PROCEEDINGS OF SCIENCE

Non-perturbative renormalization of the flavour singlet local vector current with O(a)-improved Wilson fermions

Matteo Bresciani,[a,b] **Mattia Dalla Brida,**[c] **Leonardo Giusti,**[a,b] **Michele Pepe**[b,*] **and Federico Rapuano**[a,b]

[a] *Dipartimento di Fisica, Universitá di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

[b] *INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

[c] *Theoretical Physics Department, CERN 1211 Geneva 23, Switzerland*

E-mail: Matteo.Bresciani@mib.infn.it, mattia.dalla.brida@cern.ch, Leonardo.Giusti@mib.infn.it, Michele.Pepe@mib.infn.it, Federico.Rapuano@mib.infn.it

We determine non-perturbatively the renormalization constant of the flavour singlet local vector current with O(a)-improved Wilson fermions. The renormalization constant is fixed by comparing the expectation values (one-point function) of the local vector current and of the conserved one in thermal QCD in a moving reference frame with a non-zero imaginary chemical potential and in the chiral limit. We implement the method in QCD with 3 flavours discretized by the standard Wilson action for gluons and the non-perturbatively O(a)-improved Wilson fermions. By carrying out extensive numerical simulations, the renormalization constant is determined with a permille precision for values of the bare coupling constant in the range $0.52 \leq g_0^2 \leq 1.13$.

The 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021 Zoom/Gather@Massachusetts Institute of Technology

*Speaker

PoS(LATTICE2021)421

1. Introduction

When studying QCD on the lattice one usually faces the problem of computing renormalization constants in order to extract physically relevant information from the expectation values of a lattice operator O. Moreover, when O is renormalized, it may mix with other operators O_k that have the same symmetry transformation rules as O and have the same canonical dimension or smaller

$$O^R = Z_O O + \sum_k Z_k O_k \qquad \text{with} \qquad \dim(O_k) \leq \dim(O) \tag{1}$$

where O^R is the renormalized operator, Z_O is the renormalization constant of the bare lattice operator O and Z_k are the renormalization constants that characterize its mixing with the other bare lattice operators O_k. Various renormalization schemes have been proposed and used like the RI-MOM scheme [1], the Schrödinger Functional scheme [2] and the Wilson flow scheme [3]. Another scheme that has been recently proposed and that has turned out to be a very convenient choice in several cases, is based on considering the quantum theory at finite temperature and in setting shifted boundary conditions [4, 5] along the temporal direction.

This framework has been used for the first time for the non-perturbative calculation of the renormalization constants of the energy-momentum tensor in $SU(3)$ Yang-Mills theory [6]. The generalization to QCD [7] has been possible by introducing a twist phase for fermions at the boundaries in addition to the shift. Before addressing the challenging numerical problem of renormalizing the QCD energy-momentum tensor, we have considered the simpler task of computing the renormalization constant of the flavour singlet local vector current: a quantity that has never been calculated before.

The calculation of the renormalization constants can be a demanding numerical problem since one often has to measure correlation functions of two or more operators at a physical distance. The computation is even more challenging for singlet operators due to contributions coming from the disconnected contraction of Dirac indices, which are characterized by the large statistical fluctuations of the vacuum. A unique advantage of introducing non-trivial boundary conditions is that one needs to measure one-point functions, attaining a higher accuracy at a cheaper numerical cost. For the moment, the scheme of shifted boundary conditions has been used to renormalize lattice operators that are related to symmetries of the theory and the application to other operators deserve further investigations.

In this paper we present the results of the numerical simulations that we have carried out with shifted boundary conditions to compute the renormalization constant of the flavour singlet local vector current. Although we do not discuss the issue here, our approach can be used in a very similar way also for the non-singlet case. The renormalization of the local vector current on the lattice has been a topic of interest for quite some time [8, 9] until recent investigations [10]. In particular, the flavour non-singlet vector current is an important observable for its relevance in the calculation of the Hadronic Vacuum Polarization contribution to the anomalous magnetic moment of the muon [11].

