

Remnants of large- N_f inhomogeneities in the 2-flavor chiral Gross-Neveu model

Julian J. Lenz and Michael Mandl*

*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena,
Fröbelstieg 1, D-07743 Jena, Germany*

E-mail: julian.johannes.lenz@uni-jena.de, michael.mandl@uni-jena.de

We study the $(1 + 1)$ -dimensional chiral Gross-Neveu model on the lattice. At finite density, analytic mean-field results predict the existence of inhomogeneous condensates breaking both chiral symmetry and spacetime symmetries spontaneously. We investigate the fate of these inhomogeneities for two flavors and find remnant structural order, albeit with a decaying amplitude. We also map out phase diagrams in the plane spanned by the chemical potential and temperature for different lattice spacings and physical volumes. Finally, we comment on the interpretation of our results in the light of various no-go theorems.

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*Speaker

1. Introduction

Despite tremendous efforts, only very little is known about the phase diagram of Quantum Chromodynamics (QCD), the theory governing the strong interactions, at high baryon density and low temperature. This is because in this region first-principles lattice methods cannot be applied due to the complex-action problem.

An alternative approach is to study effective theories or toy models, e.g. of the Nambu-Jona-Lasinio (NJL) [1] or Gross-Neveu (GN) [2] type. They are purely fermionic theories and, despite their apparent simplicity, share various important features with QCD, such as renormalizability (in low dimensions), asymptotic freedom, chiral symmetry and its potential spontaneous breakdown. It is the latter feature that we shall be concerned with in this contribution.

It has become clear [3–5] (see also [6] for a review) that the (1+1)-dimensional versions of a number of GN- or NJL-type models exhibit inhomogeneous condensates, indicating the breakdown of a combination of chiral symmetry and spacetime symmetries, at least for an infinite number of fermion flavors N_f (or on a mean-field level). Very recently it was revealed by a large-scale lattice study that in the conventional \mathbb{Z}_2 -symmetric GN model in 1 + 1 dimensions a similar observation also holds for finite N_f [7, 8]. In this work we present an analogous study of the $U(1)$ -symmetric chiral GN (cGN) model (see also [9]).

2. The chiral Gross-Neveu model

The cGN model is defined by its Lagrangian,

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \frac{g^2}{2N_f} \left((\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_* \psi)^2 \right), \quad (1)$$

where ψ implicitly combines N_f flavors of two-component spinors and $\gamma_* = i\gamma_0\gamma_1$ is the two-dimensional analogon of γ_5 . In addition to spacetime and flavor symmetries the theory is invariant under continuous axial $U(1)$ transformations,

$$\psi \rightarrow e^{i\alpha\gamma_*} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha\gamma_*}, \quad \alpha \in \mathbb{R}. \quad (2)$$

It is a common practice to trade the four-fermion terms in (1) for auxiliary scalar and pseudoscalar fields σ and π , giving the equivalent Lagrangian

$$\mathcal{L} = i\bar{\psi} (\not{\partial} + \sigma + i\gamma_*\pi + \mu\gamma_0) \psi + \frac{N_f}{2g^2} (\sigma^2 + \pi^2), \quad (3)$$

where we have also introduced a chemical potential μ in the usual way. The expectation values of these auxiliary fields are related via Dyson-Schwinger equations to the chiral and pseudoscalar condensates,

$$\langle \bar{\psi} \psi \rangle = \frac{iN_f}{g^2} \langle \sigma \rangle, \quad \langle \bar{\psi} \gamma_* \psi \rangle = \frac{N_f}{g^2} \langle \pi \rangle. \quad (4)$$

The cGN model has been solved analytically for finite μ and temperature T in the limit $N_f \rightarrow \infty$ [3, 4]. It shows a μ -independent critical temperature $T_c \approx 0.567\rho_0$, where $\rho_0 = \langle \rho \rangle_{T=0=\mu}$

