (OS) valence quarks[2]:

$$\langle 0|(A_0)_R|\pi\rangle = Z_V \langle 0|A_0|\pi\rangle^{\mathrm{tm}} = Z_A \langle 0|A_0|\pi\rangle^{\mathrm{OS}} + O(a^2) \, . \tag{7}$$

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[2]The two regularizations once expressed in the *physical* quark basis differ by the values of the Wilson parameters of the u, d quark fields, namely $r_u = -r_d = 1$ for tm fermions and $r_u = r_d = 1$ for OS fermions. See Ref. [3] for further details.

The two matrix elements differ only by $O(a^2)$ effects at maximal twist and therefore Z_A/Z_V is readily obtained from their ratio. The results obtained with this method using the same $N_f = 4$ gauge ensembles are reported in the third column of Table 3. The difference $(Z_A/Z_V)[\text{RI'}] - (Z_A/Z_V)[\text{tm/OS}]$ has a very good scaling with $(a/w_0)^2$ also in this case. Notice that currently the precision on Z_A/Z_V obtained in this way is much lower than for $Z_V[\text{WTI}]$. However, this can be easily improved in the future by increasing the statistics. Finally, we extract Z_A by combining the ratio Z_A/Z_V from RI'-MOM and Z_V from WTI, namely as $Z_A = (Z_A/Z_V)[\text{RI'}] \times Z_V[\text{WTI}]$. This procedure yields an overall uncertainty of $O(0.1\%)$ on Z_A, mainly driven by the RI'-MOM determination $Z_A/Z_V[\text{RI'}]$.

β	Z_A/Z_V [RI′]	Z_A/Z_V [tm/OS]
1.726	1.0599 (13)	1.036 (7)
1.778	1.0563 (11)	1.035 (5)
1.836	1.0524 (8)	1.040 (4)

Table 3: Results for the ratio Z_A/Z_V obtained in the RI'-MOM scheme using the M3 method (second column) and with the alternative hadronic method through chiral extrapolation (third column).

The scalar density   The scalar RC Z_S is obtained analogously as a combination of results from RI'-MOM and hadronic methods. In this case, since Z_P is extracted very precisely in the RI'-MOM scheme (see Sec. 3 and Ref. [1]), we extract Z_S by combining such result with the ratio $(Z_S/Z_P)[\text{tm/OS}]$ obtained from the calculation of the hadronic matrix element of the pseudoscalar density $\langle 0|P|\pi \rangle$ in the tm and OS regularizations, namely $Z_S = (Z_S/Z_P)[\text{tm/OS}] \times Z_P[\text{RI'}]$. The current precision on Z_S is $O(1\%)$ and it is dominated by the RI'-MOM determination of Z_P.

The tensor operator   The calculation of Z_T is still in progress and its accuracy has to be established. The complete analysis will be presented in a forthcoming publication [10].

5. Conclusion

In this work we have presented an overview and some preliminary results about the calculation of RCs of quark bilinear operators performed using $N_f = 4$ ETMC gauge ensembles [1]. Since many current lattice calculations require high precision level, systematic uncertainties in the determination of RCs become relevant and must be kept under control. For this reason, in our analysis we carry on a careful study of the hadronic contaminations affecting lattice estimators of RCs, evaluated non-perturbatively in the RI'-MOM scheme. The different impact of such hadronic effects on the Green functions of quark bilinear operators suggests different numerical strategies to extract the various RCs. In the case of significant hadronic contaminations, these have been identified and subtracted from the data, as it has been shown in the case of Z_V. Scale-independent combinations of RCs can also be computed using alternative methods based on ratios of suitable hadronic matrix elements. Looking at the a^2-scaling of the difference of the results from the hadronic and RI'-MOM methods we get a conservative upper bound on the residual systematic errors.

Acknowledgments

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