2. Thermal Lattice QCD in a moving frame

The study of QCD at finite temperature is usually performed to investigate thermodynamical features of the theory like, for instance, the pressure, the entropy density, the energy density as well as screening masses, trasport coefficients or other physically interesting quantities. Instead, in this paper we consider thermal QCD as a numerically efficient framework for computing the renormalization constant of the vector current on the lattice: that quantity is related to the specific definitions of the operator and of the action in the lattice regularization and not to physics.

We study QCD with 3 flavours of massless quarks on the lattice and we consider the Wilson plaquette action for the gauge sector and the O(a)-improved clover definition for Wilson fermions. We formulate the theory in a moving reference frame [4, 5] that corresponds to introducing a spatial shift $\boldsymbol{\xi}$ for the fields when setting the boundary conditions in the temporal direction. For conventions and notations as well as for the detailed definition of the action, we refer to the Appendices A and C of reference [7]. The gauge fields and the quark and the anti-quark fields satisfy the following boundary conditions

$$U_\mu(x_0', \boldsymbol{x}) = U_\mu(x_0, \boldsymbol{x}') \; ; \quad \psi(x_0', \boldsymbol{x}) = -e^{i\theta_0}\,\psi(x_0, \boldsymbol{x}') \; ; \quad \overline{\psi}(x_0', \boldsymbol{x}) = -e^{-i\theta_0}\,\overline{\psi}(x_0, \boldsymbol{x}') \tag{2}$$

respectively, where $x_0' = x_0 + L_0$ and $\boldsymbol{x}' = \boldsymbol{x} - L_0\boldsymbol{\xi}$; all fields are periodic in the spatial directions. In the above equations we have considered a more general form for shifted boundary conditions in which the fermionic fields pick up also a non trivial phase θ_0 in addition to the usual antiperiodicity [7]. By a change of variables that phase can be rewritten as an imaginary chemical potential [12] and it is interesting to note that there is an effective $2\pi/3$ periodicity of the free energy due to the mixing of θ_0 with the $\mathbb{Z}_3$ center symmetry of the $SU(3)$ pure gauge sector [13].

The non-anomalous Abelian part of chiral symmetry is not affected by the lattice regularization and it is exactly conserved also at finite lattice spacing. As a consequence, there is a flavour singlet conserved lattice operator that describes it and which is given by

$$V_\mu^c(x) = \frac{1}{2}\Big[\overline{\psi}(x + a\hat{\mu})U_\mu^\dagger(x)(\gamma_\mu + 1)\psi(x) + \overline{\psi}(x)U_\mu(x)(\gamma_\mu - 1)\psi(x + a\hat{\mu})\Big]. \tag{3}$$

Since this operator represents a conserved current on the lattice, it has a unit renormalization constant and it approaches the flavour singlet continuum vector current in the limit of vanishing lattice spacing $a \to 0$. Other definitions of the flavour singlet vector current on the lattice can also be studied like, for instance, the one that more closely resembles the continuum definition

$$V_\mu^l(x) = \overline{\psi}(x)\gamma_\mu\psi(x). \tag{4}$$

Although this latter definition requires to compute a renormalization constant Z_V of the lattice operator, it has the appealing numerical features of being ultra-local – i.e. not involving fields on different lattice points as $V_\mu^c(x)$ does – and of having smaller statistical fluctuations. Moreover, the possibility of considering a local and a conserved definition of a given operator may be useful in some cases and it can be numerically convenient to consider both operators when computing multi-point correlation functions [16]. In the thermodynamic limit, the phase θ_0 is related to the temporal component V_0^c of the conserved current by

$$\langle V_0^c \rangle = -iL_0 \frac{\partial}{\partial \theta_0} f \tag{5}$$

where f is the free energy density. The expectation value of the temporal component of the current vanishes both at zero and finite temperature, however, that is no longer the case when θ_0 takes non vanishing values. We can now define the renormalization constant of the local current as

$$Z_V(g_0^2) = \lim_{a/L_0 \to 0} \frac{\langle V_0^c \rangle_{\theta_0}}{\langle V_0^l \rangle_{\theta_0}} \tag{6}$$