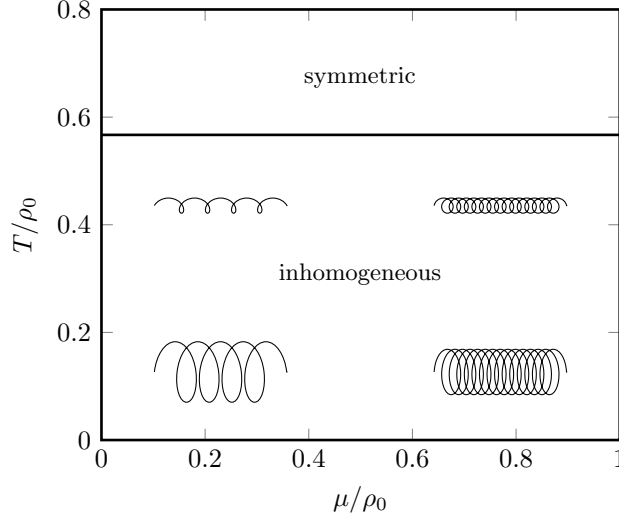


Figure 1: Large- N_f phase diagram showing a symmetric phase above and an inhomogeneous chiral spiral phase below T_c , see also [4].

and $\rho^2 = \sigma^2 + \pi^2$ [10]. Above T_c the theory is in a chirally symmetric phase with vanishing $\langle \bar{\psi}\psi \rangle$ and $\langle \bar{\psi}\gamma_*\psi \rangle$, while below T_c one finds inhomogeneous condensates,

$$\langle \bar{\psi}\psi \rangle(x) = \rho(T) \cos(2\mu x), \quad \langle \bar{\psi}\gamma_*\psi \rangle(x) = -\rho(T) \sin(2\mu x), \quad (5)$$

which together are called the *chiral spiral* [3]. Its amplitude $\rho(T)$ depends on the temperature but not on the chemical potential while its wave number $k = 2\mu$ is only μ -dependent. We depict the large- N_f phase diagram in the (μ, T) plane in Fig. 1.

This work, along with [9], serves to answer the question whether the inhomogeneous condensates are purely a large- N_f artifact or if instead there is a remnant structural order even at finite flavor number, where quantum fluctuations are no longer suppressed. For the \mathbb{Z}_2 -symmetric GN model this question has been answered in [7, 8], where the inhomogeneities were shown to survive the continuum limit even for $N_f = 2$. There are, however, numerous no-go theorems [11–13] that prohibit the spontaneous breaking of continuous symmetries in low dimensions. While the participation of external symmetries at finite density invites subtle discussions about their applicability, there is no doubt about their validity at vanishing chemical potential. All in all, the interpretation of these results is a rather delicate issue, c.f. [7].

3. Correlators and related quantities

We study the cGN model (3) on a finite two-dimensional spacetime lattice with N_t and N_s points in the temporal and spatial directions respectively, using the chiral SLAC derivative [14, 15] as the fermion discretization of choice. Since the previous works [7, 8] focused on the case $N_f = 8$, in which the results resembled the large- N_f findings in many ways, we shall be concerned with $N_f = 2$ in this contribution with the goal of better understanding the effect of fluctuations. In order

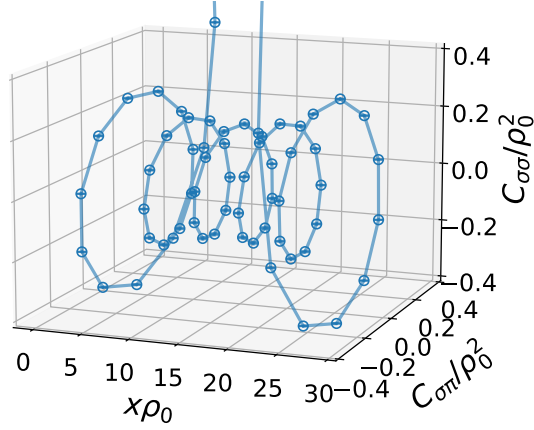


Figure 2: Chiral spiral on a lattice with $N_s = 63$, $T/\rho_0 \approx 0.030$, $\mu/\rho_0 \approx 0.88$ and $a\rho_0 \approx 0.46$.

to facilitate comparisons with large- N_f results we use the quantity

$$\rho_0 = \left\langle \sqrt{\sigma^2 + \pi^2} \right\rangle_{T \approx 0 = \mu} \quad (6)$$

to set the scale. We simulate lattices with different lattice spacings a as well as spatial extents $L = N_s a$ to study finite-size and discretization effects. We refer to previous works for details on the lattice setup [7, 9], implementation [16] and the exact scale-setting procedure [9], the latter of which is quite a subtle affair.

