

Localised Dirac eigenmodes and Goldstone's theorem at finite temperature

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I show that a finite density of near-zero localised Dirac modes can lead to the disappearance of the massless excitations predicted by the finite-temperature version of Goldstone's theorem in the chirally broken phase of a gauge theory.

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1. Introduction

Ample evidence from lattice calculations shows that the lowest modes of the Euclidean Dirac operator \mathcal{D} are localised in the high-temperature phase of QCD [1–4] and of other gauge theories [5–13] (see [14] for a recent review). Localised modes are supported essentially only in a finite spatial region whose size does not change as the system size grows. In contrast, delocalised modes extend over the whole system and keep spreading out as the system size is increased. The distinction is made quantitative by the scaling with the spatial volume V of the *inverse participation ratio* (IPR), which for a normalised eigenmode $\psi_n(x)$ reads

$$\text{IPR}_n = \int_T d^{d+1}x \|\psi_n(x)\|^4, \quad \text{with} \quad \int_T d^{d+1}x \|\psi_n(x)\|^2 = 1, \quad (1)$$

where d is the spatial dimension of the system, $\|\psi_n(x)\|^2 = \sum_{A,c} |\psi_{nA,c}(x)|^2$ is the local amplitude squared of the mode summed over colour (c) and Dirac (A) indices, and $\int_T d^{d+1}x = \int_0^{\frac{1}{T}} dt \int d^d x$, with T the temperature of the system. Assuming that $\psi_n(x)$ is non-negligible only in a region of size $O(V^\alpha)$, one can easily estimate that $\text{IPR}_n \sim V^{-\alpha}$. For localised modes $\alpha = 0$, while for delocalised modes $0 < \alpha \leq 1$.

Lattice studies show the same situation in a variety of gauge theories, with different gauge groups and in different dimensions (also using different fermion discretisations): while delocalised in the low-temperature, confined phase, low modes are localised in the high-temperature, deconfined phase up to some critical point λ_c in the spectrum, above which they are again delocalised. Localisation is a well-known phenomenon in condensed matter physics, commonly appearing in disordered systems [15]. Technically, the Dirac operator can indeed be seen as (i times) the Hamiltonian of a disordered system, with disorder provided by the fluctuations of the gauge fields. It is then not surprising that the features of localisation observed in gauge theory are analogous to those found in condensed matter systems: for example, at the “mobility edge” λ_c , where localised modes turn into delocalised modes, one finds a second-order phase transition along the spectrum (“Anderson transition”) [16] with critical spectral statistics [17] and multifractal eigenmodes [18], exactly as in condensed matter systems [19].

The physical consequences of localisation in disordered systems are clear: most notably, localisation of electron eigenmodes leads to the transition from conductor to insulator in a metal with a large amount of impurities [15]. The situation is instead not so clear for gauge theories, where the physical meaning of the localisation of Dirac modes has proved to be more elusive. There is, however, growing evidence of an intimate connection between localisation and deconfinement: in a variety of systems with a genuine deconfinement transition, localisation of the low Dirac modes appears in fact precisely at the critical point [7–13]. This is true even for the simplest model displaying a deconfinement transition, namely 2+1 dimensional \mathbb{Z}_2 gauge theory [12]. Theoretical arguments for this behaviour have also been discussed in the literature [20–22]. This connection could help in better understanding confinement and the deconfinement transition.

Still, one would like to find a more direct physical interpretation for localisation in gauge theories. This may seem a hopeless task, given that no physical meaning is attached to individual points, or even regions, of the Dirac spectrum, with observables obtained only integrating over the whole spectrum. A notable exception to this state of affairs is the chiral limit: in this case the point

$\lambda = 0$ is singled out as only near-zero modes are physically relevant, and the localisation properties of these modes may have direct physical implications. In particular, one wonders how a finite density of near-zero localised modes can affect (if at all) the usual picture of spontaneous chiral symmetry breaking and generation of Goldstone excitations. While I know of no model where such a scenario has been demonstrated, there are intriguing hints in (i) 2+1 flavour QCD towards the chiral limit, and (ii) SU(3) gauge theory with $N_f = 2$ massless adjoint fermions.