where g_0^2 is the bare coupling and the notation $\langle \cdot \rangle_{\theta_0}$ means that a non vanishing value of the fermionic phase has to be considered. In the above definition we have suppressed the dependence on x thanks to translation invariance of the expectation values. It is interesting to note that the eq. (6) is a legitimate definition for $Z_V(g_0^2)$ also for usual periodic boundary conditions, namely when there is no shift. However, we observe that when we consider the shift $\boldsymbol{\xi} = (1, 0, 0)$, lattice artifacts turn out to be particularly small.

In figure 1 we show $Z_V^{(0)}$, the value of $Z_V(g_0^2)$ at tree-level in perturbation theory, as a function of θ_0 and for several values of the lattice temporal extent L_0/a

$$Z_V^{(0)}\left(\frac{L_0}{a}\right) = \frac{\langle V_0^{c\,(0)} \rangle_{\theta_0}}{\langle V_0^{l\,(0)} \rangle_{\theta_0}} = 1 + O\left(\left(\frac{a}{L_0}\right)^2\right) \tag{7}$$

where the label (0) means that the computation is performed at tree-level in perturbation theory. Thus, when L_0/a goes large, $Z_V^{(0)}$ approaches the asymptotic unit value and the deviations from 1 are tree-level lattice artifacts. The left panel displays the results for usual periodic boundary conditions – i.e. no shift – while in the right panel we see the data for shifted boundary conditions with shift $\boldsymbol{\xi} = (1, 0, 0)$. We observe that lattice artifacts are one order of magnitude smaller in the latter case: hence, even if it is not necessary, it is numerically very convenient to perform numerical simulations with shifted boundary conditions. We noticed a similar behaviour also in the $SU(3)$ Yang-Mills theory where we found small lattice artifacts when measuring the entropy density with shifted boundary conditions with shift $\boldsymbol{\xi} = (1, 0, 0)$ [14, 15].

The local and the conserved currents – given by eqs. (3) and (4) respectively – can be improved in order to reduce the relevance of the lattice artifacts; under renormalization they mix, in the chiral limit, with a single dimension 4 operator related to the tensor current [17] and we have the following O(a)-improved definiton of the lattice operators

$$\hat{V}_\mu^{c,l}(x) = V_\mu^{c,l}(x) + a\, c_V^{c,l}\, \partial_\nu \left(\frac{1}{2}\overline{\psi}(x)\, [\gamma_\mu, \gamma_\nu]\, \psi(x)\right) + O(a^2) \tag{8}$$

where $c_V^{c,l}$ are numerical coefficients that accomplish the non-perturbative improvement when properly tuned. Thanks to translation invariance, the expectation value of the second term on the r.h.s. vanishes and, hence, both the expectation values of the local and the conserved vector currents are automatically O(a)-improved.

The tree-level calculation $Z_V^{(0)}(L_0/a)$ allows to introduce a tree-level improved definiton of the renormalization constant of the local vector current as follows