It has proven useful in the past to probe for inhomogeneous phases by studying the spatial correlation functions of the auxiliary fields,

$$\begin{aligned} C_{\sigma\sigma}(x) &= \frac{1}{N_t N_s} \sum_{t,y} \langle \sigma(t, y+x) \sigma(t, y) \rangle, \\ C_{\sigma\pi}(x) &= \frac{1}{N_t N_s} \sum_{t,y} \langle \sigma(t, y+x) \pi(t, y) \rangle, \end{aligned} \quad (7)$$

where the sums run over all lattice points. Their use avoids cancellations due to the exact preservation of all symmetries in the finite volume and instead pronounces the common structures of the generated Monte-Carlo configurations.

We show a 3D plot of typical spatial correlators we find at low $T = 1/N_t a$ and intermediate to high μ in Fig. 2. We see that there is indeed structural order at finite N_f . It is important to stress at this point that this is likely no perfect long-range order. Otherwise, that would imply spontaneous symmetry breaking which is prohibited by the no-go theorems mentioned above. Indeed, the amplitude of the chiral spiral decays with x (see also the left plot of Fig. 3), indicating some sort of *quasi-long-range* order.

Analytical studies at vanishing [17] and finite [9] density find a massive phase at finite temperature that becomes critical at $T = 0$. Thus, we expect the correlators to decay exponentially with a thermal mass that vanishes as $T \rightarrow 0$. To investigate this issue we show in Fig. 3 a comparison between the spatial correlation functions at low and high temperature. Indeed, while still oscillatory, the correlators decay more rapidly as one increases the temperature supporting the analytical

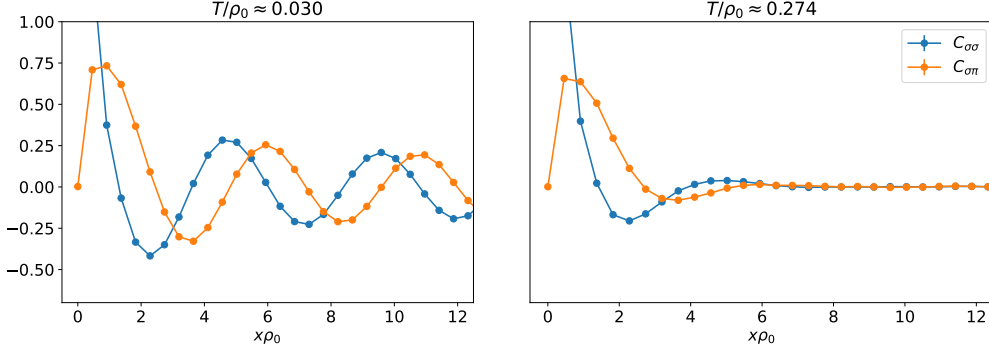


Figure 3: Spatial correlation functions $C_{\sigma\sigma}(x)$ and $C_{\sigma\pi}(x)$ on a lattice with $N_s = 63$, $\mu/\rho_0 \approx 0.88$ and $a\rho_0 \approx 0.46$ for two different temperatures.

claims. Our results are also consistent with the "quantum spin liquid" scenario discussed in [18]. We remark that this decrease of correlation functions does not occur in the large- N_f limit and is thus caused by fluctuations (see also [19]).

Our main interest in this contribution is to study the phase diagram of the (1 + 1)-dimensional cGN model at $N_f = 2$. To condense an extensive study of the correlators (7) we propose to study the two quantities

$$C_{\text{short}} = \min_x C_{\sigma\sigma}(x), \quad (8)$$

$$C_{\text{long}} = \min_x \sqrt{C_{\sigma\sigma}^2(x) + C_{\sigma\pi}^2(x)}. \quad (9)$$