- (i) A peak of localised near-zero modes has been observed in overlap spectra computed in HISQ backgrounds for near-physical light-quark mass right above the crossover temperature T_c [23]. This peak persists also for lighter-than-physical light-quark masses [24, 25], but the localisation properties are not known in that case. It is possible that this peak will survive and the localised nature of the modes will not change in the chiral limit.
- (ii) SU(3) gauge theory with $N_f = 2$ massless adjoint fermions displays an intermediate, chirally broken but deconfined phase [26, 27], where a nonzero density of near-zero Dirac modes is certainly present. As the theory is deconfined, one expects these modes to be localised.

2. Localised modes and Goldstone's theorem at zero temperature

It is instructive to discuss first the case $T = 0$. Consider a gauge theory with N_f degenerate flavours of fundamental quarks of mass m . In such a theory, as a consequence of the Banks-Casher relation [28] and of Goldstone's theorem [29], a nonzero density of near-zero modes in the chiral limit implies the spontaneous breaking of chiral symmetry down to $SU(N_f)_V$, and in turn the presence of massless pseudoscalar Goldstone bosons in the particle spectrum. However, one should say more precisely “delocalised near-zero modes”: in fact, it has been known for quite some time [30, 31] that if the near-zero modes are localised then the Goldstone bosons disappear. To see this in the case at hand, one uses the $SU(N_f)_A$ (axial nonsinglet) Ward-Takahashi (WT) identity,

$$-\langle \partial_\mu A_\mu^a(x) P^b(0) \rangle + 2m \langle P^a(x) P^b(0) \rangle = \delta^{(4)}(x) \delta^{ab} \Sigma, \quad (2)$$

where $A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 t^a \psi$, $P_\mu^a = \bar{\psi} \gamma_5 t^a \psi$, and $\Sigma = \frac{1}{N_f} \langle \bar{\psi} \psi \rangle$, with γ_μ and γ_5 the Euclidean Hermitian gamma matrices and t^a the generators of $SU(N_f)$ in the fundamental representation normalised as $2 \text{tr} t^a t^b = \delta^{ab}$, and $\langle \dots \rangle$ is the Euclidean expectation value. In momentum space Eq. (2) becomes

$$i p_\mu \mathcal{G}_{AP\mu}(p) + 2m \mathcal{G}_{PP}(p) = \Sigma, \quad (3)$$

where $\delta^{ab} \mathcal{G}_{PP}(p) = \int d^4x e^{i p \cdot x} \langle P^a(x) P^b(0) \rangle$, and similarly for $\mathcal{G}_{AP\mu}(p)$. In the limit $m \rightarrow 0$, one finds near $p = 0$ that

$$\mathcal{G}_{AP\mu}(p) \xrightarrow{p \rightarrow 0} -\frac{i p_\mu}{p^2} [\Sigma - R], \quad R \equiv \lim_{p \rightarrow 0} \lim_{m \rightarrow 0} 2m \mathcal{G}_{PP}(p), \quad (4)$$

with Σ denoting from now on the chiral condensate in the chiral limit. If $\Sigma - R \neq 0$, $\mathcal{G}_{AP\mu}$ has a pole at zero momentum implying the existence of massless bosons. If \mathcal{G}_{PP} behaves reasonably as a function of m in the chiral limit then $R = 0$, and massless bosons are present if chiral symmetry is spontaneously broken by a nonzero chiral condensate Σ . However, as I show below in Section 4,

if there is a finite density of localised near-zero modes then \mathcal{G}_{PP} generally diverges like $1/m$ in the chiral limit. This divergence leads to a nonzero R proportional to the density of localised near-zero modes, by cancelling the factor of m in a way reminiscent of how UV anomalies are formed. In particular, if a finite mobility edge is found in the chiral limit then the ‘‘anomalous remnant’’ R cancels Σ exactly, removing the pole from $\mathcal{G}_{AP\mu}$, and so the Goldstone bosons from the spectrum.

A non-vanishing anomalous remnant allows one to evade Goldstone's theorem. In fact, the anomalous remnant leads to chiral symmetry being *explicitly* broken in the chiral limit, with the resulting modification of the usual WT identity showing that the axial-vector current is not conserved. Current conservation is a fundamental hypothesis of the theorem, and since it does not hold the theorem does not apply.