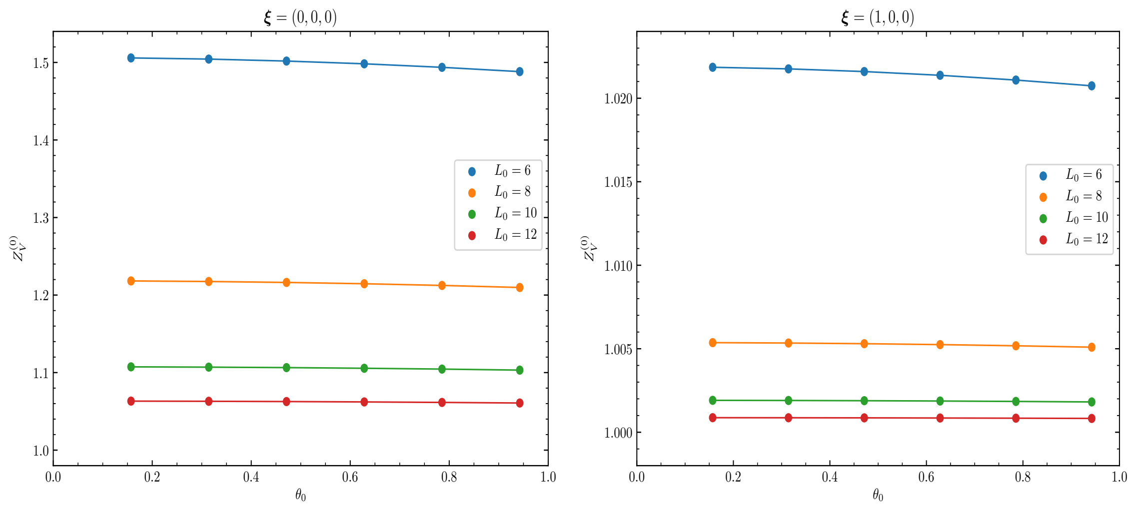

Figure 1: The lattice artifacts of the renormalization constant of the flavour singlet local vector current at tree-level in perturbation theory as a function of θ_0. Values for various sizes L_0/a of the lattice in the temporal direction are shown: the left panel refers to the case of periodic boundary conditions (no shift) and the right one to shifted boundary conditions with shift $\boldsymbol{\xi} = (1, 0, 0)$.

$$Z_V(g_0^2) = \lim_{a/L_0 \to 0} \left[\frac{\langle V_0^c \rangle}{\langle V_0^l \rangle} + 1 - Z_V^{(0)}(L_0/a) \right] \tag{9}$$

where we left unchanged the symbol with respect to the unimproved definition. We have also completed the calculation of $Z_V(g_0^2)$ at 1-loop order in perturbation theory and we plan to further improve the definiton of the renormalization constant in a forthcoming paper.

3. The numerical study

In this section we present the study that we have performed for the non-perturbative computation of the renormalization constant of the flavour singlet local vector current in QCD. The Monte Carlo simulations have been carried out with 3 flavours of massless O(a)-improved Wilson fermions and with the Wilson plaquette gauge action. We have considered 7 values of the bare gauge coupling $g_0^2 = 6/\beta$ in the range $[0.52, 1.13]$: $\beta = 5.3$, 5.65, 6.0433, 6.6096, 7.6042, 8.8727 and 11.5. For each one of those values we have run numerical simulations on lattices with size $96^3 \times L_0/a$ with $L_0/a = 4, 6, 8, 10$; we have chosen the value $\theta_0 = \frac{\pi}{6}$ for the fermionic phase and $\boldsymbol{\xi} = (1, 0, 0)$ for the shift. The critical value of the hopping parameter has been determined from ref. [18] for the two smallest and the largest values of β while for the other 4 values we have used the results of [19]. A statistics of 100 trajectories has been collected for $L_0/a = 4$ and 6 while for $L_0/a = 8$ and 10 we have generated 400 and 1000 trajectories, respectively; each trajectory is 2 Molecular Dynamics Units long. The autocorrelation time is less than 2.

In figure 2 we show the extrapolation to $a/L_0 \to 0$ at fixed value of the bare gauge coupling of the tree-level improved definition of $Z_V(g_0^2)$ for the 7 considered values of g_0^2. Linear fits in

$(a/L_0)^2$ provide a very good description of the numerical data and we obtain a final accuracy of 1% or less on the extrapolated values.