In the limit of infinite flavor number non-vanishing values of these quantities would indicate spontaneous symmetry breaking. In this scenario, the sign of C_{short} (which was denoted as C_{min} in [7–9]) would discriminate inhomogeneous from homogeneous spontaneous symmetry breaking. For $N_f = 2$, where long-range order is most unlikely, one has to be more careful in their interpretation: Whenever C_{short} or C_{long} are positive (beyond noise), the system exhibits correlated patches of space that are of the order of the lattice volume. In that sense, correlations are at least quasi-long-ranged if the signal persists in the infinite-volume limit. A negative value of C_{short} indicates further that there are inhomogeneities on some scale. This scale, however, by no means has to be large, as is nicely illustrated by the right-hand plot of Fig. 3. In fact, negative values of C_{short} will always stem from the short-range behavior, no matter if correlations persist on larger scales or not, because the correlations are largest on short scales. We summarize this discussion in Tab. 1.

One should note that neither C_{short} nor C_{long} is a local observable and it is therefore questionable if they could be considered as legitimate order parameters. However, at finite N_f there is most likely no long-range order in the strict sense to be found.

4. Phase diagrams

We study phase diagrams in the (μ, T) plane using (8) and (9). In order to quantify finite-volume effects we performed an infinite-volume extrapolation at fixed lattice spacing and show the results in Fig. 4.

Table 1: Interpretation of the quantities (8) and (9) for $N_f = \infty$, with the possibility of spontaneous symmetry breaking, and for $N_f = 2$, where most likely no strict long-range order can exist.

	C_{short}		C_{long}	
	$N_f = \infty$	$N_f = 2$	$N_f = \infty$	$N_f = 2$
> 0	hom. broken	hom. (quasi-)long-ranged	broken (any kind)	(quasi-)long-ranged
≈ 0	symmetric	short-ranged	symmetric	short-ranged
< 0	inhom. broken	any inhomogeneities	–	–

First of all, there is a region where $C_{\text{short}} > 0$ and thus homogeneous configurations dominate, i.e. the red region in Fig. 4. Its extent in the μ -direction shrinks in the infinite-volume limit, which can be understood by recalling that the chiral spiral's wavelength is inversely proportional to μ . This means that for small but non-vanishing chemical potential smaller lattices are simply not large enough to fit in a full wavelength, thus leading to a suppression of inhomogeneous configurations. The large extent of the red region in the T -direction on the smallest lattice can also be seen to be a finite-size effect.

We furthermore observe that the boundary between the "short-ranged" ($C_{\text{short}} \approx 0$) and "inhomogeneous" ($C_{\text{short}} < 0$) regions remains essentially unchanged in the infinite-volume extrapolation. This is actually expected because the appearance of $C_{\text{short}} < 0$ only depends on the two scales introduced by the temperature and the chemical potential respectively, neither of which changes in the infinite-volume limit. More precisely, negative values of C_{short} occur whenever the preferred wavelength of the oscillatory (quasi-)condensates is smaller than the thermally induced finite correlation length. From their analytically known values in an expansion in $1/N_f$ (see [9]), it is easy to anticipate the shape of this boundary – at least on a qualitative level. In contrast, C_{long} becomes smaller and smaller in the infinite-volume limit since it probes the largest scales on which we do not expect correlations. This is indeed strong evidence against spontaneous symmetry breaking and fundamentally differs from what was observed in [7].

Finally, we study discretization effects via a continuum extrapolation at roughly fixed physical volume in Fig. 5. We find remnant structural order for three strongly decreasing lattice spacings. However, due to subtleties in the scale-setting procedure (see [9]) we refrain from drawing definite conclusions at this point and instead hope to revisit the continuum limit in the future. In related models in higher dimensions the continuum limit turned out to be crucial for removing misleading lattice artifacts [20]. We do emphasize, however, that our results are consistent with the persistence of remnant inhomogeneities as a approaches zero – in accordance with the observation that in lower dimensions the UV limit usually tends to be less problematic.