3. Localised modes and Goldstone's theorem at finite temperature

The argument discussed above is not really relevant to realistic gauge theories (e.g., QCD and QCD-like theories), where no localised near-zero modes have been observed at $T = 0$. However, it suggests a general strategy to study the physical effects of localisation in the chiral limit also at finite temperature: relate the properties of the Euclidean Dirac spectrum with those of the physical spectrum using the axial nonsinglet WT identity Eq. (2), that holds also at $T \neq 0$. In this case, due to technical reasons related to the breaking of $O(4)$ invariance in the Euclidean setting, the physical spectrum is accessed more naturally by reconstructing the axial-vector-pseudoscalar spectral function ϱ^{AP} (see [32]) from the Euclidean correlators,

$$\varrho^{AP}(\omega, \vec{p}) \equiv \int d^4x e^{i(\omega t - \vec{p} \cdot \vec{x})} \langle\langle [\hat{A}_0^a(t, \vec{x}), \hat{P}^b(0)] \rangle\rangle_T, \quad (5)$$

where $\langle\langle \dots \rangle\rangle_T$ denotes the (real time) thermal expectation value, and \hat{A}_μ^a and \hat{P}^b are the Minkowskian axial-vector and pseudoscalar operators. Using the WT identity Eq. (2) and the symmetry and analyticity properties of the correlation functions,¹ one finds in the chiral limit at zero momentum [34]

$$\lim_{\vec{p} \rightarrow 0} \lim_{m \rightarrow 0} \varrho^{AP}(\omega, \vec{p}) = -2\pi[\Sigma - R]\delta(\omega) + (\text{regular at } \omega = 0), \quad (6)$$

where Σ and R are now computed at finite temperature, i.e., compactifying the Euclidean time direction to size $1/T$, and in particular

$$R = \lim_{\vec{p} \rightarrow 0} \lim_{m \rightarrow 0} 2m\mathcal{G}_{PP}(\omega = 0, \vec{p}). \quad (7)$$

The Dirac delta in Eq. (6) indicates the presence of massless *quasi-particle* excitations in the spectrum, as long as its coefficient is nonzero. Similarly to the zero-temperature case, if \mathcal{G}_{PP} is sufficiently well-behaved in the chiral limit then $R = 0$, and spontaneous breaking of chiral symmetry by a finite Σ leads to massless excitations in the spectrum. This is the finite-temperature version of Goldstone's theorem (see [35] and references therein). As shown below in Section 4, localised near-zero modes can lead to a nonzero R , which can remove these Goldstone excitations from the spectrum. Again, a finite anomalous remnant indicates explicit breaking of chiral symmetry in the massless limit, so that the axial current is not conserved and Goldstone's theorem at finite temperature is evaded.

¹It is also assumed that there is no transport peak in the pseudoscalar channel. This is expected on general grounds, and supported by numerical lattice results (see [33]).

4. Localised modes and the pseudoscalar correlator

I now show that a nonzero R is generally found in the presence of a finite density of localised near-zero modes [34]. Since UV divergences play a very limited role, the argument can be carried out safely (and more simply) in the continuum. One starts from the bare pseudoscalar correlator $\langle P_B^a(x) P_B^b(0) \rangle$ at temperature T in a finite spatial volume V and for finite (bare) mass m_B ,² written in terms of a double sum over Dirac modes,

$$\langle P_B^a(x) P_B^b(0) \rangle = -\frac{\delta^{ab}}{2} \left\langle \sum_{n,n'} \frac{O_{n'n}^{\gamma 5}(x) O_{nn'}^{\gamma 5}(0)}{(i\lambda_n + m_B)(i\lambda_{n'} + m_B)} \right\rangle \equiv -\delta^{ab} \Pi_B(x). \quad (8)$$