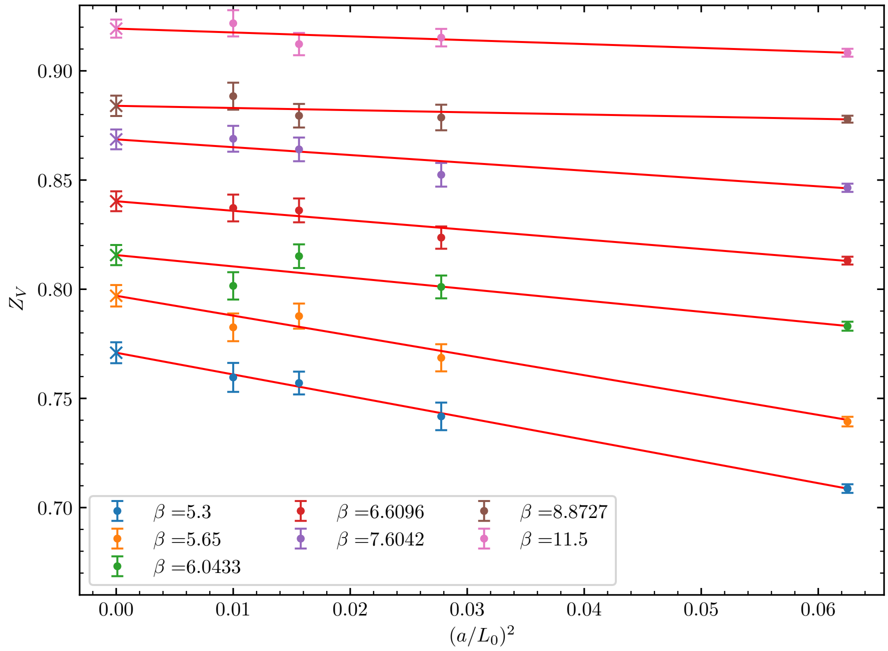

Figure 2: Extrapolations to $a/L_0 \to 0$ of the tree-level improved renormalization constant $Z_V(g_0^2)$ at the 7 values of the bare gauge coupling considered in this study.

The renormalization constant of the flavour singlet local vector current has been computed at 2-loop order in perturbation theory [20]. In figure 3 we compare the results of our non-perturbative calculation with the perturbative formula. The symbols are the data from our Monte Carlo simulations and the continuous line is the 2-loop perturbative formula: it shows up working well up to $g_0^2 \simeq 0.9$ with about a 1% accuracy.

4. Conclusions

In this study we perform the non-perturbative calculation of the renormalization constant $Z_V(g_0^2)$ of the flavour singlet local vector current of QCD with 3 flavours of massless quarks on the lattice. We show that by considering a non vanishing fermionic phase θ_0 – corresponding to an imaginary chemical potential – the renormalization constant $Z_V(g_0^2)$ can be efficiently computed by the ratio of the expectation value of one-point functions of the conserved and of the local vector currents. We have worked in the framework of shifted boundary conditions: although, in this case, one could also consider usual periodic boundary conditions, shifted ones turn out to be a more convenient choice since lattice artifacts are much smaller. Our non-perturbative results are well described by the 2-loop perturbative formula for $g_0^2 \leq 0.9$: we note that this very good agreement is quite remarkable, suggesting that both higher perturbative orders and residual discretization errors are very small.