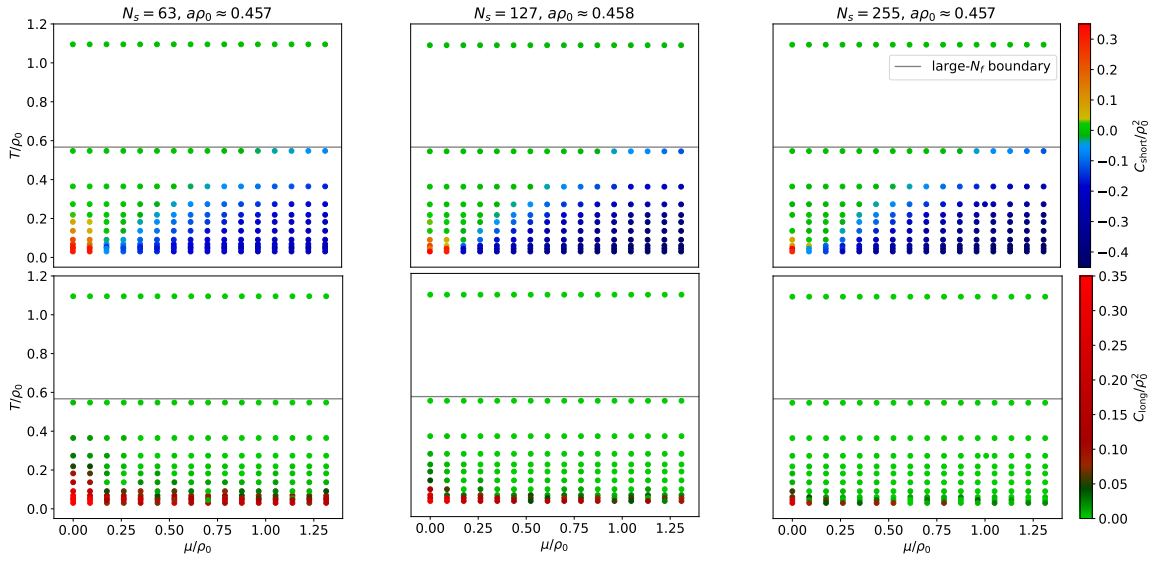


Figure 4: From left to right: infinite-volume extrapolation of the (μ, T) phase diagram at fixed lattice spacing; top row: using C_{short} (Eq. (8), taken from [9]); bottom row: using C_{long} (Eq. (9)).

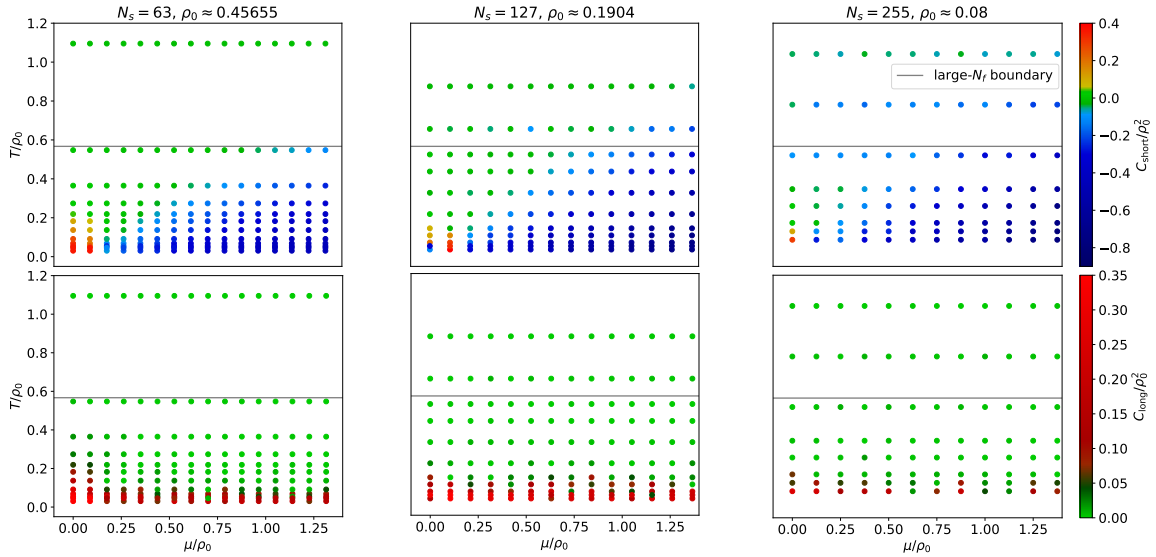


Figure 5: From left to right: continuum extrapolation of the (μ, T) phase diagram at roughly fixed physical volume; top row: using C_{short} (Eq. (8), taken from [9]); bottom row: using C_{long} (Eq. (9)).