Here $\not{D}\psi_n = i\lambda_n\psi_n$, with ψ_n obeying antiperiodic (resp. periodic) temporal (resp. spatial) boundary conditions and normalised to 1, $O_{nn'}^\Gamma(x) \equiv \sum_{c,A,B} \psi_{nA,c}(x)^* \Gamma_{AB} \psi_{n'B,c}(x)$, and a UV cutoff on $\lambda_{n,n'}$ is understood to be in place. After renormalisation of the mass, $m_B = Z_m m$, and of Π_B , $\Pi(x) = Z_m^2 [\Pi_B(x) - \text{CT}(x)]$, including the removal of the divergent contact terms CT, one can take the thermodynamic and chiral limit (in this order) to find the following expression for the coefficient of the $1/m$ divergence of $\Pi(x)$,

$$\lim_{m \rightarrow 0} 2m\Pi(x) = 2 \lim_{m \rightarrow 0} \int_0^{\frac{\mu}{m}} dz \left(\frac{C^1(mz; m; x)}{z^2 + 1} + \frac{(1 - z^2)C^{\gamma 5}(mz; m; x)}{(z^2 + 1)^2} \right), \quad (9)$$

$$C^\Gamma(\lambda; m; x) \equiv \langle \sum_n' \delta(\lambda - \lambda_n^R) O_{nn}^\Gamma(x) O_{nn}^\Gamma(0) \rangle, \quad (10)$$

where $\lambda_n^R = Z_m^{-1} \lambda_n$, $\sum_n' = \sum_{\lambda_n^R \neq 0}$, and μ is a fixed but arbitrary mass scale, which will eventually play no role. Exact zero modes have been dropped since they are negligible in the thermodynamic limit. Modes outside of a neighbourhood of $\lambda = 0$ also become negligible in the chiral limit, leading in particular to the absence of divergent contact terms.

The quantity in Eq. (9) can be nonvanishing only if C^Γ survives the thermodynamic limit, and here the localisation properties of the eigenmodes play a crucial role. In fact, using Schwarz inequality and translation invariance one can bound the eigenmode correlators entering C^Γ as follows,

$$\begin{aligned} |\langle O_{nn}^\Gamma(x) O_{nn}^\Gamma(0) \rangle| &\leq \langle \|\psi_n(x)\|^2 \|\psi_n(0)\|^2 \rangle \leq \frac{1}{2} \langle \|\psi_n(x)\|^4 + \|\psi_n(0)\|^4 \rangle \\ &= \frac{T}{V} \left\langle \int_T d^4x \|\psi_n(x)\|^4 \right\rangle = \frac{T}{V} \langle \text{IPR}_n \rangle. \end{aligned} \quad (11)$$

Making the dependence of C^Γ on V explicit by writing C_V^Γ , one then finds

$$|C_V^\Gamma(\lambda; m; x)| \leq \frac{T}{V} \langle \sum_n' \delta(\lambda - \lambda_n^R) \text{IPR}_n \rangle = \rho_V(\lambda) \overline{\text{IPR}}(\lambda), \quad (12)$$

where $\rho_V(\lambda) \equiv \frac{T}{V} \langle \sum_n' \delta(\lambda - \lambda_n^R) \rangle$ and $\overline{\text{IPR}}(\lambda) \equiv \frac{T}{V} \langle \sum_n' \delta(\lambda - \lambda_n^R) \text{IPR}_n \rangle / \rho_V(\lambda)$ are the spectral density at finite V and the average IPR computed locally in the spectrum, respectively. If modes near λ are supported in a region of size $O(V^{\alpha(\lambda)})$, one has $\overline{\text{IPR}}(\lambda) \sim V^{-\alpha(\lambda)}$, and so $C_V^\Gamma(\lambda; m; x) \rightarrow 0$

²The zero-temperature case is obtained by setting the calculation in a finite four-volume V_4 , replacing $T/V \rightarrow 1/V_4$ in the formulas below, and eventually taking the limit $V_4 \rightarrow \infty$.

in the thermodynamic limit unless $\alpha(\lambda) = 0$, i.e., unless modes near λ are localised. In the thermodynamic limit, $C^\Gamma(\lambda; m; x) = \lim_{V \rightarrow \infty} C_V^\Gamma(\lambda; m; x)$ is then nonzero only in spectral regions where localised modes are present.