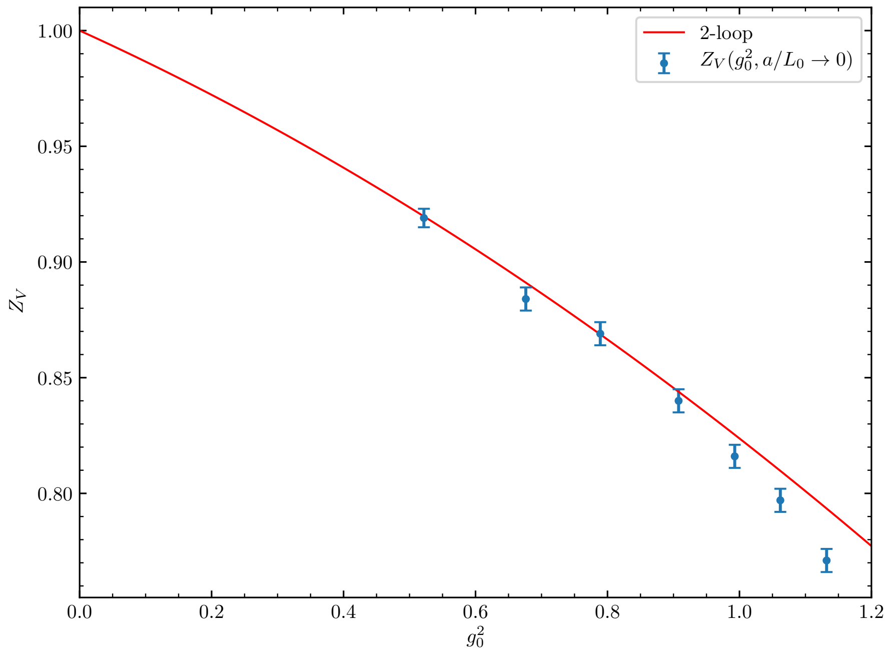

Figure 3: Comparison between the non-perturbative calculation of $Z_V(g_0^2)$ (symbols) and the 2-loop perturbative result (continuous line).

The method we exploited here to compute the renormalization constant of the flavour singlet local vector current can be applied also to the flavour non-singlet case with appropriate modifications. This study is the first application of the scheme of shifted boundary conditions to renormalize lattice operators in QCD; work is in progress to compute in this scheme the renormalization constants of the energy-momentum tensor.

Acknowledgement

We acknowledge PRACE for awarding us access to the HPC system MareNostrum4 at the Barcelona Supercomputing Center (Proposals n. 2018194651 and 2021240051). We also thanks CINECA for providing us with computer-time on Marconi (CINECA-INFN, CINECA-Bicocca agreements). The R&D has been carried out on the PC clusters Wilson and Knuth at Milano-Bicocca. We thank all these institutions for the technical support. Finally we acknowledge partial support by the INFN project “High performance data network”.

References

[1] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa and A. Vladikas, Nucl. Phys. B **445** (1995), 81-108.

[2] M. Luscher, R. Narayanan, P. Weisz and U. Wolff, Nucl. Phys. B **384** (1992), 168-228.

[3] M. Lüscher, JHEP **08** (2010), 071 [erratum: JHEP **03** (2014), 092].

[4] L. Giusti and H. B. Meyer, Phys. Rev. Lett. **106** (2011), 131601.

[5] L. Giusti and H. B. Meyer, JHEP **01** (2013), 140

[6] L. Giusti and M. Pepe, Phys. Rev. D **91** (2015), 114504.

[7] M. Dalla Brida, L. Giusti and M. Pepe, JHEP **04** (2020), 043.

[8] M. Bochicchio, L. Maiani, G. Martinelli, G. C. Rossi and M. Testa, Nucl. Phys. B **262** (1985), 331.

[9] G. Martinelli, C. T. Sachrajda and A. Vladikas, Nucl. Phys. B **358** (1991), 212-227.

[10] M. Dalla Brida, T. Korzec, S. Sint and P. Vilaseca, Eur. Phys. J. C **79** (2019) no.1, 23.

[11] T. Blum, Phys. Rev. Lett. **91** (2003), 052001.

[12] P. Hasenfratz and F. Karsch, Phys. Lett. B **125** (1983), 308-310.

[13] A. Roberge and N. Weiss, Nucl. Phys. B **275** (1986), 734-745.

[14] L. Giusti and M. Pepe, Phys. Rev. Lett. **113** (2014), 031601.

[15] L. Giusti and M. Pepe, Phys. Lett. B **769** (2017), 385-390.

[16] A. Gérardin, J. Green, O. Gryniuk, G. von Hippel, H. B. Meyer, V. Pascalutsa and H. Wittig, Phys. Rev. D **98** (2018) no.7, 074501.

[17] S. Sint and P. Weisz, Nucl. Phys. B **502** (1997), 251-268.

[18] N. Yamada *et al.* [JLQCD and CP-PACS], Phys. Rev. D **71** (2005), 054505.

[19] **ALPHA** Collaboration, P. Fritzsch and T. Korzec, internal notes ALPHA Collaboration, in preparation for publication.

[20] A. Skouroupathis and H. Panagopoulos, Phys. Rev. D **79** (2009), 094508.

PoS(LATTICE2021)421