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References

- [1] Y. Nambu and G. Jona-Lasinio, *Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I*, *Phys. Rev.* **1** (1961) 345.
- [2] D.J. Gross and A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, *Phys. Rev. D* **10** (1974) 3235.
- [3] V. Schön and M. Thies, *Emergence of the Skyrme crystal in Gross-Neveu and 't Hooft models at finite density*, *Phys. Rev. D* **62** (2000) 096002 [[arXiv:hep-th/0003195](#)].
- [4] V. Schön and M. Thies, *2D model field theories at finite temperature and density*, (2001) [[arXiv:hep-th/0008175](#)].
- [5] M. Thies and K. Urlichs, *Revised phase diagram of the Gross-Neveu model*, *Phys. Rev. D* **67** (2003) 125015 [[arXiv:hep-th/0302092](#)].
- [6] M. Thies, *From relativistic quantum fields to condensed matter and back again: updating the Gross-Neveu phase diagram*, *J. Phys. A* **39** (2006) 12707 [[arXiv:hep-th/0601049](#)].
- [7] J. Lenz, L. Pannullo, M. Wagner, B.H. Wellegehausen and A. Wipf, *Inhomogeneous phases in the Gross-Neveu model in 1+1 dimensions at finite number of flavors*, *Phys. Rev. D* **101** (2020) 094512 [[arXiv:2004.00295](#)].
- [8] J.J. Lenz, L. Pannullo, M. Wagner, B.H. Wellegehausen and A. Wipf, *Baryons in the Gross-Neveu model in 1+1 dimensions at finite number of flavors*, *Phys. Rev. D* **102** (2020) 114501 [[arXiv:2007.08382](#)].
- [9] J.J. Lenz, M. Mandl and A. Wipf, *Inhomogeneities in the 2-Flavor Chiral Gross-Neveu Model*, (2021) [[arXiv:2109.05525](#)].
- [10] G. Başar, G.V. Dunne and M. Thies, *Inhomogeneous condensates in the thermodynamics of the chiral NJL₂ model*, *Phys. Rev. D* **79** (2009) 105012 [[arXiv:0903.1868](#)].
- [11] N.D. Mermin and H. Wagner, *Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models*, *Phys. Rev. Lett.* **17** (1966) 1133.

- [12] P.C. Hohenberg, *Existence of Long-Range Order in One and Two Dimensions*, *Phys. Rev.* **158** (1967) 383.
- [13] S. Coleman, *There are no Goldstone bosons in two dimensions*, *Commun. Math. Phys.* **31** (1973) 259.
- [14] S.D. Drell, M. Weinstein and S. Yankielowicz, *Strong-coupling field theory. I. Variational approach to ϕ^4 theory*, *Phys. Rev. D* **14** (1976) 487.
- [15] G. Bergner, T. Kästner, S. Uhlmann and A. Wipf, *Low-dimensional supersymmetric lattice models*, *Annals Phys.* **323** (2008) 946 [[arXiv:0705.2212](#)].
- [16] J.J. Lenz, B.H. Wellegehausen and A. Wipf, *Absence of chiral symmetry breaking in Thirring models in 1 + 2 dimensions*, *Phys. Rev. D* **100** (2019) 054501 [[arXiv:1905.00137](#)].
- [17] J.O. Andersen, *Phase fluctuations in low-dimensional Gross-Neveu models*, (2005) [[arXiv:hep-th/0506261](#)].
- [18] R.D. Pisarski, A.M. Tsvelik and S. Valgushev, *How transverse thermal fluctuations disorder a condensate of chiral spirals into a quantum spin liquid*, *Phys. Rev. D* **102** (2020) 016015 [[arXiv:2005.10259](#)].
- [19] E. Witten, *Chiral symmetry, the $1/N$ expansion, and the $SU(N)_2$ Thirring model*, *Nucl. Phys. B* **145** (1978) 110.
- [20] M. Buballa, L. Kurth, M. Wagner and M. Winstel, *Regulator dependence of inhomogeneous phases in the (2+1)-dimensional Gross-Neveu model*, *Phys. Rev. D* **103** (2021) 034503 [[arXiv:2012.09588](#)].