The anomalous remnant R is now obtained by integrating Eq. (9) over Euclidean spacetime. Assuming that localised modes are present in the interval $[0, \lambda_c(m)]$, one obtains

$$R = - \int_T d^4x \lim_{m \rightarrow 0} 2m\Pi(x) = -\pi\xi \rho_{\text{loc}}(0), \quad (13)$$

where $\rho_{\text{loc}}(0)$ is the density of localised near-zero modes,

$$\rho_{\text{loc}}(0) \equiv \lim_{m \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \frac{T}{V} \sum_{n \in \text{loc}} \langle \delta(\lambda - \lambda_n^R) \rangle, \quad (14)$$

and

$$\xi \equiv \lim_{m \rightarrow 0} \frac{2}{\pi} \arctan \frac{\lambda_c(m)}{m} \quad (15)$$

is a function of the renormalisation-group invariant ratio $\frac{\lambda_c(m)}{m}$ in the chiral limit. In obtaining Eq. (13) one exploits the localised nature of the modes to exchange the order of integration, chiral limit, and thermodynamic limit, as well as the orthonormality of Dirac modes. As anticipated, R is proportional to the density of localised near-zero modes. The quantity $\xi \in [0, 1]$ depends on how the mobility edge scales in the chiral limit: $\xi = 0$ if it vanishes faster than m , $0 < \xi < 1$ if it vanishes like m , and $\xi = 1$ if it vanishes more slowly than m , including not vanishing at all. The arbitrary scale μ does not appear in the final expression, as expected.

5. Localised modes and Goldstone excitations

Using Eq. (13) and the Banks-Casher relation $\Sigma = -\pi\rho(0)$ [28], where $\rho(0)$ is the density of near-zero modes (localised or otherwise) in the chiral limit obtained from $\rho_V(\lambda)$ taking limits as in Eq. (14), one finds for the singular part of the spectral function in the chiral limit [34]

$$\lim_{\vec{p} \rightarrow 0} \lim_{m \rightarrow 0} \varrho^{AP}(\omega, \vec{p})|_{\text{singular}} = -2\pi[\Sigma - R]\delta(\omega) = 2\pi^2\rho(0) \left(1 - \xi \frac{\rho_{\text{loc}}(0)}{\rho(0)}\right) \delta(\omega). \quad (16)$$

One can now determine the fate of the Goldstone excitations. Since localised and delocalised modes usually do not coexist, one has $\frac{\rho_{\text{loc}}(0)}{\rho(0)} = 1$ or 0 depending on whether near-zero modes are localised or delocalised. There are four possible scenarios.

0. Near-zero modes are delocalised: Goldstone excitations are present as long as $\rho(0) \neq 0$. This is the standard scenario predicted by Goldstone's theorem.
1. Near-zero modes are localised and $\xi = 0$: Goldstone excitations are present as long as $\rho(0) \neq 0$, i.e., localisation of near-zero modes has no effect on the Goldstone excitations, and the same standard scenario is found.
2. Near-zero modes are localised and $0 < \xi < 1$: Goldstone excitations are present if $\rho(0) = \rho_{\text{loc}}(0) \neq 0$, although the coefficient of the Dirac delta is reduced compared to scenarios 0 and 1. This is qualitatively the same as the standard scenario, but differs from it quantitatively.
3. Near-zero modes are localised and $\xi = 1$: Goldstone excitations are absent even if $\rho(0) = \rho_{\text{loc}}(0) \neq 0$.

6. Conclusions

I have shown how the pseudoscalar-pseudoscalar correlator generally develops a $1/m$ divergence in the chiral limit in the presence of a finite density of localised near-zero modes. This divergence leads to a finite anomalous remnant that modifies the usual form of the axial nonsinglet Ward-Takahashi identity in the chiral limit, signaling that chiral symmetry is broken explicitly even in this limit. This indicates non-conservation of the axial-vector current, and so the inapplicability of Goldstone's theorem, both at zero and at finite temperature. Depending on the detailed behaviour of the mobility edge λ_c as a function of m , one can either recover the standard scenario with massless excitations, possibly up to a change in the coefficient of the singular term in the spectral function, or have Goldstone excitations removed from the spectrum.

So far, the presence of localised near-zero modes in the chiral limit has not been demonstrated explicitly in any model, although there are indications that it could be a feature of the chiral limit of QCD and of SU(3) gauge theory with $N_f = 2$ flavours of adjoint fermions. It would certainly be interesting to find a model with this property, especially if it realised a non-standard scenario for Goldstone modes (i.e., cases 2 and 3 above). It would also be interesting to work out the possible signatures in the finite-mass theory originating from the realisation of a non-standard scenario in the chiral limit.

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